

**Finite Element Approximations to
the System of Shallow Water
Equations, Part III: On the
Treatment of Boundary Conditions**

*Clint N. Dawson and Monica L.
Martinez*

**CRPC-TR98764
May 1998**

Center for Research on Parallel Computation
Rice University
6100 South Main Street
CRPC - MS 41
Houston, TX 77005

FINITE ELEMENT APPROXIMATIONS TO THE SYSTEM OF SHALLOW WATER EQUATIONS, PART III: ON THE TREATMENT OF BOUNDARY CONDITIONS *

CLINT N. DAWSON AND MÓNICA L. MARTÍNEZ†

May 15, 1998

Abstract. We continue our investigation of finite element approximations to the system of shallow water equations, based on the generalized wave continuity equation (GWCE) formulation. In previous work, we analyzed this system assuming Dirichlet boundary conditions on both elevation and velocity. Based on physical grounds, it is possible to not impose boundary conditions on elevation. Thus, we examine the formulation for the case of Dirichlet conditions on velocity only. The changes required to the finite element method are presented, and stability and error estimates are derived for both an approximate linear model and a full nonlinear model, assuming continuous time. Stability for a discrete time method is also shown.

Key words. boundary conditions, shallow water equations, wave shallow water equations, error estimates

AMS subject classifications. 35Q35, 35L65 65N30, 65N15

1. Introduction. In this paper, we continue our investigation of finite element methods applied to the GWCE (Generalized Wave Continuity Equation) shallow water model of Gray *et al.* This model is described in a series of papers beginning in [8]. It has served as the basis for many shallow water simulators, most notably the Advanced Circulation Model (ADCIRC) described, for example, in [7]. The method has the advantages that it allows for a weaker coupling between the continuity and momentum equations, gives rise to symmetric positive definite matrices, and helps stabilize the numerical solution. These have been supported by a large number of studies (see [5, 6] and references therein).

In previous papers [3, 4], we derived *a priori* error estimates for the method, in both continuous and discrete time, assuming Dirichlet boundary conditions on both the free surface elevation and velocity. In this paper, we will relax this assumption on the elevation and discuss the changes to the model and to the analysis. As it turns out, the assumption of Dirichlet boundary conditions on elevation allowed for a crucial substitution which substantially simplified the analysis. However, by making appropriate changes to the model, we will demonstrate that we are still able to preserve the accuracy of the method, at the cost of some additional computational work.

We will denote by $\xi(\mathbf{x}, t)$ the free surface elevation over a reference plane and by $h_b(\mathbf{x})$ the bathymetric depth under that reference plane so that $H(\mathbf{x}, t) = \xi + h_b$ is the total water column. Also, we denote by $\mathbf{u} = [u(\mathbf{x}, t) \ v(\mathbf{x}, t)]^T$ the depth-averaged horizontal velocities and we let $\mathbf{U} = \mathbf{u}H$.

We will start with the following simplified linear shallow water model:

$$(1) \quad \xi_t + \nabla \cdot \mathbf{U} = 0,$$

$$(2) \quad \mathbf{U}_t + G \nabla \xi - \mu \Delta \mathbf{U} = \mathcal{F},$$

* This work was supported in part by National Science Foundation, Project No. DMS-9408151.

† Center for Subsurface Modeling - C0200; Texas Institute for Computational and Applied Mathematics; The University of Texas at Austin; Austin, TX 78712. This work was supported by National Science Foundation, Project No. DMS-9408151.

which we solve over a domain $\Omega \times (0, T]$. Here $G > 0$ is a gravitational constant and $\mu > 0$ is the eddy viscosity coefficient.

Note that, integrating (1) over Ω ,

$$\int_{\Omega} \xi_t d\Omega + \int_{\partial\Omega} \mathbf{U} \cdot \nu ds = 0,$$

where ν is the outward normal to $\partial\Omega$. Moreover, integrating (2) over Ω ,

$$\int_{\Omega} [\mathbf{U}_t + G\nabla\xi] d\Omega - \mu \int_{\partial\Omega} \nabla\mathbf{U} \cdot \nu ds = \int_{\Omega} \mathcal{F} d\Omega.$$

Thus, it is necessary to specify some type of Dirichlet or Neumann boundary condition on \mathbf{U} , but it is not required (nor may it be desirable) to specify a boundary condition on ξ .

We will assume the Dirichlet boundary condition

$$(3) \quad \mathbf{U} = \mathbf{g},$$

on $\partial\Omega \times (0, T]$. We also assume initial conditions

$$(4) \quad \xi(x, 0) = \xi^0(x), \quad \mathbf{U}(x, 0) = \mathbf{U}^0(x).$$

The GWCE is obtained by differentiating (1) with respect to time and substituting the divergence of (2) into the result. We then obtain

$$(5) \quad \xi_{tt} - \nabla \cdot (G\nabla\xi) + \mu \nabla \cdot \Delta\mathbf{U} + \nabla \cdot \mathcal{F} = 0.$$

with the additional initial condition that

$$(6) \quad \xi_t(x, 0) = \xi_1(x) \equiv -\nabla \cdot \mathbf{U}^0.$$

The GWCE shallow water model then consists of (2) and (3)-(6).

The rest of this paper is outlined as follows. In section (2) we introduce definitions and notation. In section (3), we derive a weak formulation of the GWCE-CME system of equations and state some assumptions on the solution. In section (4), we introduce the continuous-time finite element approximation to the weak solution, and derive stability and *a priori* error estimates for this approximation. In section (5), we extend these estimates to a nonlinear shallow water model. Finally, in section (6), we discuss a discrete time approximation to the linear model given above.

2. Preliminaries.

2.1. Notation. For the purposes of our analysis, we define some notation used throughout the rest of this paper.

Let Ω be a bounded polygonal domain in \mathbb{R}^2 and $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$. Moreover, let $\bar{\Omega} = \Omega \cup \partial\Omega$, where $\partial\Omega$ is the boundary of Ω .

The \mathcal{L}^2 inner product is denoted by

$$(\varphi, \omega) = \int_{\Omega} \varphi \diamond \omega \, dx, \quad \varphi, \omega \in [\mathcal{L}^2(\Omega)]^n,$$

where “ \diamond ” here refers to either multiplication, dot product, or double dot product as appropriate. We will let $\langle \varphi, \omega \rangle$ denote an inner product on $\partial\Omega$. We denote the \mathcal{L}^2

norm by $\|\varphi\| = (\varphi, \varphi)^{1/2}$. In \mathbb{R}^n , $\alpha = (\alpha_1, \dots, \alpha_n)$ is an n -tuple with nonnegative integer components,

$$D^\alpha = D_1^{\alpha_1} \dots D_n^{\alpha_n} = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \dots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}$$

and $|\alpha| = \sum_{i=1}^n \alpha_i$.

For ℓ any nonnegative integer, let

$$\mathcal{H}^\ell \equiv \{\varphi \in \mathcal{L}^2(\Omega) \mid D^\alpha \varphi \in \mathcal{L}^2(\Omega) \text{ for } |\alpha| \leq \ell\}$$

be the Sobolev space with norm

$$\|\varphi\|_{\mathcal{H}^\ell(\Omega)} = \left(\sum_{|\alpha| \leq \ell} \|D^\alpha \varphi\|_{\mathcal{L}^2(\Omega)}^2 \right)^{1/2}.$$

Additionally, $\mathcal{H}_0^1(\Omega)$ denotes the subspace of $\mathcal{H}^1(\Omega)$ obtained by completing $\mathcal{C}_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{\mathcal{H}^1(\Omega)}$, where $\mathcal{C}_0^\infty(\Omega)$ is the set of infinitely differentiable functions with compact support in Ω .

Moreover, let

$$\mathcal{W}_\infty^\ell \equiv \{\varphi \in \mathcal{L}^\infty(\Omega) \mid D^\alpha \varphi \in \mathcal{L}^\infty(\Omega) \text{ for } |\alpha| \leq \ell\}$$

be the Sobolev space with norm

$$\|\varphi\|_{\mathcal{W}_\infty^\ell(\Omega)} = \max_{|\alpha| \leq \ell} \|D^\alpha \varphi\|_{\mathcal{L}^\infty(\Omega)}.$$

For relevant properties of these spaces, please refer to [1].

Observe, for instance, that \mathcal{H}^ℓ are spaces of \mathbb{R} -valued functions. Spaces of \mathbb{R}^n -valued functions will be denoted in boldface type, but their norms will not be distinguished. Thus, $\mathcal{L}^2(\Omega) = [\mathcal{L}^2(\Omega)]^n$ has norm $\|\varphi\|^2 = \sum_{i=1}^n \|\varphi_i\|^2$; $\mathcal{H}^1(\Omega) = [\mathcal{H}^1(\Omega)]^n$ has norm $\|\varphi\|_{\mathcal{H}^1(\Omega)}^2 = \sum_{i=1}^n \sum_{|\alpha| \leq 1} \|D^\alpha \varphi_i\|^2$; etc. For X , a normed space with norm

$\|\cdot\|_X$ and a map $f: [0, T] \rightarrow X$, define

$$\begin{aligned} \|f\|_{\mathcal{L}^2(0,T;X)}^2 &= \int_0^T \|f(\cdot, t)\|_X^2 \Delta t, \\ \|f\|_{\mathcal{L}^\infty(0,T;X)} &= \sup_{0 \leq t \leq T} \|f(\cdot, t)\|_X. \end{aligned}$$

3. Weak formulation. A weak formulation of (5) is obtained as follows. From (1), we have

$$(7) \quad (\xi_t, v) - (\mathbf{U}, \nabla v) + \langle \mathbf{g} \cdot \nu, v \rangle = 0.$$

Differentiating this equation in time and using (2) we find

$$(8) \quad (\xi_{tt}, v) + (G \nabla \xi, \nabla v) - \mu(\Delta \mathbf{U}, \nabla v) - (\mathcal{F}, \nabla v) + \langle \mathbf{g}_t \cdot \nu, v \rangle = 0, \quad v \in \mathcal{H}^1(\Omega),$$

Moreover, multiplying (2) by a test function and integrating by parts,

$$(9) \quad (\mathbf{U}_t, \mathbf{w}) + (G \nabla \xi, \mathbf{w}) + \mu(\nabla \mathbf{U}, \nabla \mathbf{w}) = (\mathcal{F}, \mathbf{w}) + \mu \langle \nabla \mathbf{U} \cdot \nu, \mathbf{w} \rangle, \quad \mathbf{w} \in \mathcal{H}^1(\Omega).$$

In our previous work, we were able to replace the term involving $\Delta \mathbf{U}$ in (8) by

$$\mu(\nabla \xi_t, \nabla v),$$

because of the assumption of Dirichlet boundary conditions on ξ . We note that, by (1),

$$\Delta \xi_t = -\Delta(\nabla \cdot \mathbf{U}) = -\nabla \cdot \Delta \mathbf{U}.$$

Multiplying by a test function and integrating by parts we find that

$$-(\nabla \xi_t, \nabla v) = (\Delta \mathbf{U}, \nabla v),$$

if the test function is zero on $\partial\Omega$. In (8), because our test function v is not zero on the boundary, we cannot make this substitution without introducing boundary terms involving ξ which are not known. In defining our method below, we handle the $\Delta \mathbf{U}$ term in (8) without requiring additional continuity on the finite element space.

3.1. Some Assumptions. We will make the following assumptions about the solutions and the data. First, we assume the domain Ω is polygonal, and for $(\mathbf{x}, t) \in \bar{\Omega} \times (0, T]$,

A1. the solutions ξ , \mathbf{U} to (2) and (3)-(6) exist and are unique,

A2. μ is a positive constant,

We make the following smoothness assumptions on the initial data and on the solutions.

A3. $\xi^0, \mathbf{U}^0(\mathbf{x}) \in \mathcal{H}^\ell(\Omega)$,

A4. $\xi \in \mathcal{H}^{\ell+1}(\Omega)$, $t > 0$,

A5. $\mathbf{U}, \mathbf{U}_t \in \mathcal{H}^\ell(\Omega)$, $t > 0$,

A6. $\Delta \mathbf{U} \in \mathcal{H}^\ell(\Omega)$, $t > 0$,

where the integer $\ell \geq 2$ is defined in the next section.

4. Galerkin Finite Element Approximation.

4.1. The Continuous-Time Galerkin Approximation. Let \mathcal{T} be a quasi-uniform triangulation of the polygonal domain Ω into elements E_i , $i = 1, \dots, m$, with $\text{diam}(E_i) = h_i$ and $h = \max_i h_i$. Let $\mathcal{S}_h(\mathcal{S}_h)$ denote a finite dimensional subspace of $\mathcal{H}^1(\Omega)$ ($\mathcal{H}^1(\Omega)$) defined on this triangulation consisting of piecewise polynomials of degree less than s_1 . Let $\mathcal{S}_h^g = \mathcal{S}_h \cap \{\mathbf{w} : \mathbf{w} = g \text{ on } \partial\Omega\}$ and $\mathcal{S}_h^0 = \mathcal{S}_h \cap \{\mathbf{w} : \mathbf{w} = 0 \text{ on } \partial\Omega\}$. Assume $\mathcal{S}_h(\mathcal{S}_h)$ satisfies the standard approximation property

$$(10) \quad \inf_{\varphi \in \mathcal{S}_h(\mathcal{S}_h)} \|v - \varphi\|_{\mathcal{H}^{s_0}(\Omega)} \leq K_0 h^{\ell-s_0} \|v\|_{\mathcal{H}^\ell(\Omega)}, \quad v \in \mathcal{H}^1(\Omega) \cap \mathcal{H}^\ell(\Omega),$$

and the inverse assumptions (see [2], Theorem 4.5.11)

$$(11) \quad \|\varphi\|_{\mathcal{L}^\infty(\Omega)} \leq K_0 \|\varphi\|_{\mathcal{L}^2(\Omega)} h^{-1}, \quad \varphi \in \mathcal{S}_h(\Omega),$$

$$(12) \quad \|\varphi\|_{\mathcal{H}^1(\Omega)} \leq K_0 \|\varphi\|_{\mathcal{L}^2(\Omega)} h^{-1}, \quad \varphi \in \mathcal{S}_h(\Omega).$$

Here, s_0 and ℓ are integers, $0 \leq s_0 \leq \ell \leq s_1$. Moreover, K_0 is a constant independent of h and v . We also define the space $\mathcal{S}_h^{\partial\Omega} = \mathcal{S}_h / \mathcal{S}_h^0$.

In defining our method, we approximate four quantities, ξ , \mathbf{U} , $\Delta \mathbf{U}$, and $\boldsymbol{\lambda} \equiv \nabla \mathbf{U} \cdot \boldsymbol{\nu}$ on $\partial\Omega$. The equations for ξ and \mathbf{U} are derived from (8) and (9). We note that by integration by parts,

$$(13) \quad (\Delta \mathbf{U}, \mathbf{w}) = -(\nabla \mathbf{U}, \nabla \mathbf{w}) + \langle \boldsymbol{\lambda}, \mathbf{w} \rangle, \quad \mathbf{w} \in \mathcal{H}^1(\Omega).$$

Moreover, by (9),

$$(14) \quad \begin{aligned} \mu \langle \boldsymbol{\lambda}, \boldsymbol{w} \rangle &= ((\boldsymbol{U}_h)_t, \boldsymbol{w}) + (G \nabla \xi_h, \boldsymbol{w}) \\ &\quad + \mu (\nabla \boldsymbol{U}_h, \nabla \boldsymbol{w}) - (\mathcal{F}, \boldsymbol{w}), \quad \boldsymbol{w} \in \mathcal{H}^1(\Omega). \end{aligned}$$

Define an approximation $\xi_h(\cdot, t) \in \mathcal{S}_h$ by

$$(15) \quad (\xi_h(\cdot, 0), v) = (\xi_h^0, v), \quad v \in \mathcal{S}_h,$$

$$(16) \quad ((\xi_h)_t(\cdot, 0), v) = (\xi_h^1, v) = -(\nabla \cdot \boldsymbol{U}_h(\cdot, 0), v), \quad v \in \mathcal{S}_h,$$

and for $t > 0$,

$$(17) \quad \begin{aligned} &((\xi_h)_{tt}, v) + (G \pi(\nabla \xi_h), \nabla v) \\ &\quad - \mu (\Delta_h \boldsymbol{U}_h, \nabla v) - (\mathcal{F}, \nabla v) + \langle \boldsymbol{g}_t \cdot \boldsymbol{\nu}, v \rangle = 0, \quad v \in \mathcal{S}_h, \end{aligned}$$

where $\pi(\nabla \xi_h)$ denotes the L^2 projection of $\nabla \xi_h$ into \mathcal{S}_h , and $\Delta_h \boldsymbol{U}_h$ is defined below.

Define an approximation $\boldsymbol{U}_h(\cdot, t) \in \mathcal{S}_h^g$ by

$$(18) \quad (\nabla \boldsymbol{U}_h(\cdot, 0), \nabla \boldsymbol{w}) = (\nabla \boldsymbol{U}^0, \nabla \boldsymbol{w}), \quad \boldsymbol{w} \in \mathcal{S}_h^0,$$

and for $t > 0$,

$$(19) \quad ((\boldsymbol{U}_h)_t, \boldsymbol{w}) + (G \nabla \xi_h, \boldsymbol{w}) + \mu (\nabla \boldsymbol{U}_h, \nabla \boldsymbol{w}) = (\mathcal{F}, \boldsymbol{w}), \quad \boldsymbol{w} \in \mathcal{S}_h^0.$$

The “discrete Laplacian” $\Delta_h \boldsymbol{U}_h \in \mathcal{S}_h$ in (17) is defined by

$$(20) \quad (\Delta_h \boldsymbol{U}_h, \boldsymbol{w}) = -(\nabla \boldsymbol{U}_h, \nabla \boldsymbol{w}) + \langle \boldsymbol{\lambda}_h, \boldsymbol{w} \rangle, \quad \boldsymbol{w} \in \mathcal{S}_h,$$

where the approximate boundary flux $\boldsymbol{\lambda}_h \in \mathcal{S}_h^{\partial\Omega}$ is defined by

$$(21) \quad \begin{aligned} \mu \langle \boldsymbol{\lambda}_h, \boldsymbol{w}_b \rangle &= ((\boldsymbol{U}_h)_t, \boldsymbol{w}_b) + (G \nabla \xi_h, \boldsymbol{w}_b) \\ &\quad + \mu (\nabla \boldsymbol{U}_h, \nabla \boldsymbol{w}_b) - (\mathcal{F}, \boldsymbol{w}_b), \quad \boldsymbol{w}_b \in \mathcal{S}_h^{\partial\Omega}. \end{aligned}$$

Thus, the system (15)–(21) yields a system of equations in four unknowns, ξ_h , \boldsymbol{U}_h , $\Delta_h \boldsymbol{U}_h$ and $\boldsymbol{\lambda}_h$.

In section (6), we will formulate a discrete time version of this scheme and show that these unknowns can be determined in a sequential manner. Moreover, all matrices which arise are symmetric, positive definite, and time-independent.

4.2. A stability estimate. As a prelude to deriving an error estimate, we study the stability of the scheme above in the case $\boldsymbol{g} = 0$ and $\mathcal{F} = 0$. Integrating (17), (19) and (21) in time, we obtain

$$(22) \quad ((\xi_h)_t, v) + \left(\int_0^t G \pi(\nabla \xi_h) ds, \nabla v \right) - \mu \left(\int_0^t \Delta_h \boldsymbol{U}_h ds, \nabla v \right) = (\xi_h^1, v), \quad v \in \mathcal{S}_h,$$

$$(23) \quad (\boldsymbol{U}_h, \boldsymbol{w}) + \left(\int_0^t G \nabla \xi_h ds, \boldsymbol{w} \right) + \mu \left(\int_0^t \nabla \boldsymbol{U}_h ds, \nabla \boldsymbol{w} \right) = (\boldsymbol{U}^0, \boldsymbol{w}), \quad \boldsymbol{w} \in \mathcal{S}_h^0,$$

$$\begin{aligned}
(24) \quad \mu \langle \int_0^t \boldsymbol{\lambda}_h ds, \mathbf{w}_b \rangle &= (\mathbf{U}_h, \mathbf{w}_b) + \left(\int_0^t G \nabla \xi_h ds, \mathbf{w}_b \right) \\
&\quad + \mu \left(\int_0^t \nabla \mathbf{U}_h ds, \nabla \mathbf{w}_b \right) - (\mathbf{U}^0, \mathbf{w}_b), \quad \mathbf{w}_b \in \mathcal{S}_h^{\partial\Omega}.
\end{aligned}$$

Adding (23) and (24) we find

$$\begin{aligned}
(25) \quad (\mathbf{U}_h, \mathbf{w} + \mathbf{w}_b) + \mu \left(\int_0^t \nabla \mathbf{U}_h ds, \nabla (\mathbf{w} + \mathbf{w}_b) \right) \\
+ \left(\int_0^t G \nabla \xi_h ds, \mathbf{w} + \mathbf{w}_b \right) - \mu \left(\int_0^t \boldsymbol{\lambda}_h ds, \mathbf{w} + \mathbf{w}_b \right) = (\mathbf{U}^0, \mathbf{w} + \mathbf{w}_b).
\end{aligned}$$

Here we have used the fact that $\mathbf{w} = 0$ on $\partial\Omega$ in the term involving $\boldsymbol{\lambda}_h$.

We now set $\mathbf{w} + \mathbf{w}_b = \pi(\nabla \xi_h)$, and we set $v = \xi_h$ in (22) to obtain

$$\begin{aligned}
(26) \quad ((\xi_h)_t, \xi_h) + \left(\int_0^t G \pi(\nabla \xi_h) ds, \nabla \xi_h \right) \\
- \mu \left(\int_0^t \Delta_h \mathbf{U}_h ds, \nabla \xi_h \right) = (\xi_h^1, \xi_h),
\end{aligned}$$

and

$$\begin{aligned}
(27) \quad (\mathbf{U}_h, \nabla \xi_h) + \left(\int_0^t G \nabla \xi_h ds, \pi(\nabla \xi_h) \right) \\
+ \mu \left(\int_0^t \nabla \mathbf{U}_h, \nabla \pi(\nabla \xi_h) \right) - \mu \left(\int_0^t \boldsymbol{\lambda}_h ds, \pi(\nabla \xi_h) \right) = (\mathbf{U}_h(\cdot, 0), \nabla \xi_h).
\end{aligned}$$

Integrating (20) in time, setting $\mathbf{w} = \pi(\nabla \xi_h)$, we find

$$\begin{aligned}
\left(\int_0^t \Delta_h \mathbf{U}_h ds, \nabla \xi_h \right) &= \left(\int_0^t \Delta_h \mathbf{U}_h ds, \pi(\nabla \xi_h) \right) \\
&= - \left(\int_0^t \nabla \mathbf{U}_h ds, \nabla \pi(\nabla \xi_h) \right) + \left(\int_0^t \boldsymbol{\lambda}_h ds, \pi(\nabla \xi_h) \right),
\end{aligned}$$

Substituting this result into (27) and subtracting from (26), we obtain

$$\begin{aligned}
(28) \quad ((\xi_h)_t, \xi_h) &= (\mathbf{U}_h, \nabla \xi_h) - \left(\int_0^t G \pi(\nabla \xi_h) ds, \nabla \xi_h \right) + \left(\int_0^t G \nabla \xi_h ds, \pi(\nabla \xi_h) \right) \\
&\quad + (\xi_h^1, \xi_h) - (\mathbf{U}_h(\cdot, 0), \nabla \xi_h) \\
&= (\mathbf{U}_h, \nabla \xi_h) + (\xi_h^1, \xi_h) - (\mathbf{U}_h(\cdot, 0), \nabla \xi_h).
\end{aligned}$$

Letting $\mathbf{w} = \mathbf{U}_h$ in (19) we obtain

$$(29) \quad ((\mathbf{U}_h)_t, \mathbf{U}_h) + \mu(\nabla \mathbf{U}_h, \nabla \mathbf{U}_h) = -(G \nabla \xi_h, \mathbf{U}_h).$$

Adding (28) and (29) and integrating by parts we find

$$\begin{aligned}
(30) \quad ((\xi_h)_t, \xi_h) + ((\mathbf{U}_h)_t, \mathbf{U}_h) + \mu(\nabla \mathbf{U}_h, \nabla \mathbf{U}_h) \\
= (\xi_h^1, \xi_h) - (\nabla \cdot \mathbf{U}_h, \xi_h) + (G \xi_h, \nabla \cdot \mathbf{U}_h) + (\nabla \cdot \mathbf{U}_h(\cdot, 0), \xi_h) \\
= -(\nabla \cdot \mathbf{U}_h, \xi_h) + (G \xi_h, \nabla \cdot \mathbf{U}_h),
\end{aligned}$$

where in the last step we've used the fact that ξ_h^1 is the L^2 projection of $-\nabla \cdot \mathbf{U}_h(\cdot, 0)$ into \mathcal{S}_h .

Integrating (30) in time from 0 to T we find

$$\begin{aligned}
 (31) \quad & \|\xi_h(\cdot, T)\|^2 + \|\mathbf{U}_h(\cdot, T)\|^2 + 2\mu \int_0^T \|\nabla \mathbf{U}_h\|^2 dt \\
 & \leq \|\xi^0\|^2 + \|\mathbf{U}^0\|^2 + \int_0^T [\mu \|\nabla \mathbf{U}_h\|^2 + C \|\xi_h\|^2] dt \\
 & \leq \|\xi^0\|^2 + \|\mathbf{U}^0\|^2 + \int_0^T [\mu \|\nabla \mathbf{U}_h\|^2 + C \|\xi_h\|^2] dt.
 \end{aligned}$$

An application of Gronwall's Lemma gives the following result.

LEMMA 4.1. *For the case $\mathbf{g} = 0$ and $\mathcal{F} = 0$, and any $T > 0$*

$$(32) \quad \|\xi_h(\cdot, T)\| + \|\mathbf{U}_h(\cdot, T)\| \leq C (\|\xi^0\| + \|\mathbf{U}^0\|).$$

4.3. An a priori error estimate. We now consider the more general case where \mathbf{g} and \mathcal{F} are not zero. In order to derive an error estimate let $\tilde{\xi}_h$ denote the L^2 projection of ξ into \mathcal{S}_h , and $\tilde{\mathbf{U}}_h$ the elliptic projection of \mathbf{U} into \mathcal{S}_h^g ; that is, $\tilde{\mathbf{U}}_h \in \mathcal{S}_h^g$ is defined by

$$(33) \quad (\nabla(\tilde{\mathbf{U}}_h - \mathbf{U}), \nabla \mathbf{w}) = 0, \quad \mathbf{w} \in \mathcal{S}_h^0.$$

Let $\psi_\xi = \xi_h - \tilde{\xi}_h$, $\psi_{\mathbf{u}} = \mathbf{U}_h - \tilde{\mathbf{U}}_h$, $\theta_\xi = \xi - \tilde{\xi}_h$, and $\theta_{\mathbf{u}} = \mathbf{U} - \tilde{\mathbf{U}}_h$.

Integrating (8) in time and combining with (22) we find

$$\begin{aligned}
 & ((\psi_\xi)_t, v) + \left(\int_0^t G \pi(\nabla \xi_h) ds, \nabla v \right) \\
 & = (\xi_h^1 - \xi_t(\cdot, 0), v) + ((\theta_\xi)_t, v) + \left(\int_0^t G \nabla \xi ds, \nabla v \right) + \mu \left(\int_0^t (\Delta_h \mathbf{U}_h - \Delta \mathbf{U}) ds, \nabla v \right).
 \end{aligned}$$

Integrating (14) in time and combining with (25) we find

$$\begin{aligned}
 & (\psi_{\mathbf{u}}, \mathbf{w} + \mathbf{w}_b) + \left(\int_0^t G \nabla \xi_h ds, \mathbf{w} + \mathbf{w}_b \right) \\
 & = (\theta_{\mathbf{u}}, \mathbf{w} + \mathbf{w}_b) - (\theta_{\mathbf{u}}(\cdot, 0), \mathbf{w} + \mathbf{w}_b) + \left(\int_0^t G \nabla \xi ds, \mathbf{w} + \mathbf{w}_b \right) \\
 (35) \quad & + \mu \left(\int_0^t (\Delta_h \mathbf{U}_h - \Delta \mathbf{U}) ds, \mathbf{w} + \mathbf{w}_b \right).
 \end{aligned}$$

Here we have used the definition of $\Delta_h \mathbf{U}_h$, (20), and (13).

Setting $v = \psi_\xi$ and $\mathbf{w} + \mathbf{w}_b = \pi(\nabla \psi_\xi)$, where $\pi(\nabla \psi_\xi)$ is the L^2 projection of $\nabla \psi_\xi$ into \mathcal{S}_h , and subtracting (35) from (34) we find

$$\begin{aligned}
 ((\psi_\xi)_t, \psi_\xi) & = (\psi_{\mathbf{u}}, \nabla \psi_\xi) - \left(\int_0^t G \nabla \xi ds, \pi(\nabla \psi_\xi) - \nabla \psi_\xi \right) \\
 & \quad + ((\theta_\xi)_t, \psi_\xi) - (\theta_{\mathbf{u}}, \pi(\nabla \psi_\xi)) + (\theta_{\mathbf{u}}(\cdot, 0), \pi(\nabla \psi_\xi)) \\
 (36) \quad & + \mu \left(\int_0^t \Delta \mathbf{U} ds, \pi(\nabla \psi_\xi) - \nabla \psi_\xi \right) + (\nabla \cdot (\mathbf{U}_h - \mathbf{U})(\cdot, 0), \psi_\xi).
 \end{aligned}$$

From (19), we find

$$(37) \quad \begin{aligned} & ((\psi_{\mathbf{u}})_t, \psi_{\mathbf{u}}) + (G\nabla\psi_{\xi}, \psi_{\mathbf{u}}) + \mu(\nabla\psi_{\mathbf{u}}, \nabla\psi_{\mathbf{u}}) \\ &= ((\theta_{\mathbf{u}})_t, \psi_{\mathbf{u}}) + (G\nabla\theta_{\xi}, \psi_{\mathbf{u}}) + \mu(\nabla\theta_{\mathbf{u}}, \nabla\psi_{\mathbf{u}}). \end{aligned}$$

Adding (36)-(37), integrating by parts, and using the definitions of the L^2 and elliptic projections, we find

$$(38) \quad \begin{aligned} & ((\psi_{\xi})_t, \psi_{\xi}) + ((\psi_{\mathbf{u}})_t, \psi_{\mathbf{u}}) + \mu(\nabla\psi_{\mathbf{u}}, \nabla\psi_{\mathbf{u}}) \\ &= -(\nabla \cdot \psi_{\mathbf{u}}, \psi_{\xi}) + \left(\int_0^t G(\nabla\xi - \pi(\nabla\xi)) ds, \pi(\nabla\psi_{\xi}) - \nabla\psi_{\xi} \right) \\ &\quad - (\theta_{\mathbf{u}}, \pi(\nabla\psi_{\xi})) + (\theta_{\mathbf{u}}(\cdot, 0), \pi(\nabla\psi_{\xi})) + ((\theta_{\mathbf{u}})_t, \psi_{\mathbf{u}}) \\ &\quad + (G(\psi_{\xi} - \theta_{\xi}), \nabla \cdot \psi_{\mathbf{u}}) + \mu \left(\int_0^t (\Delta\mathbf{U} - \pi(\Delta\mathbf{U})) ds, \pi(\nabla\psi_{\xi}) - \nabla\psi_{\xi} \right) \\ &\quad + (\nabla \cdot (\mathbf{U}_h - \mathbf{u})(\cdot, 0), \psi_{\xi}), \end{aligned}$$

where $\pi(\Delta\mathbf{U})$ is the L^2 projection of $\Delta\mathbf{U}$ into \mathcal{S}_h .

Integrating this equation in time, we find

$$(39) \quad \begin{aligned} & \|\psi_{\xi}(\cdot, T)\|^2 + \|\psi_{\mathbf{u}}(\cdot, T)\|^2 + 2\mu \int_0^T \|\nabla\psi_{\mathbf{u}}\|^2 dt \\ &\leq \mu \int_0^T \|\nabla\psi_{\mathbf{u}}\|^2 dt + Ch^{-2} \int_0^T \|\nabla\xi - \pi(\nabla\xi)\|^2 dt \\ &\quad + C \int_0^T [\|\theta_{\xi}\|^2 + \|\psi_{\xi}\|^2] dt \\ &\quad + C \int_0^T [h^{-2}\|\theta_{\mathbf{u}}\|^2 + \|(\theta_{\mathbf{u}})_t\|^2 + \|\psi_{\mathbf{u}}\|^2] dt + Ch^{-2}\|\theta_{\mathbf{u}}(\cdot, 0)\|^2 \\ &\quad + Ch^{-2} \int_0^T \|\Delta\mathbf{U} - \pi(\Delta\mathbf{U})\|^2 dt + Ch^2 \int_0^T [\|\pi(\nabla\psi_{\xi})\|^2 + \|\nabla\psi_{\xi}\|^2] dt \\ &\quad + C\|\nabla \cdot (\mathbf{U}_h(\cdot, 0) - \mathbf{U}(\cdot, 0))\|^2 \end{aligned}$$

It is easily shown that

$$\|\pi(\nabla\psi_{\xi})\| \leq \|\nabla\psi_{\xi}\|.$$

Using (40) and the inverse estimate (12), we find

$$(40) \quad \begin{aligned} & \|\psi_{\xi}(\cdot, T)\|^2 + \|\psi_{\mathbf{u}}(\cdot, T)\|^2 + \mu \int_0^T \|\nabla\psi_{\mathbf{u}}\|^2 dt \\ &\leq Ch^{2(l-1)} + C \int_0^T [\|\psi_{\xi}\|^2 + \|\psi_{\mathbf{u}}\|^2] dt. \end{aligned}$$

Applying Gronwall's inequality and the triangle inequality, we obtain the following:

THEOREM 4.2. *Let the assumptions **A1-A6** hold. Assume the finite element solutions ξ_h , \mathbf{U}_h , $\Delta_h\mathbf{U}_h$, and λ_h to (15)-(21) exist and are unique. Then there exists a constant C independent of h such that*

$$(41) \quad \|(\mathbf{U} - \mathbf{U}_h)\|_{\mathcal{L}^\infty(0,T;\mathcal{L}^2)} + \|\xi - \xi_h\|_{\mathcal{L}^\infty(0,T;\mathcal{L}^2)} \leq Ch^{l-1}.$$

We remark that this rate of convergence is the same as that obtained in our earlier paper [3].

5. A nonlinear model. Realistic shallow water models are nonlinear. For example, the term $G\nabla\xi$ in the momentum equation (2) is actually $gH\nabla\xi$, where g is gravitational acceleration (assumed constant). Moreover, an advection term $\nabla \cdot \mathbf{U}^2/H$ is also present. There are also forcing terms (Coriolis force, wind stress, tide potentials, bottom friction) present in the equation; we will assume these are known, and for simplicity lump them into the term \mathcal{F} . Thus (2) becomes

$$(42) \quad \mathbf{U}_t + gH\nabla\xi + \nabla \cdot \frac{\mathbf{U}^2}{H} - \mu\Delta\mathbf{U} = \mathcal{F},$$

and the GWCE (5) becomes

$$(43) \quad \begin{aligned} \xi_{tt} - \nabla \cdot (gH\nabla\xi) - \nabla \cdot \left(\nabla \cdot \frac{\mathbf{U}^2}{H} \right) \\ + \mu\nabla \cdot \Delta\mathbf{U} + \nabla \cdot \mathcal{F} = 0. \end{aligned}$$

Let

$$\mathbf{\Gamma} = gH\nabla\xi + \nabla \cdot \frac{\mathbf{U}^2}{H},$$

and

$$\mathbf{\Gamma}_h = gH_h\nabla\xi_h + \nabla \cdot \frac{\mathbf{U}_h^2}{H_h},$$

where

$$(44) \quad H_h = h_b + \xi_h.$$

Let $\pi\mathbf{\Gamma}_h$ ($\pi\mathbf{\Gamma}$) denote the L^2 projection of $\mathbf{\Gamma}_h$ ($\mathbf{\Gamma}$) into \mathcal{S}_h . Our finite element method is defined as follows. We choose the initial data and define the discrete Laplacian as before. Then, for $t > 0$,

$$(45) \quad \begin{aligned} ((\xi_h)_{tt}, v) + (\pi\mathbf{\Gamma}_h, \nabla v) \\ - \mu(\Delta_h \mathbf{U}_h, \nabla v) - (\mathcal{F}, \nabla v) + \langle \mathbf{g}_t \cdot \boldsymbol{\nu}, v \rangle = 0, \quad v \in \mathcal{S}_h, \end{aligned}$$

$$(46) \quad \begin{aligned} ((\mathbf{U}_h)_t, \mathbf{w}) + \mu(\nabla \mathbf{U}_h, \nabla \mathbf{w}) &= -(\mathbf{\Gamma}_h, \mathbf{w}) + (\mathcal{F}, \mathbf{w}) \\ &= -(\pi\mathbf{\Gamma}_h, \mathbf{w}) + (\mathcal{F}, \mathbf{w}), \quad \mathbf{w} \in \mathcal{S}_h^0, \end{aligned}$$

and

$$(47) \quad \begin{aligned} \mu\langle \boldsymbol{\lambda}_h, \mathbf{w}_b \rangle &= ((\mathbf{U}_h)_t, \mathbf{w}_b) + (\mathbf{\Gamma}_h, \mathbf{w}_b) \\ &\quad + \mu(\nabla \mathbf{U}_h, \nabla \mathbf{w}_b) - (\mathcal{F}, \mathbf{w}_b), \quad \mathbf{w}_b \in \mathcal{S}_h^{\partial\Omega}. \end{aligned}$$

For the error analysis below, we will assume that a constant K exists such that

A7. $\|\xi_h\|_{\mathcal{L}^\infty(0,T;\mathcal{L}^\infty)} + \|\mathbf{U}_h\|_{\mathcal{L}^\infty(0,T;\mathcal{L}^\infty)} \leq K$,
and that positive constants H_{**} , H^{**} exist such that

A8. $H_{**} < H_h < H^{**}$.

Using an induction argument as in [3], one can show that K , H_{**} and H^{**} are independent of h for h sufficiently small, for polynomials of degree two and higher. We will also assume that

A9. $\xi, h_b, \mathbf{U} \in \mathcal{W}_\infty^1(\Omega), \mathbf{\Gamma} \in \mathcal{H}^\ell(\Omega)$.

Define $\psi_\xi, \psi_{\mathbf{u}}, \theta_\xi$ and $\theta_{\mathbf{u}}$ as before. Integrate (45) in time and subtract the analogous equation obtained from (44) to find

$$(48) \quad \begin{aligned} ((\psi_\xi)_t, v) &= ((\psi_\xi)_t(\cdot, 0), v) + \left(\int_0^t \pi(\mathbf{\Gamma} - \mathbf{\Gamma}_h) ds, \nabla v \right) \\ &\quad + ((\theta_\xi)_t, v) + \mu \left(\int_0^t (\Delta_h \mathbf{U}_h - \Delta \mathbf{U}) ds, \nabla v \right) \\ &\quad + \left(\int_0^t (\mathbf{\Gamma} - \pi \mathbf{\Gamma}) ds, \nabla v \right), \quad v \in \mathcal{S}_h. \end{aligned}$$

Integrate (46) and (47) in time and subtract the analogous equation obtained from (42) to find

$$\begin{aligned} (\psi_{\mathbf{u}}, \mathbf{w}) &= \left(\int_0^t \pi(\mathbf{\Gamma} - \mathbf{\Gamma}_h) ds, \mathbf{w} \right) + \mu \left(\int_0^t (\Delta_h \mathbf{U}_h - \Delta \mathbf{U}) ds, \mathbf{w} \right) + (\theta_{\mathbf{u}}, \mathbf{w}) \\ &\quad - (\theta_{\mathbf{u}}(\cdot, 0), \mathbf{w}), \quad \mathbf{w} \in \mathcal{S}_h, \end{aligned}$$

From (46) and (42) we obtain

$$(49) \quad ((\psi_{\mathbf{u}})_t, \mathbf{w}) + \mu(\nabla \psi_{\mathbf{u}}, \nabla \mathbf{w}) = (\mathbf{\Gamma} - \mathbf{\Gamma}_h, \mathbf{w}) + ((\theta_{\mathbf{u}})_t, \mathbf{w}), \quad \mathbf{w} \in \mathcal{S}_h^0.$$

Setting $v = \psi_\xi, \mathbf{w} = \pi(\nabla \psi_\xi)$ in (49) and $\mathbf{w} = \psi_{\mathbf{u}}$ in (49). Subtracting (49) from (48) and adding (49) we obtain

$$(50) \quad \begin{aligned} &((\psi_\xi)_t, \psi_\xi) + ((\psi_{\mathbf{u}})_t, \psi_{\mathbf{u}}) + \mu(\nabla \psi_{\mathbf{u}}, \nabla \psi_{\mathbf{u}}) \\ &= ((\psi_\xi)_t(\cdot, 0), \psi_\xi) + (\psi_{\mathbf{u}}, \nabla \psi_\xi) + \mu \left(\int_0^t (\Delta \mathbf{U} - \pi(\Delta \mathbf{U})) ds, \pi(\nabla \psi_\xi) - \nabla \psi_\xi \right) \\ &\quad + \left(\int_0^t (\mathbf{\Gamma} - \pi \mathbf{\Gamma}) ds, \nabla \psi_\xi \right) + (\theta_{\mathbf{u}}, \pi \nabla \psi_\xi) - (\theta_{\mathbf{u}}(\cdot, 0), \pi \nabla \psi_\xi) \\ &\quad + (\mathbf{\Gamma} - \mathbf{\Gamma}_h, \psi_{\mathbf{u}}) + ((\theta_{\mathbf{u}})_t, \psi_{\mathbf{u}}). \end{aligned}$$

The fourth and seventh terms on the right side of (50) are handled as follows.

$$(51) \quad \begin{aligned} \left(\int_0^t (\mathbf{\Gamma} - \pi \mathbf{\Gamma}) ds, \nabla \psi_\xi \right) &\leq Ch^{-2} \left\| \int_0^t (\mathbf{\Gamma} - \pi \mathbf{\Gamma}) ds \right\|^2 + Ch^2 \|\nabla \psi_\xi\|^2 \\ &\leq Ch^{2(l-1)} + C \|\psi_\xi\|^2. \end{aligned}$$

$$(52) \quad \begin{aligned} (\mathbf{\Gamma} - \mathbf{\Gamma}_h, \psi_{\mathbf{u}}) &= (gH\nabla \xi - gH_h\nabla \xi_h, \psi_{\mathbf{u}}) + \left(\nabla \cdot \left(\frac{\mathbf{U}^2}{H} - \frac{\mathbf{U}_h^2}{H_h} \right), \psi_{\mathbf{u}} \right) \\ &= (gh_b\nabla(\xi - \xi_h), \psi_{\mathbf{u}}) + \frac{g}{2}(\nabla \xi^2 - \nabla \xi_h^2, \psi_{\mathbf{u}}) \\ &\quad - \left(\frac{\mathbf{U}^2}{H} - \frac{\mathbf{U}_h^2}{H_h}, \nabla \psi_{\mathbf{u}} \right) \\ &= -g(\xi - \xi_h, \nabla \cdot (h_b \psi_{\mathbf{u}})) - \frac{g}{2}(\xi^2 - \xi_h^2, \nabla \cdot \psi_{\mathbf{u}}) \\ &\quad - \left(\frac{\mathbf{U}^2 H_h - \mathbf{U}_h^2 H}{HH_h}, \nabla \psi_{\mathbf{u}} \right) \end{aligned}$$

$$\begin{aligned}
&= -g(\xi - \xi_h, \nabla \cdot (h_b \psi \mathbf{u})) - \frac{g}{2}(\xi^2 - \xi_h^2, \nabla \cdot \psi \mathbf{u}) \\
&\quad - \left(\frac{\mathbf{U}^2(H_h - H) - H_h(\mathbf{U}^2 - \mathbf{U}_h^2)}{H H_h}, \nabla \psi \mathbf{u} \right) \\
&\leq C \|\psi_\xi\|^2 + C \|\theta_\xi\|^2 + C \|\psi \mathbf{u}\|^2 \\
&\quad + C \|\theta \mathbf{u}\|^2 + \frac{\mu}{2} \|\nabla \psi \mathbf{u}\|^2.
\end{aligned}$$

Combining (50), (51) and (53), choosing ϵ sufficiently small, using bounds previously derived for the remaining terms, and integrating in time, we obtain

$$\begin{aligned}
(53) \quad &\|\psi_\xi(\cdot, T)\|^2 + \|\psi \mathbf{u}(\cdot, T)\|^2 + \mu \int_0^T \|\nabla \psi \mathbf{u}\|^2 dt \\
&\leq C h^{2(l-1)} + C \int_0^T [\|\psi_\xi\|^2 + \|\psi \mathbf{u}\|^2] dt.
\end{aligned}$$

Using Gronwall's Lemma we obtain the following.

THEOREM 5.1. *Assume the finite element solutions ξ_h , \mathbf{U}_h , $\Delta_h \mathbf{U}_h$, and λ_h to (15), (16), (45), (18), (46), (20) and (47) exist and are unique. Let the assumptions **A1-A9** hold and assume h is sufficiently small. Then, there exists a constant C independent of h such that*

$$(54) \quad \|\mathbf{U} - \mathbf{U}_h\|_{\mathcal{L}^\infty(0,T;\mathcal{L}^2)} + \|\xi - \xi_h\|_{\mathcal{L}^\infty(0,T;\mathcal{L}^2)} \leq C h^{l-1}.$$

6. A discrete time method. In this section, we return for simplicity to the linear model presented in section (1), with $\mathbf{g} = \mathcal{F} = 0$, and formulate a discrete time method. We extend our continuous-time stability argument presented in section (4) and show that the discrete scheme satisfies the same stability bound. We leave the derivation of error estimates for this scheme to the reader.

Choose a time step $\Delta t > 0$ and set $t^k = k\Delta t$, $k = 0, 1, \dots$. Denote $g(t^k)$ by g^k . A discrete time scheme based on (2) and (3)-(6) can be defined as follows. We define the initial approximations ξ_h^0 and \mathbf{U}_h^0 as before, see (15), (18). We enforce the initial condition (6) by

$$(55) \quad \left(\frac{\xi_h^1 - \xi_h^0}{\Delta t}, v \right) + (\nabla \cdot \mathbf{U}_h^0, v) = 0, \quad v \in \mathcal{S}_h.$$

Then, for $k = 1, 2, \dots$,

$$(56) \quad \left(\frac{\mathbf{U}_h^k - \mathbf{U}_h^{k-1}}{\Delta t}, \mathbf{w} \right) + (G \nabla \xi_h^k, \mathbf{w}) + \mu (\nabla \mathbf{U}_h^k, \nabla \mathbf{w}) = 0, \quad \mathbf{w} \in \mathcal{S}_h^0,$$

$$\begin{aligned}
(57) \quad \mu \langle \lambda_h^k, \mathbf{w}_b \rangle &= \left(\frac{\mathbf{U}_h^k - \mathbf{U}_h^{k-1}}{\Delta t}, \mathbf{w}_b \right) + (G \nabla \xi_h^k, \mathbf{w}_b) \\
&\quad + \mu (\nabla \mathbf{U}_h^k, \nabla \mathbf{w}_b), \quad \mathbf{w}_b \in \mathcal{S}_h^{\partial\Omega},
\end{aligned}$$

$$(58) \quad (\Delta_h \mathbf{U}_h^k, \mathbf{w}) = -(\nabla \mathbf{U}_h^k, \nabla \mathbf{w}) + \langle \lambda_h^k, \mathbf{w} \rangle, \quad \mathbf{w} \in \mathcal{S}_h,$$

and

$$(59) \quad \left(\frac{\xi_h^{k+1} - 2\xi_h^k + \xi_h^{k-1}}{\Delta t^2}, v \right) + (G\pi(\nabla \xi_h^k), \nabla v) - \mu(\Delta_h \mathbf{U}_h^k, \nabla v) = 0, \quad v \in \mathcal{S}_h.$$

Note that, at each step in the above procedure, the matrices which arise are symmetric and positive definite, and independent of time.

We now extend the stability argument given above for the continuous time scheme to this discrete scheme. This argument can also be used to show uniqueness (hence existence) for the solutions to the system give above.

Adding (56) and (58) and using the definition (58) of $\Delta_h \mathbf{U}_h^k$, we find

$$(60) \quad \left(\frac{\mathbf{U}_h^k - \mathbf{U}_h^{k-1}}{\Delta t}, \mathbf{w} \right) + (G\nabla \xi_h^k, \mathbf{w}) - \mu(\Delta_h \mathbf{U}_h^k, \mathbf{w}) = 0, \quad \mathbf{w} \in \mathcal{S}_h.$$

Multiplying this equation by Δt and summing on k , $k = 1, \dots, n$, for some integer $n > 0$, we find

$$(61) \quad (\mathbf{U}_h^n, \mathbf{w}) + \left(\sum_{k=1}^n G\nabla \xi_h^k \Delta t, \mathbf{w} \right) - \mu \left(\sum_{k=1}^n \Delta_h \mathbf{U}_h^k \Delta t, \mathbf{w} \right) = (\mathbf{U}_h^0, \mathbf{w}), \quad \mathbf{w} \in \mathcal{S}_h.$$

Multiplying (59) by Δt and summing on k we obtain

$$(62) \quad \left(\frac{\xi_h^{n+1} - \xi_h^n}{\Delta t}, v \right) + \left(\sum_{k=1}^n G\pi(\nabla \xi_h^k) \Delta t, \nabla v \right) - \mu \left(\sum_{k=1}^n \Delta_h \mathbf{U}_h^k \Delta t, \nabla v \right) = \left(\frac{\xi_h^1 - \xi_h^0}{\Delta t}, v \right), \quad v \in \mathcal{S}_h.$$

Setting $v = \xi_h^{n+1}$ in (62) and $\mathbf{w} = \pi(\nabla \xi_h^{n+1})$ in (61), subtracting (61) from (62) and substituting (55), we find

$$(63) \quad \left(\frac{\xi_h^{n+1} - \xi_h^n}{\Delta t}, \xi_h^{n+1} \right) = -(\mathbf{U}_h^0, \nabla \xi_h^{n+1}) - (\nabla \cdot \mathbf{U}_h^0, \xi_h^{n+1}) + (\mathbf{U}_h^n, \nabla \xi_h^{n+1}).$$

Setting $k = n$ in (56) and $\mathbf{w} = \mathbf{U}_h^n$, we obtain

$$(64) \quad \left(\frac{\mathbf{U}_h^n - \mathbf{U}_h^{n-1}}{\Delta t}, \mathbf{U}_h^n \right) + \mu \|\nabla \mathbf{U}_h^n\|^2 = -(G\nabla \xi_h^n, \mathbf{U}_h^n).$$

Adding (63) and (64), using the inequality $a(a-b) \geq (a^2 - b^2)/2$, and integrating by parts we find

$$(65) \quad \frac{\|\xi_h^{n+1}\|^2 - \|\xi_h^n\|^2}{\Delta t} + \frac{\|\mathbf{U}_h^n\|^2 - \|\mathbf{U}_h^{n-1}\|^2}{\Delta t} + \mu \|\nabla \mathbf{U}_h^n\|^2 = (G\xi_h^n, \nabla \cdot \mathbf{U}_h^n) - (\nabla \cdot \mathbf{U}_h^n, \xi_h^{n+1}).$$

Multiplying (65) by Δt and summing on n , $n = 1, 2, \dots, N$ where $N \geq 1$ is an integer, we find

$$(66) \quad \|\xi_h^{N+1}\|^2 + \|\mathbf{U}_h^N\|^2 + \mu \sum_{n=1}^N \|\nabla \mathbf{U}_h^n\|^2 \Delta t$$

$$\begin{aligned} &\leq ||\xi_h^1||^2 + ||\mathbf{U}_h^0||^2 + C \sum_{n=1}^{N+1} ||\xi_h^n||^2 \Delta t \\ &\quad + \frac{\mu}{2} \sum_{n=1}^N ||\nabla \mathbf{U}_h^n||^2 \Delta t. \end{aligned}$$

Finally, we note that, by (55), setting $v = \xi_h^1$ we find

$$(67) \quad ||\xi_h^1|| \leq ||\xi_h^0|| + \Delta t ||\nabla \cdot \mathbf{U}_h^0||.$$

Combining (67) with (66) and applying the discrete version of Gronwall's inequality we obtain the following.

LEMMA 6.1. *For the case $\mathbf{g} = 0$ and $\mathcal{F} = 0$, N a positive integer and $\Delta t > 0$,*

$$(68) \quad ||\xi_h^N|| + ||\mathbf{U}_h^N|| \leq C (||\xi_h^0|| + ||\mathbf{U}_h^0|| + \Delta t ||\nabla \cdot \mathbf{U}_h^0||).$$

REFERENCES

- [1] R. A. ADAMS, *Sobolev Spaces*, vol. 65 of Pure and Applied Mathematics, Academic Press, New York, 1978.
- [2] S. C. BRENNER AND L. R. SCOTT, *The Mathematical Theory of Finite Element Methods*, vol. 15 of Texts in Applied Mathematics, Springer-Verlag, New York, 1994.
- [3] S. CHIPPADA, C. N. DAWSON, M. L. MARTÍNEZ, AND M. F. WHEELER, *Finite element approximations to the system of shallow water equations, Part I: Continuous time a priori error estimates*, Tech. Report TICAM Report 95-18, University of Texas, Austin, TX 78712, December 1995. To appear in SINUM.
- [4] ———, *Finite element approximations to the system of shallow water equations, Part II: Discrete time a priori error estimates*, Tech. Report TICAM Report 96-34, University of Texas, Austin, TX 78712, December 1996. To appear in SINUM.
- [5] I. P. E. KINMARK, *The Shallow Water Wave Equations: Formulation, Analysis and Applications*, vol. 15 of Lecture Notes in Engineering, Springer-Verlag, New York, 1985.
- [6] R. L. KOLAR, J. J. WESTERINK, M. E. CANTEKIN, AND C. A. BLAIN, *Aspects of nonlinear simulations using shallow water models based on the wave continuity equation*, Computers and Fluids, ?? (1994), p. ??
- [7] R. A. LUETTICH, J. J. WESTERINK, AND N. W. SCHEFFNER, ADCIRC: *An advanced three-dimensional circulation model for shelves, coasts, and estuaries*, Tech. Report 1, Department of the Army, U.S. Army Corps of Engineers, Washington, D.C. 20314-1000, December 1991.
- [8] D. R. LYNCH AND W. G. GRAY, *A wave equation model for finite element tidal computations*, Computers and Fluids, 7 (1979), pp. 207–228.