Improved Energy Estimates for Interior Penalty, Constrained and Discontinuous Galerkin Methods for Elliptical Problems

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Part I. Improved Energy Estimates for Interior Penalty, Constrained and Discontinuous Galerkin Methods for Elliptic Problems

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Three Galerkin methods using discontinuous approximation spaces are introduced to solve elliptic problems. The underlying bilinear form for all three methods is the same and is nonsymmetric. In one case, a penalty is added to the form and in another, a constraint on jumps on each face of the triangulation. All three methods are locally conservative and the third one is not restricted. Optimal *a priori hp* error estimates are derived for all three procedures.
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1. Introduction

Over the last two decades there has been a collection of papers devoted to the use of approximation spaces with weak continuity for finite element approximations to elliptic and parabolic problems. The motivation for developing these methods was the flexibility afforded by local approximation spaces. These approaches allow meshes which are more general in their construction and degree of nonuniformity both in time and space than is permitted by the more conventional finite element methods. In general, numerical methods defined for discontinuous spaces have less numerical diffusion/dispersion and provide more accurate local approximations for problems with rough solutions. Another advantage that has

recently become apparent is the application of domain decomposition algorithms for the discrete solution.

In this paper, we discuss three numerical algorithms for elliptic problems which employ discontinuous approximation spaces. The three methods are called the nonsymmetric interior penalty Galerkin method (NIPG), the nonsymmetric constrained Galerkin (NCG) method, and the discontinuous Galerkin(DG) method. The three algorithms are closely related in that the underlying bilinear form for all three is the same and is nonsymmetric. Moreover, for all three methods, one can employ an unusual space $\mathbb{P}_k$ on quadrilaterals which have dimension substantially lower than $\mathbb{Q}_k$. $\mathbb{P}_k$ is the set of polynomials in two variables of total degree k, and $\mathbb{Q}_k$ is the set of polynomials of degree k in *each* variable. In addition, in all three methods, the error for the mass conservation can be retrieved element by element. In that sense, all three methods are locally mass conservative. The main advantage of the DG method is that the error in the mass conservation is zero.

In the NIPG method the bilinear form of the interior penalty Galerkin method treated by Douglas and Dupont [4], Wheeler [5], Arnold [6], Darlow and Wheeler [7] is modified. In this paper, an optimal hp error estimate is obtained in H^1 and in L^2. In particular the NIPG method only requires a positive penalty rather than one bounded below by a problem dependent constant as in the proofs described in [5].

The second approach is based on constraining the approximation spaces: jumps on each edge of the triangulation are required to have integral average zero. Here optimal hp estimates in H^1 and L^2 are derived.

The DG method with this bilinear form was first introduced by Baumann and Oden in [3],[9]. In [3], Baumann showed that the method is elementwise conservative and he proved a stability result in one dimension for polynomials of at least degree three. Numerical experiments showed that the method is robust and gives high-order accuracy where the solution is smooth. In this paper, theoretical optimal results are obtained for the DG method in H^1 for $n = 2$ and suboptimal for $n = 3$. To our knowledge these results represent the first convergence results for DG in higher dimensions.

This paper consists of four additional sections. In §2, notation and problem definition and formulation of the three methods are described. In §3, §4 and §5, the proofs of the error estimates of the three methods described in §2 are respectively given. Conclusions are described in the last section. Part II of this

paper describes computational results with the DG method.

2. Definition of the Problem, Methods

2.1. Notation and Approximation Properties

Let Ω be a polygonal domain in $I\!R^n$, $n = 1, 2, 3$ and let $\mathcal{E}_h = \{E_1, E_2, \ldots, E_{N_h}\}$ be a nondegenerate quasiuniform subdivision of Ω, where E_j is a triangle or a quadrilateral. The nondegeneracy requirement is that there exists $\rho > 0$ such that if $h_j = \mathrm{diam}(E_j)$, then E_j contains a ball of radius ρh_j in its interior. Let $h = \max\{h_j, \quad j = 1 \ldots N_h\}$. The quasiuniformity requirement is that there is $\tau > 0$ such that $\frac{h}{h_j} \leq \tau$ for all $j \in \{1, \ldots, N_h\}$. We denote the edges of the polygon by $\left\{e_1, e_2, \ldots, e_{P_h}, e_{P_{h+1}}, \ldots, e_{M_h}\right\}$ where $e_k \subset \Omega, 1 \leq k \leq P_h$,and $e_k \subset \partial\Omega$, $P_h + 1 \leq k \leq M_h$. With each edge e_k , we associate a unit normal vector ν_k. For $k > P_h$, ν_k is taken to be the unit outward vector normal to $\partial\Omega$. For $s \geq 0$, let

$$H^s(\mathcal{E}_h) = \{v \in L^2(\Omega) : v|_{E_j} \in H^s(E_j), j = 1 \ldots N_h\}.$$

We now define the average and the jump for $\phi \in H^s(\mathcal{E}_h)$, $s > \frac{1}{2}$. Let $1 \leq k \leq P_h$. For $e_k = \partial E_i \cap \partial E_j$ with ν_k exterior to E_i, set

$$\{\phi\} = \frac{1}{2}(\phi|_{E_i})|_{e_k} + \frac{1}{2}(\phi|_{E_j})|_{e_k}, \quad [\phi] = (\phi|_{E_i})|_{e_k} - (\phi|_{E_j})|_{e_k}.$$

We consider $K = (k_{ij})_{1 \leq i,j \leq 2}$, $k_{ij} \in L^\infty(\Omega), \alpha \in L^\infty(\Omega)$ where

$$0 < k_0 \leq k_{ij}(x) \leq k_1 < \infty, \quad 0 \leq \alpha(x) \quad \forall x \in \Omega.$$

The L^2 inner product is denoted by $(.,.)$. The usual Sobolev norm on $E \subset I\!R^n$ is denoted by $\|\cdot\|_{m,E}$. We define the following broken norms for m positive integer:

$$|||\Phi|||_m = \left(\sum_{j=1}^{N_h} \|\Phi\|_{m,E_j}^2\right)^{\frac{1}{2}}.$$

Let r be a positive integer. The finite element subspace is taken to be

$$\mathcal{D}_r(\mathcal{E}_h) = \prod_{j=1}^{N_h} P_r(E_j),$$

where $P_r(E_j)$ denotes the set of polynomials of (total) degree less than or equal to r on E_j.

The following *hp* approximation properties as defined in [1] and [2] are used: Let $E_j \in \mathcal{E}_h$ and $\phi \in H^s(E_j)$. Then there exists a constant C depending on s, τ, ρ but independent of ϕ, r and h and a sequence $z_r^h \in \mathbb{P}_r(E_j),\ r = 1, 2, \ldots$ such that for any $0 \leq q \leq s$

$$\|\phi - z_r^h\|_{q,E_j} \leq C \frac{h_j^{\mu-q}}{r^{s-q}} \|\phi\|_{s,E_j}, \qquad s \geq 0,$$

$$\|\phi - z_r^h\|_{0,\gamma_i} \leq C \frac{h_j^{\mu-\frac{1}{2}}}{r^{s-\frac{1}{2}}} \|\phi\|_{s,E_j}, \qquad s > \frac{1}{2},$$

where $\mu = \min(r+1, s)$ and $\gamma_i \subset \partial E_j$. Using the same technique as in [1], it can be shown that we have the additional approximation result:

$$\|\phi - z_r^h\|_{1,\gamma_i} \leq C \frac{h_j^{\mu-\frac{3}{2}}}{r^{s-\frac{3}{2}}} \|\phi\|_{s,E_j}, \qquad s > \frac{3}{2}.$$

As a corollary of the above results, the following global approximation property is obtained. Let $\phi \in H^s(\Omega)$. Let $\mathcal{E}_h$ be the subdivision described above of Ω. There exists $z_r^h \in \mathcal{D}_r(\mathcal{E}_h) \cap C^0$ such that for any $0 \leq q \leq s$,

$$|||\phi - z_r^h|||_{q,\Omega} \leq C \frac{h^{\mu-q}}{r^{s-q}} \|\phi\|_{s,\Omega}, \tag{2.1}$$

where $\mu = \min(r+1, s)$ and C is independent of ϕ, r, h and $\mathcal{E}_h$. Note that this result also holds if $z_h^r \in \mathcal{D}_r(\mathcal{E}_h)$.

2.2. Problem and Nonsymmetric Bilinear Form

The following elliptic problem is considered:

$$-\nabla \cdot (K\nabla p) + \alpha p = f \quad \text{in} \quad \Omega, \tag{2.2a}$$

$$p = p_0 \quad \text{on} \quad \Gamma_D, \tag{2.2b}$$

$$K\nabla p \cdot \nu_N = g \quad \text{on} \quad \Gamma_N, \tag{2.2c}$$

where the boundary of the domain, $\partial\Omega$, is the union of two disjoint sets Γ_D and Γ_N. We denote ν_D (respectively ν_N) the unit normal vector to each edge of Γ_D (respectively Γ_N) exterior to Ω.

If we assume that k_{ij} is Lipschitz continuous and $f \in L^2(\Omega)$, then the problem (2.1) has a unique solution in $H^1(\Omega)$ when $\Gamma_D \neq \emptyset$. On the other hand,

when $\partial\Omega = \Gamma_N$, the problem (2.2) has a solution in $H^1(\Omega)$ which is unique up to an additive constant, if f satisfies $\int_\Omega f = 0$. There is a constant C such that

$$\|p\|_{2,\Omega} \leq C\{\|f\|_{0,\Omega} + \|p\|_{0,\Omega}\},$$

for all $p \in H^2(\Omega)$ solution of (2.2).

We will consider the non-symmetric bilinear form:

$$\begin{aligned} a_{NS}(\psi,\phi) = & \sum_{j=1}^{N_h} \int_{E_j} (K\nabla\psi\nabla\phi + \alpha\psi\phi) \\ & - \sum_{k=1}^{P_h} \int_{e_k} \{K\nabla\psi \cdot \nu_k\}\,[\phi] + \sum_{k=1}^{P_h} \int_{e_k} \{K\nabla\phi \cdot \nu_k\}\,[\psi] \\ & - \int_{\Gamma_D} (K\nabla\psi \cdot \nu_D)\phi + \int_{\Gamma_D} (K\nabla\phi \cdot \nu_D)\psi. \end{aligned}$$

We define the linear form:

$$L(\phi) = (f,\phi) + \int_{\Gamma_D} (K\nabla\phi \cdot \nu_D)p_0 + \int_{\Gamma_N} \phi g$$

2.3. Methods

First, we introduce the following interior penalty term:

$$J_0^{\sigma,\beta}(\phi,\psi) = \sum_{k=1}^{P_h} \frac{\sigma_k}{|e_k|^\beta} \int_{e_k} [\phi][\psi],$$

where σ_k is a real positive number associated to the interior edge e_k, where $|e_k|$ denotes the length of e_k and where $\beta \geq 1$ is a real number. The Galerkin approximation $P^{NIPG} \in \mathcal{D}_r(\mathcal{E}_h)$ solves the following discrete problem:

$$a_{NS}(P^{NIPG}, v) + J_0^{\sigma,\beta}(P^{NIPG}, v) = L(v), \qquad \forall v \in \mathcal{D}_r(\mathcal{E}_h). \tag{2.3}$$

Lemma 2.1. Suppose that f is smooth and that the solution p to (2.1) exists and $K\nabla p \in H^1(\mathcal{E}_h)$. Then p satisfies the formulation (2.3). The converse is also true if we assume sufficiently smoothness on p.

Proof. First, suppose that p is a smooth solution of (2.1). Let v be an element in $\mathcal{D}_r(\mathcal{E}_h)$. We multiply the first equation by v, integrate on E_j and sum over all j.

$$\sum_{j=1}^{N_h} \int_{E_j} K\nabla p \nabla v + \alpha p v - \sum_{k=1}^{M_h} \int_{e_k} \{K\nabla p \cdot \nu_k\}[v] - \int_{\partial\Omega} (K\nabla p \cdot \nu) v = (f,v)$$

Using the boundary conditions, we get:

$$\sum_{j=1}^{N_h}\int_{E_j} K\nabla p\nabla v+\alpha pv-\sum_{k=1}^{P_h}\int_{e_k}\{K\nabla p\cdot\nu_k\}\,[v]-\int_{\Gamma_D}(K\nabla p\cdot\nu_D)v=(f,v)+\int_{\Gamma_N} gv$$

We add $\int_{\Gamma_D}(K\nabla v\cdot\nu_D)p$ to both sides and we note that $[p]=0$. We clearly have (2.3). To prove the converse, suppose that p is sufficiently smooth and satisfies (2.3). By the Green identity, we have:

$$\begin{aligned}\sum_{j=1}^{N_h}\int_{E_j} K\nabla p\nabla v &= -\sum_{j=1}^{N_h}\int_{E_j}(\nabla\cdot K\nabla p)v+\sum_{j=1}^{M_h}\int_{\partial E_j}(K\nabla p\cdot\nu)v\\ &= -\sum_{j=1}^{N_h}\int_{E_j}(\nabla\cdot K\nabla p)v+\int_{\partial\Omega}(K\nabla p\cdot\nu)v+\sum_{k=1}^{P_h}\int_{e_k}(K\nabla p\cdot\nu_k)[v]\end{aligned}$$

Combining this result with (2.3) and noting that the penalty term vanishes:

$$\begin{aligned}&\sum_{j=1}^{N_h}\int_{E_j}(-\nabla\cdot K\nabla pv+\alpha pv)+\int_{\Gamma_N}(K\nabla p\cdot\nu_N-g)v\\ &+\sum_{k=1}^{P_h}\int_{e_k}\{K\nabla v\cdot\nu_k\}[p]+\int_{\Gamma_D}(K\nabla v\cdot\nu_D)\,(p-p_0)=\int_\Omega fv\end{aligned}$$

By choosing appropriate test functions, we conclude that p satisfies :

$$\begin{aligned}-\nabla\cdot(K\nabla p)+\alpha p=f &\quad \text{in} \quad \Omega,\\ p=p_0 &\quad \text{on} \quad \Gamma_D,\\ K\nabla p\cdot\nu_N=g &\quad \text{on} \quad \Gamma_N.\end{aligned}$$

□

We note that on each element, the mass conservation for the NIPG method can be written as

$$\int_{E_j}\alpha P^{NIPG}-\int_{\partial E_j}\{K\nabla P^{NIPG}\cdot\nu_{\partial E_j}\}+\sum_{e_k\in\partial E_j\setminus\partial\Omega}\frac{\sigma_k}{|e_k|^\beta}\int_{e_k}[P^{NIPG}][1]=\int_{E_j} f.$$

The constrained discrete space is defined as follows:

$$\mathcal{D}_r^*(\mathcal{E}_h)=\left\{v\in\prod_{j=1}^{N_h}P_r(E_j):\int_{e_k}[v]=0\quad\forall k=1,\ldots,P_h\right\}.$$

The discrete approximation $P^{NCG}\in\mathcal{D}_r^*(\mathcal{E}_h)$ satisfies:

$$a_{NS}(P^{NCG},v)=L(v),\qquad\forall v\in\mathcal{D}_r^*(\mathcal{E}_h).\tag{2.4}$$

Lemma 2.2. Suppose that f is smooth and that the solution p to (2.1) exists and $K\nabla p \in H^1(\mathcal{E}_h)$. Then p satisfies the formulation (2.4). The converse is also true if we assume sufficiently smoothness on p.

Proof. The proof is very similar to the one given above. □

The Discontinuous Galerkin approximation $P^{DG} \in \mathcal{D}_r(\mathcal{E}_h)$ satisfies the formulation

$$a_{NS}(P^{DG}, v) = L(v), \qquad \forall v \in \mathcal{D}_r(\mathcal{E}_h). \tag{2.5}$$

The fact that this scheme is consistent with the problem (2.1) has been shown by Baumann [3].

Clearly the discrete solution of each of the three methods exists and is unique. Indeed, since it is a discrete problem, it suffices to show uniqueness of the solution. For instance for the NCG method, choose $f = 0$ and $v = P^{NCG}$. Thus, $|||K^{\frac{1}{2}}\nabla P^{NCG}|||_0 + |||\alpha^{\frac{1}{2}} P^{NCG}|||_0 = 0$. This easily implies that $P^{NCG} = 0$.

3. A priori error estimates for NIPG method

In this section, we derive an *a priori* error estimate for the Neumann problem ($\Gamma_N = \partial\Omega$).

Theorem 3.1. If $\alpha \equiv 0$, then

$$|||K^{\frac{1}{2}}\nabla(P^{NIPG} - p)|||_0 \leq C(\frac{1}{\sigma}, K)\left(\sum_{j=1}^{N_h} \frac{h_j^{2(\mu-1)}}{r^{2s-4}} \|p\|_{s,E_j}^2\right)^{\frac{1}{2}}.$$

If $\alpha \geq \alpha_0 > 0$, then

$$|||P^{NIPG} - p|||_1 \leq C(\frac{1}{\sigma}, K, \|\alpha\|_\infty)\left(\sum_{j=1}^{N_h} \frac{h_j^{2(\mu-1)}}{r^{2s-4}} \|p\|_{s,E_j}^2\right)^{\frac{1}{2}},$$

where $\mu = \min(r+1, s)$ and $s \geq 2$.

Proof. In all the proofs, C will be a generic constant with different values on different places, that is independent of h and r. Since p is continuous, we have :

$$a_{NS}(p, v) + J_0^{\sigma,\beta}(p, v) = L(v), \quad \forall v \in \mathcal{D}_r(\mathcal{E}_h).$$

Thus, we have the orthogonality equation:

$$a_{NS}(P^{NIPG}-p,v)+J_0^{\sigma,\beta}(P^{NIPG}-p,v)=0,\quad \forall v\in\mathcal{D}_r(\mathcal{E}_h).$$

Denote $\chi = P^{NIPG}-p$. Take $\tilde{p}\in C^0(\Omega)\cap\mathcal{D}_r$, the interpolant of p satisfying (2.1).

$$a_{NS}(\chi,\chi)+J_0^{\sigma,\beta}(\chi,\chi)=a_{NS}(\chi,\tilde{p}-p)+J_0^{\sigma,\beta}(\chi,\tilde{p}-p)\equiv A.$$

We note that $[\tilde{p}-p]=0$.

$$\begin{aligned}
A=&\sum_{j=1}^{N_h}\int_{E_j}K\nabla\chi\nabla(\tilde{p}-p)+\int_\Omega\alpha\chi(\tilde{p}-p)+\sum_{k=1}^{P_h}\int_{e_k}\{K\nabla(\tilde{p}-p)\cdot\nu_k\}[\chi],\\
|A|\leq&\,|\!|\!|K^{\frac12}\nabla\chi|\!|\!|_0|\!|\!|K^{\frac12}\nabla(\tilde{p}-p)|\!|\!|_0+|\!|\!|\alpha^{\frac12}\chi|\!|\!|_0|\!|\!|\alpha^{\frac12}(\tilde{p}-p)|\!|\!|_0\\
&+\sum_{k=1}^{P_h}\|\{K\nabla(\tilde{p}-p)\cdot\nu_k\}\|_{0,e_k}\|[\chi]\|_{0,e_k}\\
\leq&\,\epsilon|\!|\!|K^{\frac12}\nabla\chi|\!|\!|_0^2+\frac{1}{4\epsilon}|\!|\!|K^{\frac12}\nabla(\tilde{p}-p)|\!|\!|_0^2+\epsilon|\!|\!|\alpha^{\frac12}\chi|\!|\!|_0^2+\frac{1}{4\epsilon}|\!|\!|\alpha^{\frac12}(\tilde{p}-p)|\!|\!|_0^2\\
&+\sum_{k=1}^{P_h}\left(\frac{|e_k|^\beta}{\sigma_k}\right)^{\frac12}\|\{K\nabla(\tilde{p}-p)\cdot\nu_k\}\|_{0,e_k}\left(\frac{\sigma_k}{|e_k|^\beta}\right)^{\frac12}\|[\chi]\|_{0,e_k}.
\end{aligned}$$

If we choose $\epsilon=\frac12$, we obtain:

$$\begin{aligned}
|A|\leq&\,C|\!|\!|K^{\frac12}\nabla(\tilde{p}-p)|\!|\!|_0^2+C|\!|\!|\alpha^{\frac12}(\tilde{p}-p)|\!|\!|_0^2\\
&+\left(\sum_{k=1}^{P_h}\frac{|e_k|^\beta}{\sigma_k}\|\{K\nabla(\tilde{p}-p)\cdot\nu_k\}\|_{0,e_k}^2\right)^{\frac12}J_0^{\sigma,\beta}(\chi,\chi)^{\frac12},\\
\leq&\,C|\!|\!|K^{\frac12}\nabla(\tilde{p}-p)|\!|\!|_0^2+C|\!|\!|\alpha^{\frac12}(\tilde{p}-p)|\!|\!|_0^2\\
&+\epsilon J_0^{\sigma,\beta}(\chi,\chi)+C(\epsilon)\sum_{k=1}^{P_h}\frac{|e_k|^\beta}{\sigma_k}\|\{K\nabla(\tilde{p}-p)\cdot\nu_k\}\|_{0,e_k}^2.
\end{aligned}$$

Again, if $\epsilon=\frac12$, we have:

$$A\leq C|\!|\!|K^{\frac12}\nabla(\tilde{p}-p)|\!|\!|_0^2+C|\!|\!|\alpha^{\frac12}(\tilde{p}-p)|\!|\!|_0^2+C\sum_{k=1}^{P_h}\frac{|e_k|^\beta}{\sigma_k}\|\{K\nabla(\tilde{p}-p)\cdot\nu_k\}\|_{0,e_k}^2.$$

Using the trace theorem and an approximation result, we get:

$$\sum_{k=1}^{P_h}\frac{|e_k|^\beta}{\sigma_k}\|\{K\nabla(\tilde{p}-p)\cdot\nu_k\}\|_{0,e_k}^2\leq C(\frac{1}{\sigma},K)\sum_{j=1}^{N_h}\frac{h_j^{2\mu+\beta-3}}{r^{2s-4}}\|p\|_{s,E_j}^2,$$

where $\mu = \min(r+1, s)$. Besides, we also have by the approximation property:

$$\|K^{\frac{1}{2}}\nabla(\tilde{p}-p)\|_0 \leq C(K)\left(\sum_{j=1}^{N_h}\frac{h_j^{2(\mu-1)}}{r^{2(s-1)}}\|p\|_{s,E_j}^2\right)^{\frac{1}{2}},$$

$$\|\alpha^{\frac{1}{2}}(\tilde{p}-p)\|_0 \leq C\|\alpha\|_\infty\left(\sum_{j=1}^{N_h}\frac{h_j^{2\mu}}{r^{2s}}\|p\|_{s,E_j}^2\right)^{\frac{1}{2}}.$$

Combining all the results and assuming that $\beta \geq 1$, we conclude that:

$$\|K^{\frac{1}{2}}\nabla\chi\|_0^2 + \|\alpha\chi\|_0^2 \leq C\frac{h^{\mu-1}}{r^{s-2}}\|p\|_s\|K^{\frac{1}{2}}\nabla\chi\|_0 + C\|\alpha\|_\infty\frac{h^\mu}{r^s}\|p\|_s\|\alpha^{\frac{1}{2}}\chi\|_0,$$

where $C = C(\frac{1}{\sigma}, K)$. □

Theorem 3.2.

$$\|P^{NIPG} - p\|_{0,\Omega} \leq C\frac{h^{\mu-\frac{3}{2}+\frac{\beta}{2}(n-1)}}{r^{s-2}}\|p\|_s,$$

for $s \geq 2$ and C independent of h, r, p. In particular, optimal L^2 rates of convergence are obtained if $\beta \geq 3$ for $n = 3$ and if $\beta \geq \frac{\beta}{2}$ for $n = 3$.

Proof. Consider the dual problem

$$-\nabla\cdot K\nabla\phi + \alpha\phi = P^{NIPG} - p \quad \text{in} \quad \Omega,$$
$$K\nabla\phi\cdot\nu = 0 \quad \text{on} \quad \partial\Omega.$$

We assume that $\phi \in H^2(\Omega)$ and that there is a constant C that depends on Ω such that

$$\|\phi\|_{2,\Omega} \leq C\|P^{NIPG} - p\|_{0,\Omega}.$$

Denote $\chi = P^{NIPG} - p$.

$$\|\chi\|_{0,\Omega}^2 = (-\nabla\cdot K\nabla\phi + \alpha\phi, \chi).$$

Integrating by parts on each element yields:

$$\|\chi\|_{0,\Omega}^2 = \sum_{j=1}^{N_h}\int_{E_j} K\nabla\phi\nabla\chi + \alpha\phi\chi - \sum_{j=1}^{N_h}\int_{\partial E_j}(K\nabla\phi\cdot\nu)\chi,$$
$$= \sum_{j=1}^{N_h}\int_{E_j} K\nabla\phi\nabla\chi + \alpha\phi\chi - \sum_{k=1}^{P_h}\int_{e_k}\{K\nabla\phi\cdot\nu_k\}[\chi] - \sum_{k=1}^{P_h}\int_{e_k}[K\nabla\phi\cdot\nu_k]\{\chi\}.$$

By substracting the orthogonality equation, we obtain for $\phi^* \in \mathcal{D}_r$:

$$\begin{aligned}
\|\chi\|_{0,\Omega}^2 = & \sum_{j=1}^{N_h} \int_{E_j} K\nabla(\phi - \phi^*)\nabla\chi + \alpha(\phi - \phi^*)\chi \\
& - \sum_{k=1}^{P_h} \int_{e_k} \{K\nabla\phi \cdot \nu_k\}[\chi] - \sum_{k=1}^{P_h} \int_{e_k} [K\nabla\phi \cdot \nu_k]\{\chi\} \\
& + \sum_{k=1}^{P_h} \int_{e_k} \{K\nabla\chi \cdot \nu_k\}[\phi^*] - \sum_{k=1}^{P_h} \int_{e_k} \{K\nabla\phi^* \cdot \nu_k\}[\chi] - J_0^{\sigma,\beta}(\chi, \phi^*).
\end{aligned}$$

By noting that $\phi \in H^2(\Omega)$ and by choosing $\phi^* \in C^0$, we are left with

$$\begin{aligned}
\|\chi\|_{0,\Omega}^2 = & \sum_{j=1}^{N_h} \int_{E_j} K\nabla(\phi - \phi^*)\nabla\chi + \alpha(\phi - \phi^*)\chi \\
& - \sum_{k=1}^{P_h} \int_{e_k} \{K\nabla\phi \cdot \nu_k\}[\chi] - \sum_{k=1}^{P_h} \int_{e_k} \{K\nabla\phi^* \cdot \nu_k\}[\chi], \\
= & \sum_{j=1}^{N_h} \int_{E_j} K\nabla(\phi - \phi^*)\nabla\chi + \alpha(\phi - \phi^*)\chi \\
& + \sum_{k=1}^{P_h} \int_{e_k} \{K\nabla(\phi - \phi^*) \cdot \nu_k\}[\chi] - 2\sum_{k=1}^{P_h} \int_{e_k} \{K\nabla\phi \cdot \nu_k\}[\chi].
\end{aligned}$$

The first two terms are bounded in the following fashion

$$\begin{aligned}
\left|\sum_{j=1}^{N_h} \int_{E_j} K\nabla(\phi - \phi^*)\nabla\chi\right| & \leq C \sum_{j=1}^{N_h} \|\phi - \phi^*\|_{1,E_j} \|K^{\frac{1}{2}}\nabla\chi\|_{0,E_j}, \\
& \leq C\frac{h}{r}\|\phi\|_{2,\Omega} |\!|\!| K^{\frac{1}{2}}\nabla\chi |\!|\!|_0, \\
& \leq C\frac{h}{r}\|\chi\|_{0,\Omega} |\!|\!| K^{\frac{1}{2}}\nabla\chi |\!|\!|_0. \qquad (3.1) \\
\left|\sum_{j=1}^{N_h} \int_{E_j} \alpha(\phi - \phi^*)\chi\right| & \leq C\|\alpha\|_\infty \frac{h^2}{r^2}\|\phi\|_{2,\Omega} |\!|\!| \chi |\!|\!|_0, \\
& \leq C\|\alpha\|_\infty \frac{h^2}{r^2}\|\chi\|_{0,\Omega} |\!|\!| \chi |\!|\!|_0. \qquad (3.2)
\end{aligned}$$

The third term is bounded by using Cauchy-Schwarz inequality and the previous theorem:

$$\left|\sum_{k=1}^{P_h} \int_{e_k} \{K\nabla(\phi - \phi^*) \cdot \nu_k\}[\chi]\right| \leq C\frac{h^{\frac{1}{2}}}{r^{\frac{1}{2}}}\|\phi\|_{2,\Omega} \left(\sum_{k=1}^{P_h} \|[\chi]\|_{0,e_k}^2\right)^{\frac{1}{2}},$$

$$\leq C\frac{h^{\mu-\frac{1}{2}}|e_k|^{\frac{\beta}{2}}}{r^{s-3}}\|\chi\|_{0,\Omega}|\!|\!|p|\!|\!|_s. \tag{3.3}$$

The last term is bounded above by using trace theorems and the previous theorem

$$|\sum_{k=1}^{P_h}\int_{e_k}\{K\nabla\phi\cdot\nu_k\}[\chi]| \leq C\frac{1}{h^{\frac{1}{2}}}\|\phi\|_{2,\Omega}\left(\sum_{k=1}^{P_h}\|[\chi]\|_{0,e_k}^2\right)^{\frac{1}{2}},$$

$$\leq C\frac{h^{\mu-\frac{3}{2}}|e_k|^{\frac{\beta}{2}}}{r^{s-3}}\|\chi\|_{0,\Omega}|\!|\!|p|\!|\!|_s. \tag{3.4}$$

Combining (3.1), (3.2), (3.3) and (3.4), and assuming that $\beta \geq 1$, we obtain the final result:

$$\|\chi\|_{0,\Omega} \leq C\frac{h^{\mu-\frac{3}{2}+\frac{\beta}{2}}(n-1)}{r^{s-3}}|\!|\!|p|\!|\!|_s + C\|\alpha\|_\infty\frac{h^{\mu+1}}{r^s}|\!|\!|p|\!|\!|_s.$$

□

4. A priori error estimates for the NCG method

In this section, we derive an error estimate for the H^1 norm and the L^2 norm that are both h-optimal for the constrained Galerkin method.

Theorem 4.1. If $\alpha \equiv 0$, then

$$|\!|\!|K^{\frac{1}{2}}\nabla(P^{NCG}-p)|\!|\!|_0 \leq C(K)\frac{h^{\mu-1}}{r^{s-2}}|\!|\!|p|\!|\!|_s$$

If $\alpha \geq \alpha_0 > 0$, then

$$|\!|\!|P^{NCG}-p|\!|\!|_1 \leq C(K,\|\alpha\|_\infty)\frac{h^{\mu-1}}{r^{s-2}}|\!|\!|p|\!|\!|_s$$

where $\mu = \min(r+1,s)$, C independent of h, r, p and $s \geq 2$.

Proof. We have the following orthogonality equation:

$$a_{NS}(P^{NCG}-p,v) = 0, \quad \forall v \in \mathcal{D}_r^*(\mathcal{E}_h).$$

We can show [8] that there is an interpolant $\tilde{p} \in \mathcal{D}_r^*(\mathcal{E}_h)$ that is optimally closed to p in the H^m Sobolev norms. Let $\chi = P^{NCG} - \tilde{p}$.

$$a_{NS}(\chi,\chi) = a_{NS}(p-\tilde{p},\chi),$$

$$= \sum_{j=1}^{N_h} \int_{E_j} K\nabla(p-\tilde{p})\nabla\chi + \sum_{j=1}^{N_h} \int_{E_j} \alpha\chi(p-\tilde{p})$$
$$- \sum_{k=1}^{P_h} \int_{e_k} \{K\nabla(p-\tilde{p})\cdot\nu_k\}[\chi] + \sum_{k=1}^{P_h} \int_{e_k} \{K\nabla\chi\cdot\nu_k\}[p-\tilde{p}].$$

The first two terms are easily bounded by Cauchy-Schwarz and an approximation result:

$$|\sum_{j=1}^{N_h} \int_{E_j} K\nabla(p-\tilde{p})\nabla\chi| \leq \|K^{\frac{1}{2}}\nabla\chi\|_0 \|K^{\frac{1}{2}}\nabla(p-\tilde{p})\|_0,$$
$$\leq C\|K^{\frac{1}{2}}\nabla\chi\|_0 \left(\sum_{j=1}^{N_h} \frac{h_j^{2\mu-2}}{r^{2s-2}}\|p\|_{s,E_j}^2\right)^{\frac{1}{2}}. \tag{4.1}$$

In a similar manner, we have:

$$|\sum_{j=1}^{N_h} \int_{E_j} \alpha\chi(p-\tilde{p})| \leq \|\alpha^{\frac{1}{2}}\chi\|_0 \|\alpha(p-\tilde{p})\|_0,$$
$$\leq C\|\alpha\|_\infty^{\frac{1}{2}}\|\alpha^{\frac{1}{2}}\chi\|_0 \left(\sum_{j=1}^{N_h} \frac{h_j^{2\mu}}{r^{2s}}\|p\|_{s,E_j}^2\right)^{\frac{1}{2}}. \tag{4.2}$$

Now we try to estimate the third term. Let c_k be any constant.

$$A \equiv \sum_{k=1}^{P_h} \int_{e_k} \{K\nabla(p-\tilde{p})\cdot\nu_k\}[\chi] = \sum_{k=1}^{P_h} \int_{e_k} \{K\nabla(p-\tilde{p})\cdot\nu_k\}[\chi - c_k].$$

We have by Cauchy-Schwarz and Holder inequality:

$$|A| \leq \left(\sum_{k=1}^{P_h} \|\{K\nabla(p-\tilde{p})\cdot\nu_k\}\|_{0,e_k}^2\right)^{\frac{1}{2}} \left(\sum_{k=1}^{P_h} \|[\chi - c_k]\|_{0,e_k}^2\right)^{\frac{1}{2}}. \tag{4.3}$$

We look at one edge e_k. We assume that $e_k = \partial E^1 \cap \partial E^2$, where E^1 and E^2 are elements of $\mathcal{E}_h$. Let B_1 and B_2 be the matrices of the mappings from the reference element $\hat{E}$ onto E^1 and E^2 respectively. It is known that

$$\|B_i^{-1}\| \leq C\frac{1}{h}, \quad \|B_i\| \leq Ch, \quad |\mathrm{det}B_i| \leq Ch^n, \quad i = 1, 2.$$

Then, by the trace theorem,

$$\|\{K\nabla(p-\tilde{p})\cdot\nu_k\}\|_{0,e_k} \leq \hat{C}|e_k|^{\frac{1}{2}}(\|B_1^{-1}\|\|\nabla(p-\tilde{p})\|_{0,E^1} + \|\nabla^2(p-\tilde{p})\|_{0,E^1}$$

$$+\|B_2^{-1}\|\|\|\nabla(p-\tilde{p})\|_{0,E^2}+\|\nabla^2(p-\tilde{p})\|_{0,E^2}).$$

Summing on k,

$$\left(\sum_{k=1}^{P_h}\|\{K\nabla(p-\tilde{p})\cdot\nu_k\}\|_{0,e_k}^2\right)^{\frac{1}{2}}\leq C\frac{h^{\mu-\frac{3}{2}}}{r^{s-2}}|\!|\!|p|\!|\!|_s. \quad (4.4)$$

The other factor is bounded in the following way.

$$\|[\chi-c_k]\|_{0,e_k}\leq\|(\chi-c_k)_1\|_{0,e_k}+\|(\chi-c_k)_2\|_{0,e_k}.$$

Since $P^{NCG}\in\mathcal{D}_r^*(\mathcal{E}_h)$, we have

$$\int_{e_k}(\chi)_1 d\sigma=\int_{e_k}(\chi)_2 d\sigma.$$

Therefore, it suffices to estimate $\|(\chi-c_k)_1\|_{0,e_k}$.

$$\|(\chi-c_k)_1\|_{0,e_k}\leq h_j^{\frac{1}{2}}\|\widehat{\chi-c_k}\|_{L^2(\hat{e})}.$$

Choose

$$c=\frac{1}{|e_k|}\int_{e_k}(\chi)_1 d\sigma.$$

We note that $c=\frac{1}{|\hat{e}|}\int_{\hat{e}}(\hat{\chi})d\hat{\sigma}$ and that the mapping $\hat{f}\mapsto\hat{f}-\frac{1}{|\hat{e}|}\int_{\hat{e}}\hat{f}d\hat{\sigma}$ is continuous on $H^1(\hat{e})$ and vanishes on constant functions. Thus,

$$\|\hat{\chi}-c_k\|_{0,\hat{e}}\leq\hat{C}\|\hat{\nabla}\hat{\chi}\cdot\hat{t}\|_{0,\hat{e}}\leq\hat{C}\|\hat{\nabla}\hat{\chi}\|_{0,\hat{e}}.$$

Since the subdivision of Ω is regular and since $\hat{\nabla}\hat{\chi}$ belongs to a finite-dimensional space, on which all norms are equivalent, we get:

$$\|\hat{\chi}-c_k\|_{0,\hat{e}}\leq\hat{C}\|\hat{\nabla}\hat{\chi}\|_{0,\hat{E}}\leq\hat{C}\|\nabla\chi\|_{0,E_1}.$$

Therefore,

$$\|(\chi-c_k)_{E_1}\|_{0,e_k}\leq Ch_j^{\frac{1}{2}}\|\nabla\chi\|_{0,E_1}.$$

Thus, summing on k, we have

$$\sum_{k=1}^{P_h}\|[\chi-c_k]\|_{0,e_k}^2\leq C\sum_{j=1}^{N_h}h_j\|\nabla\chi\|_{0,E_j}^2. \quad (4.5)$$

Combining (4.3), (4.4) and (4.5), we obtain a bound for A:

$$|A|\leq C\frac{h^{\mu-1}}{r^{s-2}}|\!|\!|p|\!|\!|_s|\!|\!|K^{\frac{1}{2}}\nabla\chi|\!|\!|_0. \quad (4.6)$$

The last term is bounded in the following way:

$$|\int_{e_k}\{K\nabla\chi\cdot\nu_k\}[p-\tilde{p}]|\leq C\|\{\nabla\chi\cdot\nu_k\}\|_{0,e_k}\|[p-\tilde{p}]\|_{0,e_k}.$$

Since $\hat{\nabla}\hat{\chi}$ belongs to a finite-dimensional space, we have

$$\|\{\nabla\chi\cdot\nu_k\}\|_{0,e_k}\leq\hat{C}|e_k|^{\frac{1}{2}}\left(\|B_1^{-1}\|\|\nabla\chi\|_{0,E_1}+\|B_2^{-1}\|\|\nabla\chi\|_{0,E_2}\right).$$

The other factor is bounded by

$$\begin{aligned}\|[p-\tilde{p}]\|_{0,e_k}\leq\hat{C}|e_k|^{\frac{1}{2}}(&|\mathrm{det}B_1|^{-\frac{1}{2}}\|p-\tilde{p}\|_{0,E_1}+\|\nabla(p-\tilde{p})\|_{0,E_1}\\&+|\mathrm{det}B_2|^{-\frac{1}{2}}\|p-\tilde{p}^I\|_{0,E_2}+\|\nabla(p-\tilde{p})\|_{0,E_2}).\end{aligned}$$

Thus,

$$\begin{aligned}|\int_{e_k}\{K\nabla\chi\cdot\nu_k\}[p-\tilde{p}]|\leq\hat{C}|e_k|&\left(\|B_1^{-1}\|\|\nabla\chi\|_{0,E_1}+\|B_2^{-1}\|\|\nabla\chi\|_{0,E_2}\right)\\&(|\mathrm{det}B_1|^{-\frac{1}{2}}\|p-\tilde{p}\|_{0,E_1}+\|\nabla(p-\tilde{p})\|_{0,E_1}\\&+|\mathrm{det}B_2|^{-\frac{1}{2}}\|p-\tilde{p}\|_{0,E_2}+\|\nabla(p-\tilde{p})\|_{0,E_2}).\end{aligned}$$

Summing on k,

$$|\sum_{k=1}^{P_h}\int_{e_k}\{K\nabla\chi\cdot\nu_k\}[p-\tilde{p}]|\leq C\frac{h^{\mu-1}}{r^{s-1}}\|p\|_s\|K^{\frac{1}{2}}\nabla\chi\|_0.\tag{4.7}$$

Combining (4.1) , (4.2), (4.7) and (4.6), we obtain:

$$a_{NS}(\chi,\chi)\leq C\frac{h^{\mu-1}}{r^{s-1}}\|p\|_s\|K^{\frac{1}{2}}\nabla\chi\|_0+C\|\alpha\|_\infty^{\frac{1}{2}}\frac{h^\mu}{r^s}\|\alpha^{\frac{1}{2}}\chi\|_0\|p\|_s$$

Thus, if $\alpha=0$, we obtain an optimal bound on the H^1 semi-norm, i.e. the convergence rate of the L^2 norm of the velocities is optimal. If α is bounded below by a positive constant, the full H^1 norm is recovered. $\square$

Theorem 4.2.

$$\|P^{NCG}-p\|_{0,\Omega}\leq C\frac{h^\mu}{r^{s-2}}\|p\|_s,$$

for $s\geq 2$ and C independent of h,r,p.

Proof. We consider the auxiliary problem:

$$\begin{cases}-\nabla\cdot(K\nabla\psi)+\alpha\psi=P^{NCG}-p & \text{in}\quad\Omega,\\ K\nabla\psi\cdot\nu=0 & \text{on}\quad\partial\Omega.\end{cases}$$

Denote $\chi = P^{NCG} - p$. Thus, we have:

$$\begin{aligned}\|\chi\|_{0,\Omega}^2 &= (-(\nabla \cdot K\nabla\psi) + \alpha\psi, \chi),\\ &= \sum_{j=1}^{N_h} \int_{E_j} K\nabla\psi\nabla\chi + \alpha\psi\chi - \sum_{k=1}^{P_h} \int_{e_k} \{K\nabla\psi \cdot \nu_k\}[\chi]. \end{aligned} \tag{4.8}$$

Let ψ^* be in $\mathcal{D}_r^* \cap C^0(\Omega)$. The orthogonality condition implies that:

$$0 = \sum_{j=1}^{N_h} \int_{E_j} K\nabla\chi\nabla\psi^* + \alpha\chi\psi^* + \sum_{k=1}^{P_h} \int_{e_k} \{K\nabla\psi^* \cdot \nu_k\}[\chi]. \tag{4.9}$$

Now, we substract (4.9) to (4.8):

$$\begin{aligned}\|\chi\|_{0,\Omega}^2 &= \sum_{j=1}^{N_h} \int_{E_j} K\nabla(\psi - \psi^*)\nabla\chi + \alpha(\psi - \psi^*)\chi\\ &-2\sum_{k=1}^{P_h} \int_{e_k} \{K\nabla\psi \cdot \nu_k\}[\chi] + \sum_{k=1}^{P_h} \int_{e_k} \{K\nabla(\psi - \psi^*) \cdot \nu_k\}[\chi]. \end{aligned} \tag{4.10}$$

The first two terms are easily bounded by using Cauchy-Schwarz, Holder inequalities and the approximation property.

$$\begin{aligned}|\sum_{j=1}^{N_h} \int_{E_j} K\nabla(\psi - \psi^*)\nabla\chi| &\leq C \left(\sum_{j=1}^{N_h} \|\nabla(\psi - \psi^*)\|_{0,E_j}^2 \right)^{\frac{1}{2}} \left(\sum_{j=1}^{N_h} \|K^{\frac{1}{2}}\nabla\chi\|_{0,E_j}^2 \right)^{\frac{1}{2}},\\ &\leq C \left(\sum_{j=1}^{N_h} \frac{h_j^2}{r^2} \|\psi\|_{2,E_j}^2 \right)^{\frac{1}{2}} |||K^{\frac{1}{2}}\nabla\chi|||_0,\\ &\leq C\frac{h}{r}\|\chi\|_{0,\Omega} |||K^{\frac{1}{2}}\nabla\chi|||_0. \end{aligned}$$

In a similar manner, we have:

$$\begin{aligned}\left|\sum_{j=1}^{N_h} \int_{E_j} \alpha(\psi - \psi^*)\chi\right| &\leq C \sum_{j=1}^{N_h} \|(\psi - \psi^*)\|_{0,E_j} \|\alpha^{\frac{1}{2}}\chi\|_{0,E_j},\\ &\leq C \left(\sum_{j=1}^{N_h} \frac{h_j^4}{r^4} \|\psi\|_{2,E_j}^2 \right)^{\frac{1}{2}} \left(\sum_{j=1}^{N_h} \|\alpha^{\frac{1}{2}}\chi\|_{0,E_j}^2 \right)^{\frac{1}{2}},\\ &\leq C\frac{h^2}{r^2}\|\chi\|_{0,\Omega} |||K^{\frac{1}{2}}\nabla\chi|||_0. \end{aligned}$$

Then, we will bound the third term in (4.10). Let $\tilde{p}$ be an element of $\mathcal{D}_r^*(\mathcal{E}_h)\cap C^0$ and $\vec{c}$ be any constant vector.

$$2\sum_{k=1}^{P_h}\int_{e_k}\{K\nabla\psi\cdot\nu_k\}\,[\chi] = 2\sum_{k=1}^{P_h}\int_{e_k}\{K\nabla\psi\cdot\nu_k\}\left[P^{NCG}-\tilde{p}\right],$$
$$= 2\sum_{k=1}^{P_h}\int_{e_k}\{(K\nabla\psi-\vec{c})\cdot\nu_k\}\,[P^{NCG}-\tilde{p}].$$

As it was proved in Theorem 4.1, we have:

$$\sum_{k=1}^{P_h}\|[P^{NCG}-\tilde{p}]\|_{0,e_k}^2 \le C\sum_{j=1}^{N_h} h_j\|\nabla(P^{NCG}-\tilde{p})\|_{0,E_j}^2$$

By the triangle inequality and Theorem 4.1, we have

$$\sum_{j=1}^{N_h}\|\nabla(P^{NCG}-\tilde{p})\|_{0,E_j}^2 \le 2\sum_{j=1}^{N_h}\|\nabla(P^{NCG}-p)\|_{0,E_j}^2 + 2\sum_{j=1}^{N_h}\|\nabla(p-\tilde{p})\|_{0,E_j}^2,$$
$$\le C\frac{h^{2\mu-2}}{r^{2s-4}}\sum_{j=1}^{N_h}\|p\|_{s,E_j}^2.$$

On the other hand, we have

$$\|\{(K\nabla\psi-\vec{c})\cdot\nu_k\}\|_{0,e_k} \le \|\{K\nabla\psi-\vec{c}\}\|_{0,e_k} \le Ch^{\frac{1}{2}}\|\nabla(K\nabla\psi)\|_{0,E^1\cup E^2}.$$

If we assume that $K\nabla\psi\in H^1(E_j)$ with

$$\sum_{j=1}^{N_h}\|\nabla(K\nabla\psi)\|_{0,E_j}^2 \le \|\chi\|_{0,\Omega}^2,$$

then

$$|2\sum_{k=1}^{P_h}\int_{e_k}\{K\nabla\psi\cdot\nu_k\}\,[\chi]| \le C\frac{h^\mu}{r^{s-2}}\|\chi\|_{0,\Omega}|\!|\!|p|\!|\!|_s$$

Let A denote the last term in (4.10). We have by Cauchy-Schwarz and Holder inequality:

$$|A| \le C\left(\sum_{k=1}^{P_h}\|\{\nabla(\psi-\psi^*)\cdot\nu_k\}\|_{0,e_k}^2\right)^{\frac{1}{2}}\left(\sum_{k=1}^{P_h}\|[\chi]\|_{0,e_k}^2\right)^{\frac{1}{2}}.$$

By the approximation property:

$$\sum_{k=1}^{P_h} \|\{\nabla(\psi-\psi^*)\cdot\nu_k\}\|_{0,e_k}^2 \leq C\sum_{j=1}^{N_h}\frac{h_j}{r}\|\psi\|_{2,E_j}^2 \leq C\frac{h}{r}\|\chi\|_{0,\Omega}.$$

We also have

$$\sum_{k=1}^{P_h} \|[\chi]\|_{0,e_k}^2 \leq C\frac{h}{r}\|\nabla\chi\|_0 \leq C\frac{h^{2\mu-1}}{r^{2s-1}}\|p\|_s$$

Thus, we obtain:

$$|A| \leq \frac{h^\mu}{r^s}\|\chi\|_{0,\Omega}\|p\|_s.$$

The theorem is obtained by combining all the results. □

5. A priori error estimate for the DG method

In this section, we derive an *a priori* optimal error estimate for the Dirichlet or mixed boundary conditions problem. The estimate still holds for the Neumann problem. We make several assumptions:

- $p \in H^s(E_j), \quad \forall j = 1,\ldots,N_h \quad \text{and} \quad s \geq 2.$
- $K = (k_{i_0,j_0}), \quad k_{i_0j_0} \in \mathcal{C}^\infty(E_j), \quad \forall j = 1,\ldots,N_h.$
- $K \in [W^{1,\infty}(E_j)]^{2\times 2}, \quad \forall j = 1,\ldots,N_h.$
- We denote $\bar{K} = (\bar{k}_{ij})$, where $\bar{k}_{ij} = \frac{1}{|e_k|}\int_{e_k}\{k_{ij}\}$. We assume that $\bar{K}$ is symmetric positive definite (this is true if K is symmetric positive definite).

We first prove an approximation result that holds for $n = 2$.

Lemma 5.1. Let $p \in H^s(\mathcal{E}_h)$, for $s \geq 2$ and let $r \geq 2$. Let $\bar{K} = (\bar{k}_{ij})$ where $\bar{k}_{ij}$ are positive constants, bounded above by k_1 and below by $k_0 > 0$. There is $\tilde{p}^I \in \mathcal{D}_r(\mathcal{E}_h)$ interpolant of p satisfying

$$\int_{e_k}\{\bar{K}\nabla(\tilde{p}^I - p)\cdot\nu_k\} = 0, \quad \forall k = 1,\ldots,P_h$$

$$\int_{e_k}\bar{K}\nabla(\tilde{p}^I - p)\cdot\nu_k = 0, \quad \forall k = P_h+1,\ldots,M_h$$

$$\|\tilde{p}^I - p\|_{0,\Omega} \leq C\frac{h^\mu}{r^{s-1}}\|p\|_s,$$

$$\|\nabla(\tilde{p}^I - p)\|_0 \leq C\frac{h^{\mu-1}}{r^{s-1}}\|p\|_s,$$

where $\mu = \min(r+1, s)$ and $C = C(\frac{k_1}{k_0})$.

Proof. Let E be a triangle or a quadrilateral. We will show that given $f \in H^s(E)$ with $s \geq 2$ and given an edge e of the element E and ν a unit normal vector to e, there is a polynomial $q \in \mathbb{P}_2(E)$ such that q vanishes on the other sides of E and that $\int_e \nabla(q - f) \cdot \nu = 0$.

First, we assume that E is a triangle, and we denote a_1, a_2 the vertices of e. There is a transformation that maps the reference element $\hat{E}$ onto E such that the vertex $(0,0)$ is mapped onto a_1. Take λ_1 and λ_2 the barycentric coordinates of a_1 and a_2 in E and choose

$$q(x) = 4\lambda_1\lambda_2 q(a_{12}),$$

where a_{12} is the midpoint of e. Let B be the matrix of the transformation that maps $\hat{E}$ onto E. If $\hat{x}$ denotes the coordinates on a point of $\hat{E}$, the following relations hold:

$$\begin{aligned}
x &= B\hat{x} + a_1 \\
\widehat{\nabla q \cdot \nu} &= (B^t)^{-1}\hat{\nabla}\hat{q} \cdot \|B^t\nu\|(B^t)^{-1}\hat{\nu} \\
&= \|B^t\nu\|((B^t)^{-1} \cdot \hat{\nabla}\hat{q}, (B^t)^{-1}\hat{\nu}) \\
&= \|B^t\nu\|(B^{-1}(B^t)^{-1} \cdot \hat{\nabla}\hat{q}, \hat{\nu}) \\
&= \|B^t\nu\|((B^tB)^{-1} \cdot \hat{\nabla}\hat{q}, \hat{\nu}),
\end{aligned}$$

where $\hat{\nu} = (1,0)^t$. We have

$$\begin{aligned}
\frac{\partial \hat{q}}{\partial \hat{x}_1} &= 4\hat{q}(0, \frac{1}{2})\frac{\partial}{\partial \hat{x}_1}(\hat{\lambda}_1\hat{\lambda}_2) \\
&= -4\hat{q}(0, \frac{1}{2})\hat{x}_2 \\
\frac{\partial \hat{q}}{\partial \hat{x}_2} &= 4\hat{q}(0, \frac{1}{2})\frac{\partial}{\partial \hat{x}_2}(\hat{\lambda}_1\hat{\lambda}_2) \\
&= -4\hat{q}(0, \frac{1}{2})(1 - \hat{x}_1 - 2\hat{x}_2)
\end{aligned}$$

The coordinates of the vertex a_i are denoted (a_i^1, a_i^2).

$$B = \begin{pmatrix} a_3^1 - a_1^1 & a_2^1 - a_1^1 \\ a_3^2 - a_1^2 & a_2^2 - a_1^2 \end{pmatrix}.$$

$$B^tB = \begin{pmatrix} \|a_3 - a_1\|^2 & * \\ (a_2^1 - a_1^1)(a_3^1 - a_1^1) + (a_2^2 - a_1^2)(a_3^2 - a_1^2) & \|a_2 - a_1\|^2 \end{pmatrix}.$$

Thus,

$$\bar{K}\widehat{\nabla q \cdot} \nu = \frac{\|B^t \nu\|}{\det(B^t B)}\{-\bar{k}_{11}\|a_2 - a_1\|^2 \hat{x}_2 - \bar{k}_{12}(1-2\hat{x}_2)((a_2^1 - a_1^1)(a_3^1 - a_1^1) + (a_2^2 - a_1^2)(a_3^2 - a_1^2))\}$$

But when integrated from 0 to 1, the second term gives 0 so that

$$\int_e \bar{K}\nabla q \cdot \nu = -2|e|\bar{k}_{11}\hat{q}(0, \frac{1}{2})\frac{\|B^t\nu\|\|a_2 - a_1\|^2}{\det(B^t B)}$$

But,

$$\|B^t \nu\| \geq \frac{\|\nu\|}{\|(B^{-1})^t\|} = \frac{1}{\|B^{-1}\|} \geq C\rho_E$$

Thus, the coefficient of $\hat{q}(0, \frac{1}{2})$ is bounded below by $\frac{\hat{C}\rho_E^4}{h_E^4}k_0 \geq \frac{\hat{C}}{\sigma_E^4}k_0$, a constant that is independent of h when the triangulation is regular. Now, if E is a quadrilateral, the process is easily modified. Let $f = p - p^I$, where $p^I \in I\!P_r(E)$ is the standard Lagrange interpolant ot p. Then,

$$q(a_{12}) = -\frac{|\det B|^2}{2\bar{k}_{11}\|a_2 - a_1\|^3\|B^t\nu\|}\int_e \bar{K}\nabla(p - p^I)\cdot\nu$$

$$\begin{aligned}
|\int_e \bar{K}\nabla(p - p^I)\cdot \nu| &\leq |e|^{\frac{1}{2}}k_1\|\nabla(p - p^I)\|_{0,e} \\
&\leq |e|k_1\|B^{-1}\|\|\hat{\nabla}(\hat{p} - \hat{p}^I)\|_{0,\hat{e}} \\
&\leq k_1\hat{C}|e|\|B^{-1}\|(\|\hat{\nabla}(\hat{p} - \hat{p}^I)\|_{0,\hat{E}} + \|\hat{\nabla}^2(\hat{p} - \hat{p}^I)\|_{0,\hat{E}}) \\
&\leq k_1\hat{C}|e|\|B^{-1}\||\det B|^{-\frac{1}{2}}\|B\|(\|\nabla(p - p^I)\|_{0,E} + \|B\|\|\nabla^2(p - p^I)\|_{0,E})
\end{aligned}$$

Thus,

$$\begin{aligned}
|q(a_{12})| &\leq \frac{k_1}{k_0}\hat{C}\frac{|\det B|^{\frac{3}{2}}\|B^{-1}\|\|B\|}{|e|^2\|B^t\nu\|}(\|\nabla(p - p^I)\|_{0,E} + \|B\|\|\nabla^2(p - p^I)\|_{0,E}) \\
&\leq \frac{k_1}{k_0}\hat{C}(\|\nabla(p - p^I)\|_{0,E} + \|B\|\|\nabla^2(p - p^I)\|_{0,E})
\end{aligned}$$

$$\begin{aligned}
\|q\|_{0,E} &\leq |\det B|^{\frac{1}{2}}\hat{C}|q(a_{12})| \\
&\leq \frac{k_1}{k_0}\hat{C}|\det B|^{\frac{1}{2}}(\|\nabla(p - p^I)\|_{0,E} + \|B\|\|\nabla^2(p - p^I)\|_{0,E})
\end{aligned}$$

and

$$\begin{aligned}
\|\nabla q\|_{0,E} &\leq |\det B|^{\frac{1}{2}}\|B^{-1}\|\hat{C}|q(a_{12})| \\
&\leq \frac{k_1}{k_0}\hat{C}(\|\nabla(p - p^I)\|_{0,E} + \|B\|\|\nabla^2(p - p^I)\|_{0,E})
\end{aligned}$$

So, if we consider $\tilde{p}^I = q + p^I$, then

$$\begin{aligned}||\tilde{p}^I - p||_{0,\Omega} &\leq ||q||_{0,\Omega} + ||p^I - p||_{0,\Omega} \\ &\leq \hat{C}h\frac{k_1}{k_0}(||\nabla(p^I - p)||_{0,\Omega} + h||\nabla^2(p^I - p)||_{0,\Omega}) + ||p^I - p||_{0,\Omega},\end{aligned}$$

which has the same order of approximation as $||p^I - p||_{0,\Omega}$ and

$$\begin{aligned}||\nabla(\tilde{p}^I - p)||_{0,\Omega} &\leq ||\nabla q||_{0,\Omega} + ||\nabla(p^I - p)||_{0,\Omega} \\ &\leq \hat{C}\frac{k_1}{k_0}(||\nabla(p^I - p)||_{0,\Omega} + h||\nabla^2(p^I - p)||_{0,\Omega}) + ||\nabla(p^I - p)||_{0,\Omega},\end{aligned}$$

which is also of the same order as $||\nabla(p^I - p)||_{0,\Omega}$.

□

Theorem 5.2. Assume $\Omega \in I\!R^2$. If $\alpha \equiv 0$ and $K = \bar{K}$, then there is a constant C independent of h, r, p such that for $s \geq 2$

$$|||K^{\frac{1}{2}}\nabla(p - P^{DG})|||_0 \leq C(K)\frac{h^{\mu-1}}{r^{s-1}}|||p|||_s.$$

If $\alpha \geq \alpha_0 > 0$, then the following inequality holds

$$|||p - P^{DG}|||_1 \leq C(K, ||\alpha||_\infty)\frac{h^{\mu-1}}{r^{s-1}}|||p|||_s,$$

where $\mu = \min(r+1, s)$. If $\Omega \in I\!R^3$, then similar results hold with the suboptimal bound $\frac{h^{\mu-2}}{r^{s-1}}|||p|||_s$.

Proof. P^{DG} and p satisfy:

$$a_{NS}(P^{DG} - p, v) = 0, \quad \forall v \in \mathcal{D}_r(\mathcal{E}_h).$$

We first prove the result for $n = 2$. We take $v = P^{DG} - \tilde{p}^I$, where $\tilde{p}^I$ is the interpolant of p constructed such that

$$\int_{e_k}\{\bar{K}\nabla(p - \tilde{p}^I)\cdot\nu_k\} = 0, \quad \forall k = 1, \ldots, P_h$$

The interpolant $\tilde{p}^I$ has the same approximation error in L^2 and in H^1 as the standard Lagrange interpolant. Denote $\chi = P^{DG} - \tilde{p}^I$.

$$\begin{aligned}a_{NS}(\chi,\chi) = &\sum_{j=1}^{N_h}\int_{E_j}(K\nabla(p - \tilde{p}^I)\cdot\nabla\chi + \alpha(p - \tilde{p}^I)\chi) \\ &- \sum_{k=1}^{P_h}\int_{e_k}\{K\nabla(p - \tilde{p}^I)\cdot\nu_k\}[\chi] + \sum_{k=1}^{P_h}\int_{e_k}\{K\nabla\chi\cdot\nu_k\}[p - \tilde{p}^I]\end{aligned}$$

$$-\int_{\Gamma_D}(K\nabla(p-\tilde{p}^I)\cdot\nu_D)\chi+\int_{\Gamma_D}(K\nabla\chi\cdot\nu_D)(p-\tilde{p}^I)$$

$$\begin{aligned}a_{NS}(\chi,\chi)=&\sum_{j=1}^{N_h}\int_{E_j}(K\nabla(p-\tilde{p}^I)\cdot\nabla\chi+\alpha(p-\tilde{p}^I)\chi)\\&-\sum_{k=1}^{P_h}\int_{e_k}\{(K-\bar{K})\nabla(p-\tilde{p}^I)\cdot\nu_k\}[\chi]-\sum_{k=1}^{P_h}\int_{e_k}\{\bar{K}\nabla(p-\tilde{p}^I)\cdot\nu_k\}([\chi]-c_k)\\&+\sum_{k=1}^{P_h}\int_{e_k}\{K\nabla\chi\cdot\nu_k\}[p-\tilde{p}^I]\\&-\sum_{e_k\in\Gamma_D}\int_{e_k}(K\nabla(p-\tilde{p}^I)\cdot\nu_k)(\chi-c_k)+\int_{\Gamma_D}(K\nabla\chi\cdot\nu_D)(p-\tilde{p}^I)\end{aligned}\tag{5.1}$$

where c_k is any constant depending on e_k.

The first two terms in (5.1) can be bounded in the following way:

$$|\sum_{j=1}^{N_h}\int_{E_j}K\nabla(p-\tilde{p}^I)\cdot\nabla\chi|\leq C|||\nabla(p-\tilde{p}^I)|||_0|||K^{\frac{1}{2}}\nabla\chi|||_0,$$

$$|\sum_{j=1}^{N_h}\int_{E_j}\alpha(p-\tilde{p}^I)\chi|\leq C\|\alpha\|_\infty|||p-\tilde{p}^I|||_0|||\alpha^{\frac{1}{2}}\chi|||_0.$$

To bound the other terms, we consider the contribution from each interior edge. We assume that $e_k=\partial E^1\cap\partial E^2$, where E^1 and E^2 are elements of $\mathcal{E}_h$ and we denote by B_1 and B_2 the matrices of the mappings from the reference element $\hat{E}$ onto E^1 and E^2 respectively.

The third term in (5.1) is bounded by

$$\begin{aligned}|\int_{e_k}\{(K-\bar{K})\nabla(p-\tilde{p}^I)\cdot\nu_k\}[\chi]|\leq&\frac{1}{2}\|K-\bar{K}\|_{\infty,E^1}\int_{e_k}|\nabla(p-\tilde{p}^I)\cdot\nu_k||[\chi]|\\&+\frac{1}{2}\|K-\bar{K}\|_{\infty,E^2}\int_{e_k}|\nabla(p-\tilde{p}^I)\cdot\nu_k||[\chi]|,\\\leq&\frac{1}{2}Ch_1\int_{e_k}|\nabla(p-\tilde{p}^I)\cdot\nu_k||[\chi]|\\&+\frac{1}{2}Ch_2\int_{e_k}|\nabla(p-\tilde{p}^I)\cdot\nu_k||[\chi]|.\end{aligned}$$

We only look at the element E^1 since a similar result holds for E^2.

$$\|\nabla(p-\tilde{p}^I)\cdot\nu_k\|_{0,e_k}\leq|e_k|^{\frac{1}{2}}(\|B_1^{-1}\|\|\nabla(p-\tilde{p}^I)\|_{0,E^1}+\|B_1^{-1}\|\|B_1\|\|\nabla^2(p-\tilde{p}^I)\|_{0,E_1}),$$

$$\|[\chi]\|_{0,e_k}\leq\hat{C}|e_k|^{\frac{1}{2}}(\|B_1^{-1}\|\|\chi\|_{0,E^1}+\|B_2^{-1}\|\|\chi\|_{E^2}+\|\nabla\chi\|_{0,E^1\cup E^2}).$$

Thus,

$$|\sum_{k=1}^{P_h}\int_{e_k}\{(K-\bar{K})\nabla(p-\tilde{p}^I)\cdot\nu_k\}[\chi]|\leq C\frac{h^{\mu-1}}{r^{s-1}}|\!|\!|p|\!|\!|_s|\!|\!|K^{\frac{1}{2}}\nabla\chi|\!|\!|_0.$$

The fourth term can also be bounded as follows:

$$\left|\int_{e_k}\{\bar{K}\nabla(p-\tilde{p}^I)\cdot\nu_k\}([\chi]-c_k)\right|\leq\|\{\bar{K}\nabla(p-\tilde{p}^I)\cdot\nu_k\}\|_{0,e_k}\|[\chi]-c_k\|_{0,e_k},$$

By the trace theorem,

$$\begin{aligned}\|\{\bar{K}\nabla(p-\tilde{p}^I)\cdot\nu_k\}\|_{0,e_k}\leq\hat{C}\bar{k_1}|e_k|^{\frac{1}{2}}(\|B_1^{-1}\|\|\nabla(p-\tilde{p}^I)\|_{0,E^1}+\|\nabla^2(p-\tilde{p}^I)\|_{0,E^1}\\+\|B_2^{-1}\|\|\nabla(p-\tilde{p}^I)\|_{0,E^2}+\|\nabla^2(p-\tilde{p}^I)\|_{0,E^2}).\end{aligned}$$

Take c_k as follows:

$$c_k=\frac{1}{|e_k|}\int_{e_k}[P^{DG}-\tilde{p}^I].$$

$$\begin{aligned}\|[\chi]-c_k\|_{0,e_k}&\leq|e_k|^{\frac{1}{2}}\|[\hat{\chi}]-c_k\|_{0,\hat{e}}\\&\leq|e_k|^{\frac{1}{2}}\|\frac{d}{d\hat{\sigma}}[\hat{\chi}]\|_{0,\hat{e}}\leq\hat{C}|e_k|^{\frac{1}{2}}r^2\left(\|\hat{\nabla}\hat{\chi}\|_{0,\hat{E}^1}+\|\hat{\nabla}\hat{\chi}\|_{0,\hat{E}^2}\right)\\&\leq\hat{C}\||e_k|^{\frac{1}{2}}r^2\left(\|K^{\frac{1}{2}}\nabla\chi\|_{0,E^1}+\|K^{\frac{1}{2}}\nabla\chi\|_{0,E^2}\right).\end{aligned}$$

Thus,

$$\begin{aligned}|\int_{e_k}\{\bar{K}\nabla(p-\tilde{p}^I)\cdot\nu\}[\chi]|\leq C(K)\hat{C}|e_k|(\|B_1^{-1}\|\|\nabla(p-\tilde{p}^I)\|_{0,E^1}+\|\nabla^2(p-\tilde{p}^I)\|_{0,E^1\cup E^2}\\+\|B_2^{-1}\|\|\nabla(p-\tilde{p}^I)\|_{0,E^2})\left(\|K^{\frac{1}{2}}\nabla\chi\|_{0,E^1}+\|K^{\frac{1}{2}}\nabla\chi\|_{0,E^2}\right).\end{aligned}$$

Combining the contributions from all the interior edges, we can bound the fourth term of (5.1):

$$\sum_{k=1}^{P_h}\int_{e_k}\{\bar{K}\nabla(p-\tilde{p}^I)\cdot\nu_k\}([\chi]-c_k)\leq C(K)\frac{h^{\mu-1}}{r^{s-1}}|\!|\!|K^{\frac{1}{2}}\nabla\chi|\!|\!|_0|\!|\!|p|\!|\!|_s.$$

Now, we bound the fith term in (5.1):

$$|\int_{e_k}\{K\nabla\chi\cdot\nu_k\}[p-\tilde{p}^I]|\leq k_1\|\{K\nabla\chi\cdot\nu_k\}\|_{0,e_k}\|[p-\tilde{p}^I]\|_{0,e_k}.$$

Since $\hat{\nabla}\hat{\chi}$ belongs to a finite-dimensional space, we have

$$\begin{aligned}\|\{\nabla\chi\cdot\nu_k\}\|_{0,e_k}&\leq\hat{C}|e_k|^{\frac{1}{2}}r^2\left(\|B_1^{-1}\|\|\hat{\nabla}(\hat{\chi})\|_{0,\hat{E}_1}+\|B_2^{-1}\|\|\hat{\nabla}(\hat{\chi})\|_{0,\hat{E}_2}\right)\\&\leq\hat{C}|e_k|^{\frac{1}{2}}r^2\left(\|B_1^{-1}\|\|\nabla\chi\|_{0,E_1}+\|B_2^{-1}\|\|\nabla\chi\|_{0,E_2}\right)\end{aligned}$$

The other factor is bounded by

$$\begin{aligned}\|[p-\tilde{p}^I]\|_{0,e_k} &\leq |e_k|^{\frac{1}{2}}\|[\widehat{p-\tilde{p}^I}]\|_{0,\hat{e}}\\ &\leq \hat{C}|e_k|^{\frac{1}{2}}\left(\|\widehat{p-\tilde{p}^I}\|_{0,\hat{E}_1\cup\hat{E}_2}+\|\hat{\nabla}(\widehat{p-\tilde{p}^I})\|_{0,\hat{E}_1\cup\hat{E}2}\right)\\ &\leq \hat{C}|e_k|^{\frac{1}{2}}(|\mathrm{det}B_1|^{-\frac{1}{2}}\|p-\tilde{p}^I\|_{0,E_1}+\|\nabla(p-\tilde{p}^I)\|_{0,E_1}\\ &\quad +|\mathrm{det}B_2|^{-\frac{1}{2}}\|p-\tilde{p}^I\|_{0,E_2}+\|\nabla(p-\tilde{p}^I)\|_{0,E_2})\end{aligned}$$

Thus,

$$\begin{aligned}|\int_{e_k}\{K\nabla\chi\cdot\nu_k\}[p-\tilde{p}^I]| \leq \hat{C}k_1|e_k|r^2\left(\|B_1^{-1}\|\|\nabla\chi\|_{0,E_1}+\|B_2^{-1}\|\|\nabla\chi\|_{0,E_2}\right)\\ (|\mathrm{det}B_1|^{-\frac{1}{2}}\|p-\tilde{p}^I\|_{0,E_1}+\|\nabla(p-\tilde{p}^I)\|_{0,E_1}\\ +|\mathrm{det}B_2|^{-\frac{1}{2}}\|p-\tilde{p}^I\|_{0,E_2}+\|\nabla(p-\tilde{p}^I)\|_{0,E_2})\end{aligned}$$

Or,

$$|\sum_{k=1}^{P_h}\int_{e_k}\{K\nabla\chi\cdot\nu_k\}[p-\tilde{p}^I]| \leq C(K)\frac{h^{\mu-1}}{r^{s-1}}|||p|||_s|||K^{\frac{1}{2}}\nabla\chi|||_0$$

Let $e_k\in\Gamma_D$.

$$|\int_{e_k}(K\nabla(p-\tilde{p}^I)\cdot\nu_k)(\chi-c_k)| \leq \|K\nabla(p-\tilde{p}^I)\|_{0,e_k}\|\chi-c_k\|_{0,e_k}$$

Take $c_k=\frac{1}{|e_k|}\int_{e_k}\chi$. As before, we have

$$\begin{aligned}\|K\nabla(p-\tilde{p}^I)\|_{0,e_k} &\leq \hat{C}|e_k|^{\frac{1}{2}}\|B_1^{-1}\|(\|\nabla(p-\tilde{p}^I)\|_{0,E^1}+\|B_1\|\|\nabla^2(p-\tilde{p}^I)\|_{0,E^1})\\ \|\chi-c_k\|_{0,e_k} &\leq C|e_k|^{\frac{1}{2}}r^2\|K^{\frac{1}{2}}\nabla\chi\|_{0,E^1}\end{aligned}$$

Thus,

$$|\int_{\Gamma_D}(K\nabla(p-\tilde{p}^I)\cdot\nu_k)(\chi-c_k)| \leq C\frac{h^{\mu-1}}{r^s}|||p|||_s$$

The last term is also easily bounded and the theorem is obtained by combining all the results together.

Now, we look at the case $\Omega\in I\!\!R^3$. We show that the interior penalty approximation P^{NIPG} is very close to the discontinuous Galerkin approximation P^{DG}. Denote $\xi=P^{NIPG}-P^{DG}$ and $\chi=P^{NIPG}-p$. ξ satisfies the following orthogonality equation:

$$a_{NS}(\xi,v)+J_0^{\sigma,\beta}(P^{NIPG},v)=0,\quad \forall v\in\mathcal{D}_r(\mathcal{E}_h).$$

Take $v = \xi$.

$$a_{NS}(\xi,\xi) = -J_0^{\sigma,\beta}(P^{NIPG},\xi).$$

Since p is continuous, we obtain:

$$J_0^{\sigma,\beta}(P^{NIPG},\xi) = J_0^{\sigma,\beta}(\chi,\xi) = \sum_{k=1}^{P_h} \frac{\sigma_k}{|e_k|^\beta} \int_{e_k} [\chi][\xi] \leq Ch^{-\beta} \sum_{k=1}^{P_h} \|[\chi]\|_{0,e_k} \|[\xi]\|_{0,e_k} \tag{5.2}$$

By the Theorem 3.1, we have:

$$\sum_{k=1}^{P_h} \sigma_k \|[\chi]\|_{0,e_k}^2 \leq C(\frac{1}{\sigma}) \frac{h^{2\mu-2+\beta}}{r^{2s-4}} \|p\|_s.$$

By the trace theorem, we have

$$\sum_{k=1}^{P_h} \|\xi\|_{0,e_k}^2 \leq Ch^{-1} \|K^{\frac{1}{2}} \nabla \xi\|_0^2.$$

Thus,

$$a_{NS}(\xi,\xi) \leq C(\sigma) \frac{h^{\mu-\frac{3}{2}-\frac{\beta}{2}}}{r^{s-2}} \|p\|_s \|K^{\frac{1}{2}} \nabla \xi\|_0.$$

The theorem is obtained, since $\beta \geq 1$. □

6. Conclusion

In this paper we have presented *hp* convergence results for three methods for modeling elliptic problems with discontinuous spaces. Unlike the interior penalty methods, which were shown to be effective for modeling sharp fronts arising in miscible displacement in porous media, the NIPG schemes do not require problem dependent penalties to be defined. Even though we have obtained optimal hp convergence results for the constrained NCG method, this procedure is more complicated to implement than the DG method. The latter is locally conservative. Computational results for the DG method are described in Part II of this paper.

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