Stability and Error Analysis of the Finite Element Models of the 1-D Shallow Water Equations

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Abstract

Interest in the numerical solutions of shallow water equations has grown in recent years. Shallow water equations predict tidal elevation and velocities in bodies such as bays and estuaries. The circulations patterns obtained can be used to determine contaminant propagation in coastal areas. In this paper, we provide stability and error analysis for the 1-dimensional linearized shallow water equations. We derive $L^\infty(L^2)$ stability estimates for the Galerkin finite element approximation of the wave formulation in continuous time. We also derive $L^\infty(L^2)$ error estimates for elevation and velocity which are optimal in $L^\infty(H^1)$. In addition, we successfully develop a Chorin-type projection operator-splitting scheme, using finite difference time-stepping, for the primitive formulation. We also develop a stability estimate which seems to be the first of its kind and a significant literary contribution.

1 Introduction

Shallow water equations model the fluid flow in vertically well mixed fluids experiencing external forces such as tidal and atmospheric forcing. Therefore we can determine circulations patterns and tidal amplitudes in the region subject to these forces at the open boundaries of the region. The applications of shallow water equations are numerous, consequently leading to a growing interest in their numerical solutions. For environmental purposes, circulations patterns obtained from the shallow water equations can be used to determine contaminant propagation in coastal areas under varied conditions in order to evaluate different courses of remediation. Shallow water equations are also used to estimate the impact of structures such as dams on hydrological conditions of bays [6]. In addition, data on wave heights and tidal forces can be used to determine loads on offshore structures such as oilwell platforms.

The numerical procedures used to solve the shallow water equations must ensure the accuracy of the numerical solution while maintaining the physics of the problem. Many early finite element simulations of the primitive [1]shallow water equations were plagued with spurious oscillations. Many attempts to eliminate these oscillations did not resolve the oscillations successfully [5]. Consequently, Lynch and Gray [3] reformulated the continuity equation of the shallow water equations into a second-order wave equation. The resulting generalized wave continuity equation (GWCE) successfully suppressed these oscillations without using numerical damping.

In Section 2, we introduce the 1-dimensional primitive equation formulation and wave equation formulation of the shallow water equations. We then establish our linearization of these two formulations and the numerical methods we used to solve them.

Then in Section 3, we prove a stability result in the L^2 and L^∞ spatial norms of the linearized 1-dimensional GWCE in continuous time. The motivation in developing this stability result is that stability analysis is more general than the traditional Fourier analysis. We derive an error estimate for a Galerkin finite element method approximation to these shallow water equations. Error estimates for the 1-dimensional case are devoid in the literature. Martínez [7] analyzed the 2-dimensional shallow water equations, but did not explore the 1-dimensional case. Martínez derived $L^\infty(L^2)$ and $L^2(H^1)$ *a pri-*

[1] "Primitive" is understood to reflect the pure governing equation.

ori error estimates for both the continuous-time and discrete-time Galerkin approximation to the nonlinear 2-dimensional wave model. Unable to circumvent the nonlinearities of the model using either an L^2 or elliptic comparison function, Martínez was only able to prove $H^1(\Omega)$ optimality. The error estimates presented in this paper, give us insight into developing a comparison function, based on the solution of the linearized model, to improve optimality in the fully nonlinear error estimates in 2 dimensions.

In Section 4, we analyze the stability of the Chorin projection of the primitive shallow water equations. The Chorin projection is a procedure used to decouple surface elevations and velocity in the momentum equation. The prevailing issue is whether we can develop a scheme to yield larger Δt than the traditional time-stepping methods. The Chorin projection in a Navier-Stokes setting seemed to yield larger time-steps [2]. However, can the Chorin projection in the incompressible Navier-Stokes be generalized to a compressible flow application? The answer is yes. There has been some work done in combustion applications [8]. However, until now, no analysis had been done. We will provide the first analysis of the Chorin Projection in the setting of the primitive formulation of the shallow water equations.

Finally, in Section 5 we discuss our analysis and conclude in Section 6 with future work.

2 The Model

2.1 Preliminaries

Let us define the notation we use throughout the this paper. Recall the usual Sobolev spaces:

- $L^\infty(\Omega) = \{v : \Omega \to \Re : |v(x)| \leq K\, a.e.\, on\, \Omega\, for\, some\, constant\, K\}$;
- $L^2(\Omega) = \{v : \Omega \to \Re : \int_\Omega |v(x)|^2 dx < \infty\}$;
- $H^1(\Omega) = \{v : \Omega \to \Re : \int_\Omega |v(x)|^2 dx < \infty\, and\, \int_\Omega |D^\alpha v|^2 dx < +\infty$ for $|\alpha| = 1\}$;
- $H^1_0(\Omega) = H^1(\Omega) \bigcap \{v : v = 0\ on\ \partial\Omega\}$.

Define the norms associated with these spaces:

- $\|v\|_{L^\infty(\Omega)} = \text{ess sup}_{x\epsilon\Omega}|v(x)|$;

- $\|v\|_{L^2(\Omega)} = (\int_\Omega |v(x)|^2 dx)^{1/2} = \|v\|$;
- $\|v\|_{H^1(\Omega)} = (\sum_{0\le|\alpha|\le 1} \|D^\alpha v\|^2_{L^2(\Omega)})^{1/2}$;
- $\|v\|_{W^1_\infty(\Omega)} = \max\,(\sum_{|\alpha|\le 1} \|D^\alpha_w v\|^\alpha_{L^\infty(\Omega)})$.

Define the time-space norms:

- $\|v\|_{L^\infty(X)} = \max_{0\le t\le T} \|v(\cdot,t)\|_X$;
- $\|v\|_{L^2(L^2)} = \int_0^T \|v(\cdot,t)\|^2_{L^2(\Omega)} dt$;
- $\|v\|_{L^\infty(L^\infty)} = \max_{0\le t\le T} \|v(\cdot,t)\|_{L^\infty(\Omega)}$.

Define the inner product as:

$$(f,g) = \int_\Omega fg\ dx.$$

Recall the Cauchy-Schwartz inequality:

$$\|vw\| \le \|v\| \cdot \|w\|,$$

and the arithmetic geometric mean inequality:

$$ab \le \frac{1}{4\varepsilon}a^2 + \varepsilon b^2, \quad \varepsilon > 0.$$

We use the following standard tools frequently throughout this paper:

Gronwall's Lemma 2.1 *If f, g, h are piecewise-continuous, non-negative functions and g is non-decreasing, and*

$$f(t) + h(t) \le g(t) + \int_a^t f(s)ds,\ \forall t \in [a,b],$$

then

$$f(t) + h(t) \le e^{t-a} g(t).$$

The analogous general discrete version from Heywood and Rannacher [4] is:

Gronwall's Lemma 2.2 *Let Δt, B, $\mathcal{K}$ and a^n, b^n, c^n, γ^n (for integers $n \geq 0$) be non-negative numbers such that*

$$a^N + \Delta t \sum_{n=0}^{N} b^n \leq \Delta t \sum_{n=0}^{N} \gamma^n a^n + \Delta t \sum_{n=0}^{N} c^n + B \;\; for \;\; n \geq 0. \qquad (1)$$

Suppose that $\Delta t \gamma^n < 1$ for all n, and set $\sigma^n \equiv (1 - \Delta t \gamma^n)^{-1}$. Then,

$$a^n + \Delta t \sum_{n=0}^{N} b^n \leq exp\left(\Delta t \sum_{n=0}^{N} \sigma^n \gamma^n\right) + \left\{\Delta t \sum_{n=0}^{N} c^n + B\right\} \;\; for \;\; n \geq 0. \qquad (2)$$

Remark: If the sum on the right in (1) extends only up to N-1, then estimate (2) holds for all $\Delta t > 0$ with $\sigma^n \equiv 1$.

2.2 The Physical Model

Shallow water equations model flow in water bodies under the following assumptions as considered in Weiyan [9].

- The change in the underwater topography is not abrupt.
- The water body is shallow. The total water depth $H \ll$ the wave length or the characteristic length of water body, L.
- The value of the horizontal space scale is between 1 and 1000 meters.

In addition, we define the following as in the literature:

- ξ is the free surface elevation above the geoid,
- h_b is the bathymetry,
- $H = h_b + \xi$ is the total fluid depth,
- $\mu > 0$ is the viscosity,
- $\tau = c\sqrt{u}/h_b > 0$ is the bottom friction where is c is a friction coefficient,
- $\tau_\circ$ is a tuning parameter, ($\tau_\circ > 0$) ,
- g is the gravitational acceleration,
- F corresponds to external body forces which we will assume to be bounded in $L^2(L^2)$,
- u is the depth-averaged fluid velocity.

2.3 The Mathematical Model

The shallow water equations describe conservation of mass and momentum in a fluid. They are derived by depth-integration of the incompressible Navier-Stokes equations under the assumption of hydrostatic pressure distribution. The 1-dimensional shallow water equations model the propagation of planar free surface waves. We focus our attention on the primitive equation formulation and the wave equation formulation [5]. The primitive shallow water equations formulation (hereafter referred to as P-SWE) are expressed as the continuity equation:

$$\xi_t + (uH)_x = 0 \tag{3}$$

and the non-conservative momentum equation:

$$u_t + uu_x + g\xi_x - \frac{\mu}{H}(uH)_{xx} + \tau u + F = 0. \tag{4}$$

Let $U = uH$, then P-SWE may be expressed as (3) and the conservative momentum equation:

$$U_t + \left(\frac{UU}{H}\right)_x + Hg\xi_x - \mu U_{xx} + \tau U + HF = 0. \tag{5}$$

The P-SWE was reformulated into the wave shallow water equations [3] (hereafter referred to as W-SWE) by replacing the primitive continuity equation by the wave continuity equation:

$$\xi_{tt} + \tau_\circ \xi_t - \left(\frac{UU}{H}\right)_x - (gH\xi_x)_x - \mu\xi_{txx} - (\tau - \tau_\circ)U_x - (HF)_x = 0. \tag{6}$$

The W-SWE may be expressed by (4)-(6) or (5)-(6).

2.4 The Linearized Models

We restrict our attention to the linearized form of both the P-SWE and W-SWE. We use (3)-(5) as the primitive formulation and (4)-(6) as the wave formulation. In linearized form, (4) and (5) are shown to be equivalent. Before we present the linearized forms, consider an example of a simple linearization of u^2. We linearize u^2 as $u \cdot f(u)$, where $f(u)$ can be expressed as follows:

$$f(u) = \alpha, \quad \alpha \text{ a constant}$$

or

$$f(u) = f^*, \quad f^* \text{ independent of u}$$

Applying similar linearizations to P-SWE, we retain the continuity equation (3) without modification and linearize the nonconservative momentum equation in the following manner:

$$U_t + (Uf^*)_x + gH^*\xi_x - \mu U_{xx} + h_b F = 0, \tag{7}$$

where f^* is given and is independent of $\frac{U}{H}$. Similarly we assume H^* is given data in the original $gH\xi_x$ term, $\tau U = 0$. Finally, HF is linearized as $h_b F$.

To linearize the wave formulation, we write (6) as:

$$\xi_{tt} + \tau_\circ \xi_t - (gh_b\xi_x)_x + ((\tau_\circ - \tau)uh_b)_x - \mu\xi_{txx} - (HF)_x = 0, \tag{8}$$

and (4) as:

$$u_t + g\xi_x - \mu u_{xx} + \tau u + F = 0, \tag{9}$$

where the bottom friction is linearized as $\tau = c\frac{u}{h_b}$. Notice in our linearization of (6) to (8) simplify by letting $\left(\frac{UU}{H}\right)_x = 0$ (only in the wave formulation) whereas in the linearization of (4) to (9) we simplify by letting $uu_x = 0$.

2.5 Numerical Methods

Applying standard numerical methods to solve the shallow water equations, we use the Galerkin finite element method for the spatial discretization of both formulations. The finite element method is a general technique for constructing approximate solutions to boundary-value problems. This method involves dividing the domain of the solution into a finite number of subdomains or elements. Concepts of the variational form of differential equations can then be used to construct an approximate solution over the collection of finite elements [1].

The difference in our numerical methods for solving P-SWE versus W-SWE comes in the temporal discretization. For the W-SWE, we assume continuous time and hence no time-stepping. In solving the P-SWE, we apply a finite difference time-stepping method. Furthermore, we utilize a Chorin projection scheme to decouple elevation and velocity in the momentum equations. The innovation in our approach to solve the P-SWE is precisely the use of the Chorin projection which has been shown to allow larger time-steps than if we were to use a finite-difference time-stepping scheme alone.

3 Wave Formulation

3.1 Stability Analysis

We now analyze the stability of the wave continuity equation in the L^2 space. Stability analysis in a functional space refers to the measurement in the norm, in this case the L^2 norm, of the dependence of the weak solution at some final time on the initial conditions, the forcing data and the boundary conditions.

Theorem 3.1 *Let ξ and u be solutions to the linearized W-SWE (8)-(9). Let $\frac{\tau_o}{2} - 1 > 0$, $H \in L^\infty(L^\infty)$ with $\tau_\circ$ bounded above, and $\mathcal{K}$ a positive constant. If $\xi \in L^2(H^2(\Omega))$, $\xi_t \in L^2(H^2(\Omega))$, $\xi_{tt} \in L^2(H^2(\Omega))$, $u \in L^2(H^2(\Omega))$, and $u_t \in L^2(H^2(\Omega))$, and given homogeneous Dirchlet boundary conditions, then:*

$$\|\xi_t(T)\|^2 + \|u(T)\|^2 + 4(\frac{\tau_\circ}{2} - 1)\|\xi(T)\|^2 + \mu\|\xi_x(T)\|^2 + \|\sqrt{gh_b}\xi_x(T)\|^2$$
$$+\|\xi_t\|^2_{L^2(L^2)} + \mu\|\xi_{tx}\|^2_{L^2(L^2)} + \|\sqrt{gh_b}\,\xi_x\|^2_{L^2(L^2)} + \mu\|u_x\|^2_{L^2(L^2)} + \|\sqrt{\tau}\,u\|^2_{L^2(L^2)}$$
$$\leq \mathcal{K}\left\{\|\xi_t(0)\|^2 + \|\xi_x(0)\|^2 + \|\xi(0)\|^2 + \|u(0)\|^2 + \|F\|^2_{L^2(L^2)}\right\}.$$

Proof: In proving Theorem 3.1, we begin by taking the weak form of equations (8) -(9) and obtain:

$$(\xi_{tt}\,, v)+(\tau_\circ\xi_t\,, v)-((\tau_\circ-\tau)uh_b\,, v_x)+\mu(\xi_{tx}\,, v_x)+(HF, v_x)+(gh_b\xi_x\,, v_x) = 0\ \forall v \in H^1_0; \tag{10}$$

$$(u_t, w) + (g\xi_x, w) + \mu(u_x, w_x) + (\tau u, w) + (F, w) = 0\ \forall w \in H^1_0. \tag{11}$$

Set the test function $v = \xi$ in equation (10), then:

$$(\xi_{tt}, \xi)+\frac{\tau_\circ}{2}\frac{d}{dt}\|\xi\|^2-((\tau_\circ-\tau)uh_b\,, \xi_x)+\frac{\mu}{2}\frac{d}{dt}\|\xi_x\|^2+(HF, \xi_x)+\|\sqrt{gh_b}\,\xi_x\|^2 = 0.$$

Integrate this equation over time, from 0 to T, and integrate the first term by parts. Apply the Cauchy-Schwartz inequality to arrive at the following inequality:

$$\frac{\tau_\circ}{2}\|\xi(T)\|^2 + \frac{\mu}{2}\|\xi_x(T)\|^2 + \|\sqrt{gh_b}\xi_x\|^2_{L^2(L^2)}$$
$$\leq \frac{\tau_\circ}{2}\|\xi(0)\|^2 + \frac{\mu}{2}\|\xi_x(0)\|^2 + \|\xi_t(T)\|\,\|\xi(T)\|$$
$$+\|\xi_t(0)\|\,\|\xi(0)\| + \|\xi_t\|^2_{L^2(L^2)} + \|(\tau_\circ - \tau)h_b\|_{L^\infty(L^\infty)}\,\|u\|_{L^2(L^2)}\,\|\xi_x\|_{L^2(L^2)}$$
$$+\|H\|_{L^\infty(L^\infty)}\,\|F\|_{L^2(L^2)}\,\|\xi_x\|_{L^2(L^2)}. \tag{12}$$

Taking the test function $v = \xi_t$ in equation (10), integrating over time and applying the Cauchy-Schwartz inequality:

$$\frac{1}{2}\|\xi_t(T)\|^2 + \tau_\circ\|\xi_t\|^2_{L^2(L^2)} + \mu\|\xi_{tx}\|^2_{L^2(L^2)} + \frac{1}{2}\|\sqrt{gh_b}\,\xi_x(T)\|^2$$
$$\leq \frac{1}{2}\|\xi_t(0)\|^2 + \frac{1}{2}\|\sqrt{gh_b}\,\xi_x(0)\|^2 + \|(\tau_\circ - \tau)h_b\|_{L^\infty(L^\infty)}\,\|u\|_{L^2(L^2)}\,\|\xi_{tx}\|_{L^2(L^2)}$$
$$+\|H\|_{L^\infty(L^\infty)}\,\|F\|_{L^2(L^2)}\,\|\xi_{tx}\|_{L^2(L^2)}. \tag{13}$$

Similarly, taking $w = u$ in equation (11):

$$\frac{1}{2}\|u(T)\|^2 + \mu\|u_x\|^2_{L^2(L^2)} + \|\tau^{1/2}u\|_{L^2(L^2)}$$
$$\leq \frac{1}{2}\|u(0)\|^2 + \|g\xi_x\|_{L^2(L^2)}\,\|u\|_{L^2(L^2)} + \|F\|_{L^2(L^2)}\,\|u\|_{L^2(L^2)}. \tag{14}$$

Next add equations (12), (13) and (14), and apply the arithmetic geometric mean inequality to appropriate terms. Collecting terms, we arrive at the following inequality assuming $\frac{\tau_\circ}{2} - 1 > 0$:

$$\frac{1}{4}\|\xi_t(T)\|^2 + \frac{1}{2}\|u(T)\|^2 + (\frac{\tau_\circ}{2} - 1)\|\xi(T)\|^2 + \frac{\mu}{2}\|\xi_x(T)\|^2 + \frac{1}{2}\|\sqrt{gh_b}\,\xi_x(T)\|^2$$
$$+(\tau_\circ - 1)\|\xi_t\|^2_{L^2(L^2)} + \frac{\mu}{2}\|\xi_{tx}\|^2_{L^2(L^2)} + \frac{1}{4}\|\sqrt{gh_b}\,\xi_x\|^2_{L^2(L^2)} + \mu\|u_x\|^2_{L^2(L^2)} + \|\tau^{1/2}u\|^2_{L^2(L^2)}$$
$$\leq \frac{1}{2}\|\xi_t(0)\|^2 + \frac{1}{2}\|\sqrt{gh_b}\,\xi_x(0)\|^2 + \|\xi_t(0)\|\,\|\xi(0)\| + \frac{\tau_\circ}{2}\|\xi(0)\|^2 + \frac{\mu}{2}\|\xi_x(0)\|^2 + \frac{1}{2}\|u(0)\|^2$$
$$+\left[\frac{1}{\mu}\|(\tau_\circ - \tau)h_b\|^2_{L^\infty} + \frac{1}{2}\|\sqrt{\frac{g}{h_b}}\|^2_{L^\infty} + \frac{1}{2} + 2\|\frac{1}{\sqrt{gh_b}}\|^2_{L^\infty}\,\|(\tau_\circ - \tau)h_b\|^2_{L^\infty}\right]\|u\|^2_{L^2(L^2)}$$
$$+\left[\frac{1}{\mu}\|H\|^2_{L^\infty} + \frac{1}{2} + 2\|\frac{H}{\sqrt{gh_b}}\|^2_{L^\infty}\right]\|F\|^2_{L^2(L^2)}.$$

Applying Gronwall's Lemma 2.1 and collecting terms with K_1 and K_2 positive constants, our theorem is proved since:

$$\|\xi_t(T)\|^2 + \|u(T)\|^2 + 4(\frac{\tau_\circ}{2} - 1)\|\xi(T)\|^2 + \mu\|\xi_x(T)\|^2 + \|\sqrt{gh_b}\xi_x(T)\|^2$$
$$+\|\xi_t\|^2_{L^2(L^2)} + \mu\|\xi_{tx}\|^2_{L^2(L^2)} + \|\sqrt{gh_b}\,\xi_x\|^2_{L^2(L^2)} + \mu\|u_x\|^2_{L^2(L^2)} + \|\sqrt{\tau}\,u\|^2_{L^2(L^2)}$$
$$\leq e^{K_1T}\left\{3\|\xi_t(0)\|^2 + 2(gh_b + \tau_\circ)\|\xi_x(0)\|^2 + \|\xi(0)\|^2 + 2\|u(0)\|^2 + K_2\|F\|^2_{L^2(L^2)}\right\},$$

and

$$\|\xi_t(T)\|^2 + \|u(T)\|^2 + 4(\frac{\tau_0}{2} - 1)\|\xi(T)\|^2 + \mu\|\xi_x(T)\|^2 + \|\sqrt{gh_b}\xi_x(T)\|^2$$

$$+\|\xi_t\|^2_{L^2(L^2)} + \mu\|\xi_{tx}\|^2_{L^2(L^2)} + \|\sqrt{gh_b}\,\xi_x\|^2_{L^2(L^2)} + \mu\|u_x\|^2_{L^2(L^2)} + \|\sqrt{\tau}\,u\|^2_{L^2(L^2)}$$

$$\mathcal{K}\left\{\|\xi_t(0)\|^2 + \|\xi_x(0)\|^2 + \|\xi(0)\|^2 + \|u(0)\|^2 + \|F\|^2_{L^2(L^2)}\right\}.$$

This result indicates that the velocity (or elevation) at the final time depends, up to some constant, on the velocity at initial time as well as initial spatial variations in elevation and forcing terms.

3.2 Error Estimate of Wave Equations

Using finite element methods to solve for approximate solutions raises the question of the degree of accuracy of the approximation. The error is the difference between the exact solution and the approximate solution. The errors $(u_h - u)$ and $(\xi_h - \xi)$ are difficult to determine directly when u and ξ are not known. Therefore, we separate the errors into two parts:

$$u_h - u = (u_h - \psi) + (\psi - u),$$

$$\xi_h - \xi = (\xi_h - \phi) + (\phi - \xi)$$

where ϕ and ψ are comparison functions in S_h, a finite dimensional subspace contained in H_0^1, and ξ_h and u_h are the Galerkin finite element approximations to ξ and u contained in S_h. We can then estimate $(u_h - u)$ and $(\xi_h - \xi)$ in the L^2 norm:

$$\|u_h - u\|_{L^2} \leq \|u_h - \psi\|_{L^2} + \|u - \psi\|_{L^2}, \tag{15}$$

$$\|\xi_h - \xi\|_{L^2} \leq \|\xi_h - \phi\|_{L^2} + \|\xi - \phi\|_{L^2}. \tag{16}$$

We choose ϕ and ψ so that they give us good approximations of u and ξ onto the approximating spaces and so that we know $\|\xi - \phi\|_{L^2}$ and $\|u - \psi\|_{L^2}$ from approximation theory results. In particular, we use the following approximation theory result:

Approximation Theory Result 3.2 *For a function $v \in H^s(\Omega)$, there exists $\phi \in S_h^{r,k}$ satisfying*

$$\|v - \phi\|_{H^j} \leq C(k,r)h^{q-j}\|v\|_{H^s}.$$

where r is the polynomial degree of the basis functions, k is the continuity imposed on the basis functions (in the finite element approximation), $C(k,r)$ is a constant depending on k and r, $q = min\{s, r+1\}$, and $0 \leq j \leq k+1$.

Although the actual error can not be calculated unless the exact solution is known, an estimate of the error can provide useful information. In deriving our error estimate, we want an estimate of the closeness of u_h to u and ξ_h to ξ as the mesh is refined (i.e. as $h \to 0$.) This information is helpful in determining the accuracy one would expect if the number of elements are increased. Therefore we can obtain experimental rates of convergence.

Theorem 3.3 *Let ξ and u be solutions to the linearized W-SWE, (8)-(9). Let $\xi_h \in S_h^{r+1}$ and $u_h \in S_h^r$ be the Galerkin finite element approximations to ξ and u respectively, where S_h is a finite dimensional subspace of $H_0^1(\Omega)$. If $\frac{\tau_o}{2} - 1 > 0$, $2\tau_o - \mu > 0$, $\tau_h = \tau =$ given data, $F_h = F =$ given data, and $H_h = \xi_h + h_b$ with $\xi \in L^2(H^{r+2}(\Omega))$, $\xi_t \in L^2(H^{r+2}(\Omega))$, $\xi_{tt} \in L^2(H^{r+2}(\Omega))$, $\xi \in L^\infty(H^{r+2}(\Omega))$, $u \in L^2(H^{r+1}(\Omega))$, $u_t \in L^2(H^{r+1}(\Omega))$, and $u \in L^\infty(H^{r+1}(\Omega))$, then:*

$$\begin{gathered}
\|(\xi_h - \xi)(T)\| + \sqrt{2}\|\sqrt{gh_b}\,(\xi_h - \xi)_x(T)\| + 2\sqrt{\frac{\tau_o}{2} - 1}\ \|(\xi_h - \xi(T)\| \\
+\sqrt{2\mu}\,\|(\xi_h - \xi)_x(T)\| + \sqrt{2}\|(u_h - u)(T)\| + \sqrt{4\tau_o - 2\mu}\ \|\xi_h - \xi\|_{L^2(L^2)} \\
+\|\sqrt{gh}\,(\xi_h - \xi)_x\|_{L^2(L^2)} + \sqrt{2\mu}\|(u_h - u)_x\|_{L^2(L^2)} + 2\|\sqrt{\tau}(u_h - u)\|_{L^2(L^2)} \\
\leq Kh^r \approx \mathcal{O}(h^r)
\end{gathered}$$

for K a positive constant.

Proof: First recall the weak formulation of the wave equations, (8)-(9) as well as the following Galerkin finite element approximation with $S_h \subset H_0^1$ where S_h is a finite dimensional subspace:

$$\begin{gathered}
((\xi_h)_{tt}\,, v_h) + \tau_o((\xi_h)_t\,, v_h) - ((\tau_o - \tau)u_h h_b\,, (v_h)_x) + \mu((\xi_h)_{tx}\,, (v_h)_x) \\
+(H_h F, (v_h)_x) + (gh_b(\xi_h)_x\,, (v_h)_x) = 0 \ \forall v_h \in S_h^{r+1};
\end{gathered}$$

$$((u_h)_t, w_h) + (g(\xi_h)_x, w) + \mu((u_h)_x, (w_h)_x) + (\tau u_h, w_h) + (F, w_h) = 0 \ \forall w_h \in S_h^r.$$

We write an error equation resulting from subtracting the weak formulation from the discrete problem associated with the Galerkin Finite Element approximation. Letting $F_h = F$, $\tau_h = \tau$, and $H_h = \xi_h + h_b$, the resulting error equations are:

$$((\xi_h-\xi)_{tt}\,,v_h)+\tau_\circ((\xi_h-\xi)_t\,,v_h)-(\tau_\circ h_b(u_h-u)\,,(v_h)_x)+(\tau h_b(u_h-u)\,,(v_h)_x)$$
$$+\mu((\xi_h-\xi)_{tx}\,,(v_h)_x)+((\xi_h-\xi)F,(v_h)_x)+(gh_b(\xi_h-\xi)_x\,,(v_h)_x)=0;$$

$$((u_h-u)_t,w_h)+(g(\xi_h-\xi)_x,w)+\mu((u_h-u)_x,(w_h)_x)+(\tau(u_h-u),w_h)=0.$$

Then we rewrite the weak form so that we separate the error into two parts and choose our comparison functions, ϕ and ψ, appropriately. In particular, we let ϕ be the L^2 projection of ξ onto S_h so that $(\xi,v)=(\phi,v)$ and ψ be the elliptic projection of u onto S_h so that $(u_x,w_x)=(\psi_x,w_x)$. Then we have:

$$\begin{aligned}&((\xi_h-\phi)_{tt}\,,v_h)+\tau_\circ((\xi_h-\phi)_t\,,v_h)-(\tau_\circ h_b(u_h-\psi)\,,(v_h)_x)+(\tau h_b(u_h-\psi)\,,(v_h)_x)\\&\quad+\mu((\xi_h-\phi)_{tx}\,,(v_h)_x)+((\xi_h-\phi)F,(v_h)_x)+(gh_b(\xi_h-\phi)_x\,,(v_h)_x)\\&\quad=-(\tau_\circ h_b(u-\psi)\,,(v_h)_x)+(\tau h_b(u-\psi)\,,(v_h)_x)\\&\quad+\mu((\xi-\phi)_{tx}\,,(v_h)_x)+((\xi-\phi)F,(v_h)_x)+(gh_b(\xi-\phi)_x\,,(v_h)_x);\end{aligned}\tag{17}$$

$$\begin{aligned}&((u_h-\psi)_t,w_h)+(g(\xi_h-\phi)_x,w)+\mu((u_h-\psi)_x,(w_h)_x)+(\tau(u_h-\psi),w_h)\\&\quad=((u-\psi)_t,w_h)+(g(\xi-\phi)_x,w)+(\tau(u-\psi),w_h).\end{aligned}\tag{18}$$

Next, choose appropriate test functions v_h and w_h. In equation (17), first let $v_h=(\xi_h-\phi)_t$ then in the same equation let $v_h=(\xi_h-\phi)$. Apply the Cauchy-Schwartz inequality and the arithmetic geometric mean inequality. Collecting terms, we obtain:

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|(\xi_h-\phi)_t\|^2+\tau_\circ\|(\xi_h-\phi)_t\|^2+\frac{\mu}{2}\|(\xi_h-\phi)_{tx}\|^2+\frac{1}{2}\frac{d}{dt}\|\sqrt{gh_b}\,(\xi_h-\phi)_x\|^2\\&\quad\le\left(\frac{4}{\mu}\|\tau_\circ\|^2_{L^\infty}+\frac{4}{\mu}\|\tau h_b\|^2_{L^\infty}\right)\|(u_h-\psi)\|^2+\frac{4}{\mu}\|(\xi_h-\phi)F\|^2\\&\quad+\left(\frac{4}{\mu}\|\tau_\circ h_b\|^2_{L^\infty}+\frac{4}{\mu}\|\tau h_b\|^2_{L^\infty}\right)\|(u-\psi)\|^2+\frac{4}{\mu}\|(\xi-\phi)F\|^2\\&\quad+4\mu\|(\xi-\phi)_{tx}\|^2+\frac{4}{\mu}\|gh_b(\xi-\phi)_x\|^2,\end{aligned}\tag{19}$$

and

$$
\begin{aligned}
&((\xi_h-\phi)_{tt},(\xi_h-\phi))+\frac{\tau_\circ}{2}\frac{d}{dt}\|(\xi_h-\phi)\|^2\\
&+\frac{\mu}{2}\frac{d}{dt}\|(\xi_h-\phi)_x\|^2+\frac{5}{9}\|\sqrt{gh_b}(\xi_h-\phi)_x\|^2\\
&\quad\le\left(\frac{9}{2}\|\frac{\tau_\circ h_b}{\sqrt{gh_b}}\|^2_{L^\infty}+\frac{9}{2}\|\frac{\tau h_b}{\sqrt{gh_b}}\|^2_{L^\infty}\right)\|(u_h-\psi)\|^2+\frac{9}{2}\|\frac{F}{\sqrt{gh_b}}(\xi_h-\phi)\|^2\\
&\quad+\left(\frac{9}{2}\|\frac{\tau_\circ h_b}{\sqrt{gh_b}}\|^2_{L^\infty}+\frac{9}{2}\|\frac{\tau h_b}{\sqrt{gh_b}}\|^2_{L^\infty}\right)\|(u-\psi)\|^2+\frac{9}{2}\|\frac{F}{\sqrt{gh_b}}(\xi-\phi)\|^2\\
&\quad+\frac{9}{2}\mu^2\|\frac{1}{\sqrt{gh_b}}(\xi-\phi)_{tx}\|^2+\frac{9}{2}\|\sqrt{gh_b}(\xi-\phi)_x\|^2. \qquad (20)
\end{aligned}
$$

Similarly, setting $w_h=u_h-\psi$ in equation (18), we get:

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|u_h-\psi\|^2+\mu\|(u_h-\psi)_x\|^2+\|\sqrt{\tau}(u_h-\psi)\|^2\\
&\qquad\le\frac{1}{18}\|\sqrt{gh_b}\,(\xi_h-\phi)_x\|^2+\frac{1}{4}\|(u-\psi)_t\|^2+\frac{1}{4}\|g(\xi-\phi)_x\|^2\\
&\qquad+\frac{1}{4}\|\tau(u-\psi)\|^2+\left(3+\frac{9}{2}\|\sqrt{\frac{g}{h_b}}\|^2_{L^\infty}\right)\|u_h-\psi\|^2. \qquad (21)
\end{aligned}
$$

We assume $\frac{\tau_\circ}{2}-1>0$, add equations (19), (20), and (21), integrate over time and apply Gronwall's Lemma 2.2 with K_1 a positive constant. The resulting inequality is as follows:

$$
\begin{aligned}
&\|(\xi_h)_t(T)\|^2+\left(2\|\sqrt{gh_b}\|^2_{L^\infty}+2\mu\right)\|(\xi_h-\phi)_x(T)\|^2+4(\frac{\tau_\circ}{2}-1)\|(\xi_h-\phi)(T)\|^2\\
&\quad+2\|(u_h-\psi)(T)\|^2+(4\tau_\circ+2\mu)\|(\xi_h-\phi)_{tx}\|^2_{L^2(L^2)}\\
&\quad+2\|\sqrt{gh_b}\,(\xi_h-\phi)_x\|^2_{L^2(L^2)}+4\mu\|(u_h-\psi)_x\|^2_{L^2(L^2)}+4\|\sqrt{\tau}\,(u_h-\psi)\|^2_{L^2(L^2)}\\
&\quad\le e^{K_4T}\{\left(16\mu+\frac{9}{2}\|\frac{1}{\sqrt{gh_b}}\|^2_{L^\infty}\right)\|(\xi-\phi)_{tx}\|^2_{L^2(L^2)}\\
&\quad+\left(\frac{16}{\mu}\|gh_b\|^2_{L^\infty}+\frac{9}{2}\|\sqrt{gh_b}\|^2_{L^\infty}+\frac{1}{4}\|g\|^2_{L^\infty}\right)\|(\xi-\phi)_x\|^2_{L^2(L^2)}\\
&\quad+\left(\frac{16}{\mu}\|\tau_\circ h_b\|^2_{L^\infty}+\frac{16}{\mu}\|\tau h_b\|^2_{L^\infty}+18\|\frac{\tau_\circ}{\sqrt{gh_b}}\|^2_{L^\infty}+\frac{9}{2}\|\frac{\tau h_b}{\sqrt{gh_b}}\|^2_{L^\infty}+\frac{1}{4}\|\tau\|^2_{L^\infty}\right)\|u-\psi\|^2_{L^2(L^2)}\\
&\quad+\|F\|^2_{L^\infty}\left(\frac{16}{\mu}+\frac{9}{2}\|\frac{1}{\sqrt{gh_b}}\|^2_{L^\infty}\right)\|(\xi-\phi)\|^2_{L^2(L^2)}+\frac{1}{4}\|(u-\psi)_t\|^2_{L^2(L^2)}\}. \qquad (22)
\end{aligned}
$$

Equation (22) gives us a bound on $u_h - \psi$ and $\xi_h - \phi$. Now we apply the triangle inequality from (15) and (16), and use the our approximation theory result with piecewise continuous basis functions. This yields the following inequality with $\mathcal{C}$ and K constants, and our theorem is proved.

$$
\begin{aligned}
&\|(\xi_h - \xi)(T)\| + \sqrt{2}\|\sqrt{gh_b}\,(\xi_h - \xi)_x(T)\| + 2\sqrt{\frac{\tau_\circ}{2} - 1}\|(\xi_h - \xi)(T)\| \\
&\quad + \sqrt{2\mu}\|(\xi_h - \xi)_x(T)\| + \sqrt{2}\|(u_h - u)(T)\| + \sqrt{4\tau_\circ - 2\mu}\|(\xi_h - \xi)\|_{L^2(L^2)} \\
&\quad + \|\sqrt{gh_b}(\xi_h - \xi)_x\|_{L^2(L^2)} + \sqrt{2\mu}\|(u_h - u)_x\|_{L^2(L^2)} + 2\|\sqrt{\tau}(u_h - u)\|_{L^2(L^2)} \\
&\quad \leq \mathcal{C}h^r \left(\|\xi_t\|_{L^2(H^{r+2})} + \|\xi\|_{L^2(H^{r+2})} + \|u\|_{L^2(H^{r+1})} + \|u_t\|_{L^2(H^{r+1})}\right) \\
&\quad + \mathcal{C}h^r \left(\|\xi\|_{L^\infty(H^{r+2})} + \|u\|_{L^\infty(H^{r+1})}\right) + \mathcal{C}h^{r+1}\|\xi\|_{L^2(H^{r+2})} \\
&\quad \leq Kh^r + Kh^{r+1} \approx \mathcal{O}(h^r)
\end{aligned}
$$

4 Chorin Projection for the Primitive Formulation

The Chorin projection is a particular type of Helmholtz decomposition of the velocity variable. In practice, it looks like a predictor-corrector of the velocity variable. It was first introduced by A. J. Chorin [2] in the incompressible Navier-Stokes setting. In 1997, Najm and Wyckoff [8] applied this projection method to compressible combustion applications. We will give an analysis of the Chorin projection for the primitive formulation. This projection allows the decoupling of the wave elevation and velocity terms in the momentum equation. The implications are two fold. First, by decoupling the wave elevation and velocity terms, we are not forced to solve the simultaneous equations thereby decreasing computational expense. Second, and more importantly, the Chorin projection, in conjunction with a finite-difference time-stepping scheme, should allow us to take larger time steps, Δt than traditional time-stepping methods.

Using this projection scheme we arrive at the following results:

Theorem 4.1 *Let* $1-\frac{4\Delta t}{\mu} > 0$ *and* $1-\mathcal{Q}\Delta t > 0$ *where* $\mathcal{Q} = \frac{\mu}{8}\|g(H^*)^{n-1}\|_{W^1_\infty(\Omega)} + 2\mathcal{K} + \frac{2\mathcal{K}^2}{\mu} + 1 > 0$ *a constant. In addition, let* H^* *and* f^* *be in* W^1_∞, $\alpha = min\ \{1 - \frac{4\Delta t}{\mu}, 1 - \mathcal{Q}\Delta t\}$ *and* $\mathcal{K}^* > 0$ *be a constant. Assume the following boundary conditions:* $\tilde{U} = 0$ *and* $\xi_x = 0$ *at the land boundary, and*

$\tilde{U}_x = 0$ *and* $\xi(t) = 0$ *at the ocean boundary, then:*

$$\alpha\left[\|\tilde{U}^{N+1}\|^2 + \|\xi^{N+1}\|^2\right] + \sum_{n=0}^{N} 2(\Delta t)^2 \|(g(H^*)^n)^{1/2}\xi_x^{n+1}\|^2 + \sum_{n=0}^{N} \mu\Delta t\|\tilde{U}^{n+1}\|^2$$

$$\leq exp\{\mathcal{K}^*\Delta t\}\left(\sum_{n=0}^{N}\|h_b\|^2_{L^\infty(L^\infty)}\|F^n\|^2\Delta t + \|\tilde{U}^0\|^2 + \left(1+\frac{2\Delta t}{\mu}\right)\|\xi^0\|^2\right).$$

Proof: Consider the linearized P-SWE equations (3) and (7) and recall that $U = uH$. Let $U^{n+1} = U(x, t^{n+1})$ where $t^{n+1} = (n+1)\Delta t$. Then we set up the projection scheme writing (7) as two equations:

$$\frac{\tilde{U}^{n+1} - U^n}{\Delta t} + (\tilde{U}^{n+1} f)_x - \mu\tilde{U}^{n+1}_{xx} + h_b F^n = 0 \tag{23}$$

and

$$\frac{U^{n+1} - \tilde{U}^{n+1}}{\Delta t} + gH^{*n}\xi_x^{n+1} = 0. \tag{24}$$

Notice, (24) contains the coupled term, ξ, and (23) does not. Next, we take the spatial derivative of (24):

$$U_x^{n+1} - \tilde{U}_x^{n+1} + \Delta t(g(H^*)^n\xi_x^{n+1})_x = 0. \tag{25}$$

Then approximate ξ_t in (3) by a first-order Taylor expansion:

$$\frac{\xi^{n+1} - \xi^n}{\Delta t} = -U_x^{n+1} \tag{26}$$

and substitute(25) into (26):

$$\xi^{n+1} - \xi^n = -\Delta t\tilde{U}_x^{n+1} + (\Delta t)^2(g(H^*)^n\xi_x^{n+1})_x.$$

Assuming $\xi_x = 0$ at the land boundary and $\xi(t) = 0$ at the ocean boundary, we multiply by a test function, v, and integrate by parts obtaining:

$$(\xi^{n+1} - \xi^n, v) + (\Delta t)^2(g(H^*)^n\xi_x^{n+1}, v_x) = -\Delta t(\tilde{U}_x^{n+1}, v), \;\; \forall v \in H^1_{0\,(ocean)}.$$

Set $v = \xi^{n+1}$, multiply by 2 and apply the Cauchy-Schwartz inequality.

$$\|\xi^{n+1}\|^2 - \|\xi^n\|^2 + 2(\Delta t)^2\|(g(H^*)^n)^{1/2}\xi_x^{n+1}\|^2 \leq 2\Delta t\|\tilde{U}_x^{n+1}\|\,\|\xi^{n+1}\|. \tag{27}$$

In (23), let $\tilde{U}_x = 0$ on the ocean boundary and $\tilde{U} = 0$ on the land boundary. Multiplying by test function, w, and integrating by parts, we get the following:

$$\left(\frac{\tilde{U}^{n+1} - U^n}{\Delta t}, w\right) + ((\tilde{U}^{n+1} f^*)_x, w) + \mu(\tilde{U}_x^{n+1}, w_x) = -(h_b F^n, w) \;\; \forall w \in H^1_{0\,(land)}. \tag{28}$$

Note from (24), we have:

$$\left(\frac{U^n}{\Delta t}, v\right) = \left(\frac{\tilde{U}^n}{\Delta t}, v\right) - (g(H^*)^{n-1}\xi_x^n, v) \;\; \forall v \in H^1_{0\,(ocean)}.$$

Integrating by parts and recalling the land boundary condition $\xi(t) = 0$ we obtain the following expression:

$$\left(\frac{U^n}{\Delta t}, v\right) = \left(\frac{\tilde{U}^n}{\Delta t}, v\right) + ((g(H^*)^{n-1}v)_x, \xi^n),$$

which can by substituted into (28) to arrive at:

$$\left(\frac{\tilde{U}^{n+1} - \tilde{U}^n}{\Delta t}, w\right) - ((g(H^*)^{n-1}w)_x, \xi^n) + ((\tilde{U}^{n+1} f^*)_x, w) + \mu(\tilde{U}_x^{n+1}, w_x) = -(h_b F^n, w).$$

Setting $w = \tilde{U}^{n+1}$, multiplying by $2\Delta t$, and applying the Cauchy-Schwartz inequality:

$$\begin{aligned} &\|\tilde{U}^{n+1}\|^2 - \|\tilde{U}^n\|^2 + 2\mu\Delta t\|\tilde{U}_x^{n+1}\|^2 \\ &\quad \leq 2\Delta t\|g(H^*)^{n-1}\|_{W^1_\infty(\Omega)} \|\tilde{U}^{n+1}\|_1 \|\xi^n\| + 2\Delta t\mathcal{C}(f^*_{W^1_\infty(\Omega)})(\|\tilde{U}^{n+1}\|^2 + \|\tilde{U}_x^{n+1}\| \, \|\tilde{U}^{n+1}\|) \\ &\quad +2(\Delta t)\|h_b\|_{L^\infty(L^\infty)}\|F^n\| \, \|\tilde{U}^{n+1}\|. \end{aligned} \tag{29}$$

Let $\mathcal{K} = \mathcal{C}(f^*_{W^1_\infty(\Omega)})$. Add (27) and (29), apply the arithmetic geometric mean inequality, and collect terms. Then let $\mathcal{Q} = \frac{\mu}{8}\|g(H^*)^{n-1}\|_{W^1_\infty(\Omega)} + 2\mathcal{K} + \frac{2\mathcal{K}^2}{\mu} + 1$:

$$\begin{aligned} &\|\tilde{U}^{n+1}\|^2 + \|\xi^{n+1}\|^2 - \left[\|\tilde{U}^n\|^2 + \|\xi^n\|^2\right] + 2(\Delta t)^2\|(g(H^*)^n)^{1/2}\xi_x^{n+1}\|^2 + \mu\Delta t\|\tilde{U}_x^{n+1}\|^2 \\ &\quad \leq \frac{2\Delta t}{\mu}\left[\|\xi^{n+1}\|^2 + \|\xi^n\|^2\right] + \mathcal{Q}\Delta t\|\tilde{U}^{n+1}\|^2 + (\Delta t) + \|h_b\|^2_{L^\infty(L^\infty)}\|F^n\|^2. \end{aligned}$$

Summing over time we get:

$$\|\tilde{U}^{N+1}\|^2 + \|\xi^{N+1}\|^2 + 2(\Delta t)^2 \sum_{n=0}^{N} \|(g(H^*)^n)^{1/2}\xi_x^{n+1}\|^2 + \mu\Delta t \sum_{n=0}^{N} \|\tilde{U}_x^{n+1}\|^2$$
$$\leq \frac{2}{\mu}\Delta t \sum_{n=0}^{N} \left[\|\xi^{n+1}\|^2 + \|\xi^n\|^2\right] + \mathcal{Q}\Delta t \sum_{n=0}^{N} \|\tilde{U}^{n+1}\|^2$$
$$+\Delta t \sum_{n=0}^{N} \|h_b\|^2_{L^\infty(L^\infty)}\|F^n\|^2 + \|\tilde{U}^0\|^2 + \|\xi^0\|^2.$$

Since $\|\xi^n\|^2$ and $\|\tilde{U}^n\|^2$ are positive for all n, the following inequality holds for Δt sufficiently small:

$$\|\tilde{U}^{N+1}\|^2 + \|\xi^{N+1}\|^2 + 2(\Delta t)^2 \sum_{n=0}^{N} \|(g(H^*)^n)^{1/2}\xi_x^{n+1}\|^2 + \mu\Delta t \sum_{n=0}^{N} \|\tilde{U}_x^{n+1}\|^2$$
$$\leq \frac{4}{\mu}\Delta t \sum_{n=0}^{N+1} \|\xi^n\|^2 + \mathcal{Q}\Delta t \sum_{n=0}^{N+1} \|\tilde{U}^n\|^2$$
$$+\Delta t \sum_{n=0}^{N} \|h_b\|^2_{L^\infty(L^\infty)}\|F^n\|^2 + \|\tilde{U}^0\|^2 + \left(1 + \frac{2\Delta t}{\mu}\right)\|\xi^0\|^2.$$

Begin to set up the inequality to use Gronwall's Lemma 2.2 by letting $\alpha = min\ \{1 - \frac{4\Delta t}{\mu}, 1 - \mathcal{Q}\Delta t\}$:

$$\|\tilde{U}^{N+1}\|^2 + \|\xi^{N+1}\|^2 + \frac{2(\Delta t)^2}{\alpha} \sum_{n=0}^{N} \|(g(H^*)^n)^{1/2}\xi_x^{n+1}\|^2 + \frac{\mu\Delta t}{\alpha} \sum_{n=0}^{N} \|\tilde{U}_x^{n+1}\|^2$$
$$\leq \frac{4\Delta t}{\mu\alpha} \sum_{n=0}^{N} \|\xi^n\|^2 + \frac{\mathcal{Q}\Delta t}{\alpha} \sum_{n=0}^{N} \|\tilde{U}^n\|^2$$
$$+\frac{\Delta t}{\alpha} \sum_{n=0}^{N} \|h_b\|^2_{L^\infty(L^\infty)}\|F^n\|^2 + \frac{1}{\alpha}\|\tilde{U}^0\|^2 + \frac{1}{\alpha}\left(1 + \frac{2\Delta t}{\mu}\right)\|\xi^0\|^2.$$

Then let $\mathcal{K}^* = max\{\frac{4}{\alpha\mu}, \frac{\mathcal{Q}}{\alpha}\}$ and:

$$\|\tilde{U}^{N+1}\|^2 + \|\xi^{N+1}\|^2 + \frac{2(\Delta t)^2}{\alpha} \sum_{n=0}^{N} \|(g(H^*)^n)^{1/2}\xi_x^{n+1}\|^2 + \frac{\mu\Delta t}{\alpha} \sum_{n=0}^{N} \|\tilde{U}_x^{n+1}\|^2$$
$$\leq \mathcal{K}^*\Delta t \sum_{n=0}^{N} \left[\|\xi^n\|^2 + \|\tilde{U}^n\|^2\right] + \frac{\Delta t}{\alpha} \sum_{n=0}^{N} \|h_b\|^2_{L^\infty(L^\infty)}|F^n\|^2$$

$$+\frac{1}{\alpha}\|\tilde{U}^0\|^2+\frac{1}{\alpha}\left(1+\frac{2\Delta t}{\mu}\right)\|\xi^0\|^2.$$

Finally, we apply Gronwall's Lemma 2.2. Since the sums extend up to N (as opposed to $N+1$), then for all $\Delta t>0$ and for all $\mathcal{K}^*>0$ with $\sigma^n\equiv 1$, then our result is proved:

$$\alpha\left[\|\tilde{U}^{N+1}\|^2+\|\xi^{N+1}\|^2\right]+2(\Delta t)^2\sum_{n=0}^{N}\|(g(H^*)^n)^{1/2}\xi_x^{n+1}\|^2+\mu\Delta t\sum_{n=0}^{N}\|\tilde{U}_x^{n+1}\|^2$$
$$\leq exp\{\mathcal{K}^*\Delta t\}\left(\Delta t\sum_{n=0}^{N}\|h_b\|^2_{L^\infty(L^\infty)}\|F^n\|^2+\|\tilde{U}^0\|^2+\left(1+\frac{2\Delta t}{\mu}\right)\|\xi^0\|^2\right).$$

5 Discussion

In deriving stability and error results, there are two things we must consider. Are our assumptions valid and what do our results mean? In stability Theorem 3.1, is our assumption that $\frac{\tau_\circ}{2}-1>0$ reasonable? That is, does it make sense that $\tau_\circ>2$. Furthermore, Theorem 3.1 tell us velocity and elevation at final time depend up to some constant on the velocity at initial time, forcing data, and initial spatial variations in elevations. That is:

$$\|u(T)\|^2\leq K\left\{\|\xi(0)_t\|^2+\|\xi_x(0)\|^2+\|\xi(0)\|^2+\|u(0)\|^2+\|F\|^2_{L^2(L^2)}\right\}$$

and

$$\|\xi(T)\|^2\leq K\left\{\|\xi(0)_t\|^2+\|\xi_x(0)\|^2+\|\xi(0)\|^2+\|u(0)\|^2+\|F\|^2_{L^2(L^2)}\right\}.$$

In Theorem 3.3, we have the additional condition that $2\tau_\circ-\mu>0$. Not only must $\tau_\circ>2$ but now, $\tau_\circ>\frac{\mu}{2}$. It is also important to notice that u_h converges like h^{r+1} to the true solution u while ξ_h converges like h^r to its true solution ξ. We address the spatial derivatives in u by using an elliptic projection. In general the elliptic projection yields higher order convergence than L^2 projection. Then why not use the elliptic projection of ξ on $H^{(h)}$? Nothing is gained by using the elliptic projection of ξ. The spatial derivative terms will not vanish, and there will still be first order h terms. Nonetheless, this result tells us that u_h converges faster than ξ_h to its true solution.

Finally, in the stability result Theorem 4.1, we have two conditions on Δt: $1-\frac{4\Delta t}{\mu}>0$ and $1-\mathcal{Q}\Delta t>0$. These are relative constraints. We could give a

better analysis by using non-dimensionalization. The non-dimensionalization allows us to give a measure of size to all the units. Consequently, if we were to use dimensionless equations, then we could get a reference for the size of Δt. We will leave this for future work. Again, this stability result, tells us that the velocity and elevation at final time depend on the initial velocity and elevation and forcing data. In particular:

$$\|\tilde{U}^{n+1}\|^2 \leq K \left\{ \sum_{n=0}^{N} \|F^n\|^2 \Delta t + \|\tilde{U}^0\|^2 + \|\xi^0\|^2 \right\}$$

and

$$\|\xi^{n+1}\|^2 \leq K \left\{ \sum_{n=0}^{N} \|F^n\|^2 \Delta t + \|\tilde{U}^0\|^2 + \|\xi^0\|^2 \right\}.$$

6 Future Work

The analysis presented here is a start to much work that has yet to be done. The next step would be to perform numerical simulations and compare them to the theoretical results we obtained for both the linearized W-SWE and the linearized P-SWE. In addition to analyzing the dimensionless equations for the Chorin projection, we would also like to derive an error estimate for this scheme. Eventually, a more general analysis includes assuming general boundary conditions rather than the homogeneous Dirchlet boundary conditions we assumed for the linearized W-SWE. Once the linearized W-SWE and linearized P-SWE have been completely explored, we can then begin to analyze the fully nonlinear 1-dimensional equations. For convenience in our analysis we linearized the advection terms by setting them equal to 0. By using the mixed finite element method for our spatial discretization instead of the Galerkin finite element method we may be able to handle the advection dominated terms more easily.

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