Finite Element Approximations to the System of Shallow Water Equations, Part III: On the Treatment of Boundary Conditions

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FINITE ELEMENT APPROXIMATIONS TO THE SYSTEM OF
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Abstract. We continue our investigation of finite element approximations to the system of shallow water equations, based on the generalized wave continuity equation (GWCE) formulation. In previous work, we analyzed this system assuming Dirichlet boundary conditions on both elevation and velocity. Based on physical grounds, it is possible to not impose boundary conditions on elevation. Thus, we examine the formulation for the case of Dirichlet conditions on velocity only. The changes required to the finite element method are presented, and stability and error estimates are derived for both an approximate linear model and a full nonlinear model, assuming continuous time. Stability for a discrete time method is also shown.

Key words. boundary conditions, shallow water equations, wave shallow water equations, error estimates

AMS subject classifications. 35Q35, 35L65 65N30, 65N10

1. Introduction. In this paper, we continue our investigation of finite element methods applied to the GWCE (Generalized Wave Continuity Equation) shallow water model of Gray et al. This model is described in a series of papers beginning in [8]. It has served as the basis for many shallow water simulators, most notably the Advanced Circulation Model (ADCIIRC) described, for example, in [7]. The method has the advantages that it allows for a weaker coupling between the continuity and momentum equations, gives rise to symmetric positive definite matrices, and helps stabilize the numerical solution. These have been supported by a large number of studies (see [5, 6] and references therein).

In previous papers [3, 4], we derived a priori error estimates for the method, in both continuous and discrete time, assuming Dirichlet boundary conditions on both the free surface elevation and velocity. In this paper, we will relax this assumption on the elevation and discuss the changes to the model and to the analysis. As it turns out, the assumption of Dirichlet boundary conditions on elevation allowed for a crucial substitution which substantially simplified the analysis. However, by making appropriate changes to the model, we will demonstrate that we are still able to preserve the accuracy of the method, at the cost of some additional computational work.

We will denote by $\xi(x, t)$ the free surface elevation over a reference plane and by $h_b(x)$ the bathymetric depth under that reference plane so that $H(x, t) = \xi + h_b$ is the total water column. Also, we denote by $u = [u(x, t) \ v(x, t)]^T$ the depth-averaged horizontal velocities and we let $U = uH$.

We will start with the following simplified linear shallow water model:

\begin{equation}
\xi_t + \nabla \cdot U = 0,
\end{equation}

\begin{equation}
U_t + G \nabla \xi - \mu \Delta U = \mathcal{F},
\end{equation}

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which we solve over a domain $\Omega \times (0, T]$. Here $G > 0$ is a gravitational constant and $\mu > 0$ is the eddy viscosity coefficient.

Note that, integrating (1) over $\Omega$,

$$\int_{\Omega} \xi_t d\Omega + \int_{\partial \Omega} U \cdot \nu ds = 0,$$

where $\nu$ is the outward normal to $\partial \Omega$. Moreover, integrating (2) over $\Omega$,

$$\int_{\Omega} \left( U_t + G \nabla \xi \right) d\Omega - \mu \int_{\partial \Omega} \nabla U \cdot \nu ds = \int_{\Omega} \mathcal{F} d\Omega.$$

Thus, it is necessary to specify some type of Dirichlet or Neumann boundary condition on $U$, but it is not required (nor may it be desirable) to specify a boundary condition on $\xi$.

We will assume the Dirichlet boundary condition

(3) \quad U = g,

on $\partial \Omega \times (0, T]$. We also assume initial conditions

(4) \quad \xi(x, 0) = \xi^0(x), \quad U(x, 0) = U^0(x).

The GWCE is obtained by differentiating (1) with respect to time and substituting the divergence of (2) into the result. We then obtain

(5) \quad \xi_{tt} - \nabla \cdot (G \nabla \xi) + \mu \nabla \cdot \Delta U + \nabla \cdot \mathcal{F} = 0.

with the additional initial condition that

(6) \quad \xi_t(x, 0) = \xi_1(x) \equiv -\nabla \cdot U^0.

The GWCE shallow water model then consists of (2) and (3)-(6).

The rest of this paper is outlined as follows, In section (2) we introduce definitions and notation, In section (3), we derive a weak formulation of the GWCE-CME system of equations and state some assumptions on the solution, In section (4), we introduce the continuous-time finite element approximation to the weak solution, and derive stability and a priori error estimates for this approximation, In section (5), we extend these estimates to a nonlinear shallow water model. Finally, in section (6), we discuss a discrete time approximation to the linear model given above.

2. Preliminaries.

2.1. Notation. For the purposes of our analysis, we define some notation used throughout the rest of this paper.

Let $\Omega$ be a bounded polygonal domain in $\mathbb{R}^2$ and $x = (x_1, x_2) \in \mathbb{R}^2$. Moreover, let $\bar{\Omega} = \Omega \cup \partial \Omega$, where $\partial \Omega$ is the boundary of $\Omega$.

The $L^2$ inner product is denoted by

$$\langle \varphi, \omega \rangle = \int_{\Omega} \varphi \cdot \omega dx, \quad \varphi, \omega \in [L^2(\Omega)]^n,$$

where "·" here refers to either multiplication, dot product, or double dot product as appropriate. We will let $\langle \varphi, \omega \rangle$ denote an inner product on $\partial \Omega$. We denote the $L^2$
norm by $||\varphi|| = (\varphi, \varphi)^{1/2}$. In $\mathbb{R}^n$, $\alpha = (\alpha_1, \ldots, \alpha_n)$ is an $n$-tuple with nonnegative integer components,

$$D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n} = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}$$

and $|\alpha| = \sum_{i=1}^{n} \alpha_i$.

For $\ell$ any nonnegative integer, let

$$\mathcal{H}^\ell \equiv \{ \varphi \in \mathcal{L}^2(\Omega) \mid D^\alpha \varphi \in \mathcal{L}^2(\Omega) \text{ for } |\alpha| \leq \ell \}$$

be the Sobolev space with norm

$$||\varphi||_{\mathcal{H}^\ell(\Omega)} = \left( \sum_{|\alpha| \leq \ell} ||D^\alpha \varphi||^2_{\mathcal{L}^2(\Omega)} \right)^{1/2}.$$ 

Additionally, $\mathcal{H}^\ell_0(\Omega)$ denotes the subspace of $\mathcal{H}^\ell(\Omega)$ obtained by completing $\mathcal{C}_0^\infty(\Omega)$ with respect to the norm $||\cdot||_{\mathcal{H}^\ell(\Omega)}$, where $\mathcal{C}_0^\infty(\Omega)$ is the set of infinitely differentiable functions with compact support in $\Omega$.

Moreover, let

$$\mathcal{W}^\ell_{\infty} \equiv \{ \varphi \in \mathcal{L}^\infty(\Omega) \mid D^\alpha \varphi \in \mathcal{L}^\infty(\Omega) \text{ for } |\alpha| \leq \ell \}$$

be the Sobolev space with norm

$$||\varphi||_{\mathcal{W}^\ell_{\infty}(\Omega)} = \max_{|\alpha| \leq \ell} ||D^\alpha \varphi||_{\mathcal{L}^\infty(\Omega)}.$$ 

For relevant properties of these spaces, please refer to [1].

Observe, for instance, that $\mathcal{H}^\ell$ are spaces of $\mathbb{R}$-valued functions. Spaces of $\mathbb{R}^n$-valued functions will be denoted in boldface type, but their norms will not be distinguished. Thus, $\mathcal{L}^2(\Omega) = [\mathcal{L}^2(\Omega)]^n$ has norm $||\varphi||^2 = \sum_{i=1}^{n} ||\varphi_i||^2$; $\mathcal{H}^1(\Omega) = [\mathcal{H}^1(\Omega)]^n$ has norm $||\varphi||^2_{\mathcal{H}^1(\Omega)} = \sum_{|\alpha| \leq 1} ||D^\alpha \varphi_i||^2$, etc. For $X$, a normed space with norm $||\cdot||_X$ and a map $f : [0, T] \to X$, define

$$||f||^2_{\mathcal{L}^2(0,T;X)} = \int_0^T ||f(\cdot, t)||^2_X dt,$$

$$||f||_{\mathcal{L}^\infty(0,T;X)} = \sup_{0 \leq t \leq T} ||f(\cdot, t)||_X.$$

3. **Weak formulation.** A weak formulation of (5) is obtained as follows. From (1), we have

$$\mathbf{u}(t) - (\mathbf{U}, \nabla v) + (\mathbf{g} \cdot \nu, v) = 0.$$ 

Differentiating this equation in time and using (2) we find

$$\mathbf{u}_t(t) + (G \nabla \mathbf{u}, \nabla v) - \mu(\Delta \mathbf{U}, \nabla v) - (\mathbf{F}, \nabla v) + (\mathbf{g}, \cdot \nu, v) = 0. \quad v \in \mathcal{H}^1(\Omega),$$

Moreover, multiplying (2) by a test function and integrating by parts,

$$\mathbf{u}(t) + (G \nabla \mathbf{w}, \nabla v) + \mu(\nabla \mathbf{U}, \nabla \mathbf{w}) = (\mathbf{F}, \mathbf{w}) + \mu(\nabla \mathbf{U} \cdot \nu, \mathbf{w}), \quad \mathbf{w} \in \mathcal{H}^1(\Omega).$$

In our previous work, we were able to replace the term involving $\Delta \mathbf{U}$ in (8) by

$$\mu(\nabla \xi, \nabla v),$$
because of the assumption of Dirichlet boundary conditions on $\xi$. We note that, by (1),

$$\Delta \xi_t = -\Delta (\nabla \cdot U) = -\nabla \cdot \Delta U.$$ 

Multiplying by a test function and integrating by parts we find that

$$-(\nabla \xi_t, \nabla v) = (\Delta U, \nabla v),$$

if the test function is zero on $\partial \Omega$. In (8), because our test function $v$ is not zero on the boundary, we cannot make this substitution without introducing boundary terms involving $\xi$ which are not known. In defining our method below, we handle the $\Delta U$ term in (8) without requiring additional continuity on the finite element space.

3.1. Some Assumptions. We will make the following assumptions about the solutions and the data. First, we assume the domain $\Omega$ is polygonal, and for $(x, t) \in \Omega \times (0, T]$,

A1. the solutions $\xi, U$ to (2) and (3)-(6) exist and are unique,

A2. $\mu$ is a positive constant,

We make the following smoothness assumptions on the initial data and on the solutions,

A3. $\xi^0, U^0(x) \in H^\ell(\Omega),$

A4. $\xi \in H^{\ell+1}(\Omega)$, $t > 0$,

A5. $U, U_t \in H^\ell(\Omega)$, $t > 0$,

A6. $\Delta U \in H^\ell(\Omega)$, $t > 0$,

where the integer $\ell \geq 2$ is defined in the next section.

4. Galerkin Finite Element Approximation.

4.1. The Continuous-Time Galerkin Approximation. Let $T$ be a quasi-uniform triangulation of the polygonal domain $\Omega$ into elements $E_i$, $i = 1, \ldots, m$, with $\operatorname{diam}(E_i) = h_i$ and $h = \max h_i$. Let $\mathcal{S}_h (\mathcal{S}_h^0)$ denote a finite dimensional subspace of $H^\ell(\Omega)$ ($H^\ell(\Omega)$) defined on this triangulation consisting of piecewise polynomials of degree less than $s_1$. Let $\mathcal{S}_h^0 = \mathcal{S}_h \cap \{ w : w = g \text{ on } \partial \Omega \}$ and $\mathcal{S}_h^0 = \mathcal{S}_h \cap \{ w : w = 0 \text{ on } \partial \Omega \}$. Assume $\mathcal{S}_h (\mathcal{S}_h^0)$ satisfies the standard approximation property

$$\inf_{\varphi \in \mathcal{S}_h (\mathcal{S}_h^0)} \| v - \varphi \|_{H^{\ell+1}(\Omega)} \leq K_0 h^{\ell-\varphi} \| v \|_{H^{\ell+1}(\Omega)}, \quad v \in H^\ell(\Omega) \cap H^{\ell+1}(\Omega),$$

and the inverse assumptions (see [2], Theorem 4.5.11)

$$\| \varphi \|_{L^\infty(\Omega)} \leq K_0 \| \varphi \|_{L^1(\Omega)} h^{-1}, \quad \varphi \in \mathcal{S}_h (\mathcal{S}_h^0),$$

(12) $$\| \varphi \|_{H^1(\Omega)} \leq K_0 \| \varphi \|_{L^1(\Omega)} h^{-1}, \quad \varphi \in \mathcal{S}_h (\mathcal{S}_h^0).$$

Here, $s_0$ and $\ell$ are integers, $0 \leq s_0 \leq \ell < s_1$. Moreover, $K_0$ is a constant independent of $h$ and $v$. We also define the space $\mathcal{S}_h^0 = \mathcal{S}_h \cap \mathcal{S}_h^0$. In defining our method, we approximate four quantities, $\xi, U, \Delta U,$ and $\lambda \equiv \nabla U \cdot \nu$ on $\partial \Omega$. The equations for $\xi$ and $U$ are derived from (8) and (9). We note that by integration by parts,

(13) $$(\Delta U, w) = -(\nabla U, \nabla w) + \langle \lambda, w \rangle, \quad w \in H^\ell(\Omega).$$
Moreover, by (9),
\[
\mu(\lambda, w) = ((U_h)_t, w) + (G\nabla \xi_h, w) + \mu(\nabla U_h, \nabla w) - (F, w), \quad w \in H^1(\Omega).
\]
(14)

Define an approximation \(\xi_h(\cdot, t) \in S_h\) by
\[
(\xi_h(\cdot, 0), v) = (\xi_0, v), \quad v \in S_h,
\]
(15)

\[
((\xi_h)_t(\cdot, 0), v) = (\xi_0^1, v), \quad v \in S_h,
\]
(16)
and for \(t > 0\),
\[
((\xi_h)_t, v) + (G\pi(\nabla \xi_h), \nabla v) - \mu(\Delta_h U_h, \nabla v) - (F, \nabla v) + \langle g_t, v \rangle = 0, \quad v \in S_h,
\]
(17)
where \(\pi(\nabla \xi_h)\) denotes the \(L^2\) projection of \(\nabla \xi_h\) into \(S_h\), and \(\Delta_h U_h\) is defined below.

Define an approximation \((U_h)_t(\cdot, t) \in S^0_h\) by
\[
(\nabla U_h(\cdot, 0), \nabla w) = (\nabla U^0, \nabla w), \quad w \in S^0_h
\]
(18)

and for \(t > 0\),
\[
((U_h)_t, w) + (G\nabla \xi_h, w) + \mu(\nabla U_h, \nabla w) = (F, w), \quad w \in S^0_h
\]
(19)

The “discrete Laplacian” \(\Delta_h U_h \in S_h\) in (17) is defined by
\[
(\Delta_h U_h, w) = -(\nabla U_h, \nabla w) + \langle \lambda_h, w \rangle, \quad w \in S_h,
\]
(20)
where the approximate boundary flux \(\lambda_h \in S^\Omega_h\) is defined by
\[
\mu(\lambda_h, w_b) = ((U_h)_t, w_b) + (G\nabla \xi_h, w_b) + \mu(\nabla U_h, \nabla w_b) - (F, w_b), \quad w_b \in S^\Omega_h.
\]
(21)

Thus, the system (15)–(21) yields a system of equations in four unknowns, \(\xi_h, U_h, \Delta_h U_h\) and \(\lambda_h\).

In section (6), we will formulate a discrete time version of this scheme and show that these unknowns can be determined in a sequential manner. Moreover, all matrices which arise are symmetric, positive definite, and time-independent.

4.2. A stability estimate. As a prelude to deriving an error estimate, we study the stability of the scheme above in the case \(g = 0\) and \(F = 0\). Integrating (17), (19) and (21) in time, we obtain
\[
((\xi_h)_t, v) + \left( \int_0^t G \pi(\nabla \xi_h) ds, \nabla v \right) - \mu \left( \int_0^t \Delta_h U_h ds, \nabla v \right) = \langle \xi_h^1, v \rangle, \quad v \in S_h,
\]
(22)

\[
((U_h)_t, w) + \left( \int_0^t G \nabla \xi_h ds, w \right) + \mu \left( \int_0^t \nabla U_h ds, \nabla w \right) = (U_0^0, w), \quad w \in S^0_h
\]
(23)
(24) \[ \mu(\int_0^t \lambda_\theta ds, w_b) = (U_h, w_b) + (\int_0^t G \nabla \xi_h ds, w_b) \]
\[ + \mu(\int_0^t \nabla U_h ds, \nabla w_b) - (U^0, w_b), \quad w_b \in S^0_h. \]

Adding (23) and (24) we find

(25) \[ (U_h, w + w_b) + \mu(\int_0^t \nabla U_h ds, \nabla (w + w_b)) \]
\[ + (\int_0^t G \nabla \xi_h ds, w + w_b) - \mu(\int_0^t \lambda_\theta ds, w + w_b) = (U^0, w + w_b). \]

Here we have used the fact that \( w = 0 \) on \( \partial \Omega \) in the term involving \( \lambda_\theta \).

We now set \( w + w_b = \pi(\nabla \xi_h) \), and we set \( v = \xi_h \) in (22) to obtain

\[ ((\xi_h)_t, \xi_h) + (\int_0^t G \pi(\nabla \xi_h) ds, \nabla \xi_h) \]
\[ - \mu(\int_0^t \Delta_h U_h ds, \nabla \xi_h) = (\xi^i_h, \xi_h), \]

and

\[ (U_h, \nabla \xi_h) + (\int_0^t G \nabla \xi_h ds, \pi(\nabla \xi_h)) \]
\[ + \mu(\int_0^t \nabla U_h, \nabla \pi(\nabla \xi_h)) - \mu(\int_0^t \lambda_\theta ds, \pi(\nabla \xi_h)) = (U_h(\cdot, 0), \nabla \xi_h). \]

Integrating (20) in time, setting \( w = \pi(\nabla \xi_h) \), we find

\[ (\int_0^t \Delta_h U_h ds, \nabla \xi_h) = (\int_0^t \Delta_h U_h ds, \pi(\nabla \xi_h)) \]
\[ = - (\int_0^t \nabla U_h ds, \nabla \pi(\nabla \xi_h)) + (\int_0^t \lambda_\theta ds, \pi(\nabla \xi_h)), \]

Substituting this result into (27) and subtracting from (26), we obtain

(28) \[ ((\xi_h)_t, \xi_h) = (U_h, \nabla \xi_h) - (\int_0^t G \pi(\nabla \xi_h) ds, \nabla \xi_h) + (\int_0^t G \nabla \xi_h ds, \pi(\nabla \xi_h)) \]
\[ + (\xi^i_h, \xi_h) - (U_h(\cdot, 0), \nabla \xi_h) \]
\[ = (U_h, \nabla \xi_h) + (\xi^i_h, \xi_h) - (U_h(\cdot, 0), \nabla \xi_h). \]

Letting \( w = U_h \) in (19) we obtain

(29) \[ ((U_h)_t, U_h) + \mu(\nabla U_h, \nabla U_h) = -(G \nabla \xi_h, U_h). \]

Adding (28) and (29) and integrating by parts we find

(30) \[ ((\xi_h)_t, \xi_h) + ((U_h)_t, U_h) + \mu(\nabla U_h, \nabla U_h) \]
\[ = (\xi^i_h, \xi_h) - (\nabla \cdot U_h, \xi_h) + (G \xi_h, \nabla \cdot U_h) + (\nabla \cdot U_h(\cdot, 0), \xi_h) \]
\[ = -(\nabla \cdot U_h, \xi_h) + (G \xi_h, \nabla \cdot U_h), \]
where in the last step we’ve used the fact that $\xi_h$ is the $L^2$ projection of $-\nabla \cdot U_h(\cdot, 0)$ into $S_h$.

Integrating (30) in time from 0 to $T$ we find

$$
\begin{align*}
||\xi_h(\cdot, T)||^2 + ||U_h(\cdot, T)||^2 + 2\mu \int_0^T ||\nabla U_h||^2 dt \\
\leq ||\xi^0||^2 + ||U^0||^2 + \int_0^T [\mu ||\nabla U_h||^2 + C ||\xi_h||^2] dt \\
\leq ||\xi^0||^2 + ||U^0||^2 + \int_0^T [\mu ||\nabla U_h||^2 + C ||\xi_h||^2] dt.
\end{align*}
$$

An application of Gronwall’s Lemma gives the following result.

**Lemma 4.1.** For the case $g = 0$ and $F = 0$, and any $T > 0$

$$
||\xi_h(\cdot, T)|| + ||U_h(\cdot, T)|| \leq C \left(||\xi^0|| + ||U^0||\right).
$$

4.3. An *a priori* error estimate. We now consider the more general case where $g$ and $F$ are not zero. In order to derive an error estimate let $\xi_h$ denote the $L^2$ projection of $\xi$ into $S_h$, and $\tilde{U}_h$ the elliptic projection of $U$ into $S^0_h$; that is, $\tilde{U}_h \in S^0_h$ is defined by

$$
(\nabla (\tilde{U}_h - U), \nabla w) = 0, \quad w \in S^0_h.
$$

Let $\psi_\xi = \xi - \tilde{\xi}_h$, $\psi_u = U - \tilde{U}_h$, $\theta_\xi = \xi - \tilde{\xi}_h$, and $\theta_u = U - \tilde{U}_h$.

Integrating (8) in time and combining with (22) we find

$$
(\partial T, v) + (\int_0^T G\pi(\nabla \xi_h)ds, \nabla v) = (\xi^0 - \xi(\cdot, 0), v) + ((\theta_\xi), v) + (\int_0^T G\nabla \xi ds, \nabla v) + \mu(\int_0^T (\Delta_h U - \Delta U)ds, \nabla v).
$$

Integrating (14) in time and combining with (25) we find

$$
(\psi_u, w + w_h) + (\int_0^T G\nabla \xi dh, w + w_h) = (\theta_u, w + w_h) - (\theta_u(\cdot, 0), w + w_h) + (\int_0^T G\nabla \xi ds, w + w_h)
$$

$$
+ \mu(\int_0^T (\Delta_h U_h - \Delta U)ds, w + w_h).
$$

Here we have used the definition of $\Delta_h U_h$, (20), and (13).

Setting $v = \psi_\xi$ and $w + w_h = \pi(\nabla \psi_\xi)$, where $\pi(\nabla \psi_\xi)$ is the $L^2$ projection of $\nabla \psi_\xi$ into $S_h$, and subtracting (35) from (34) we find

$$
([\psi_\xi], \psi_\xi) = (\psi_u, \nabla \psi_\xi) - (\int_0^T G\nabla \xi ds, \pi(\nabla \psi_\xi) - \nabla \psi_\xi)
$$

$$
+ ((\theta_u, \psi_\xi) - (\theta_u(\cdot, 0), \pi(\nabla \psi_\xi))) + (\theta_u(\cdot, 0), \pi(\nabla \psi_\xi))
$$

$$
+ \mu(\int_0^T \Delta U ds, \pi(\nabla \psi_\xi) - \nabla \psi_\xi) + (\nabla \cdot (U_h - U)(\cdot, 0), \psi_\xi).
$$

(36)
From (19), we find

\[ ((\psi u), \psi u) + (G\nabla \psi, \psi u) + \mu(\nabla \psi u, \nabla \psi u) = ((\theta u), \psi u) + (G\nabla \psi, \psi u) + \mu(\nabla \theta u, \nabla \psi u). \]

Adding (36)-(37), integrating by parts, and using the definitions of the $L^2$ and elliptic projections, we find

\[ ((\psi \xi), \psi \xi) + ((\psi u), \psi u) + \mu(\nabla \psi u, \nabla \psi u) \]
\[ = - (\nabla \cdot \psi u, \psi \xi) + \int_0^T G(\nabla \xi - \pi(\nabla \xi))ds, \pi(\nabla \psi) - \nabla \psi \xi) \]
\[ - (\theta u, \pi(\nabla \psi)) + (\theta u (\cdot , 0), \pi(\nabla \psi)) + ((\theta u), \psi u) \]
\[ + (G(\psi \xi - \theta \xi), \nabla \cdot \psi u) + \mu(\int_0^T (\Delta U - \pi(\Delta U))ds, \pi(\nabla \psi) - \nabla \psi \xi) \]
\[ + (\nabla \cdot (U_h - u)(\cdot, 0), \psi \xi), \]

where $\pi(\Delta U)$ is the $L^2$ projection of $\Delta U$ into $S_h$.

Integrating this equation in time, we find

\[ ||\psi \xi(\cdot, T)||^2 + ||\psi u(\cdot, T)||^2 + 2\mu \int_0^T ||\nabla \psi u||^2 dt \]
\[ \leq \mu \int_0^T ||\nabla \psi u||^2 dt + C h^{-2} \int_0^T ||\nabla \xi - \pi(\nabla \xi)||^2 dt \]
\[ + C \int_0^T [|\theta \xi|]^2 + ||\psi \xi||^2 dt \]
\[ + C \int_0^T [h^{-2}||\theta u||^2 + ||(\theta u)\xi||^2 + ||\psi u||^2] dt + C h^{-2}||\theta u(\cdot, 0)||^2 \]
\[ + C h^{-2} \int_0^T ||\Delta U - \pi(\Delta U)||^2 dt + C h^2 \int_0^T [||\pi(\nabla \psi)\xi||^2 + ||\nabla \psi \xi||^2] dt \]
\[ + C ||\nabla \cdot (U_h(\cdot, 0) - U(\cdot, 0))||^2 \]

It is easily shown that

\[ ||\pi(\nabla \psi \xi)|| \leq ||\nabla \psi \xi||. \]

Using (40) and the inverse estimate (12), we find

\[ ||\psi \xi(\cdot, T)||^2 + ||\psi u(\cdot, T)||^2 + \mu \int_0^T ||\nabla \psi u||^2 dt \]
\[ \leq C h^2(l^{-1}) + C \int_0^T [||\psi \xi||^2 + ||\psi u||^2] dt. \]

Applying Gronwall’s inequality and the triangle inequality, we obtain the following:

**Theorem 4.2.** Let the assumptions A1-A6 hold. Assume the finite element solutions $\xi_h, U_h, \Delta_h U_h,$ and $\lambda_h$ to (15)-(21) exist and are unique. Then there exists a constant $C$ independent of $h$ such that

\[ ||(U - U_h)||_{L^\infty(0,T;L^2)} + ||K - \xi_h||_{L^\infty(0,T;L^2)} \leq C h^{l-1}. \]

We remark that this rate of convergence is the same as that obtained in our earlier paper [3].
5. A nonlinear model. Realistic shallow water models are nonlinear. For example, the term $g H \nabla \xi$ in the momentum equation (2) is actually $g H \nabla \xi$, where $g$ is gravitational acceleration (assumed constant). Moreover, an advection term $\nabla \cdot U^2 / H$ is also present. There are also forcing terms (Coriolis force, wind stress, tide potentials, bottom friction) present in the equation; we will assume these are known, and for simplicity lump them into the term $\mathcal{F}$. Thus (2) becomes

\begin{equation}
U_t + g H \nabla \xi + \nabla \cdot \frac{U^2}{H} - \mu \Delta U = \mathcal{F},
\end{equation}

and the GWCE (5) becomes

\begin{equation}
\xi_{tt} - \nabla \cdot (g H \nabla \xi) - \nabla \cdot \left( \nabla \cdot \frac{U^2}{H} \right) + \mu \nabla \cdot \Delta U + \nabla \cdot \mathcal{F} = 0.
\end{equation}

Let

\[
\Gamma = g H \nabla \xi + \nabla \cdot \frac{U^2}{H},
\]

and

\[
\Gamma_h = g H_h \nabla \xi_h + \nabla \cdot \frac{U^2}{H_h},
\]

where

\begin{equation}
H_h = h + \xi_h.
\end{equation}

Let $\pi \Gamma_h$ ($\pi \Gamma$) denote the $L^2$ projection of $\Gamma_h$ ($\Gamma$) into $\mathcal{S}_h$. Our finite element method is defined as follows. We choose the initial data and define the discrete Laplacian as before. Then, for $t > 0$,

\begin{align}
((\xi_h)_{tt}, v) + (\pi \Gamma_h, \nabla v) & - \mu (\Delta U h, \nabla v) - (\mathcal{F}, \nabla v) + (g_h \cdot \nu, v) = 0, \quad v \in \mathcal{S}_h,

((U_h)_t, w) + \mu (\nabla U h, \nabla w) & = -(\Gamma_h, w) + (\mathcal{F}, w) = -((\pi \Gamma_h, w) + (\mathcal{F}, w), \quad w \in \mathcal{S}_h^0,

\text{and}

\begin{align}
\mu (\lambda_h, w_h) & = ((U_h)_t, w_h) + (\Gamma_h, w_h) \\
& + \mu (\nabla U h, \nabla w_h) - (\mathcal{F}, w_h), \quad w_h \in \mathcal{S}_h^0.
\end{align}

For the error analysis below, we will assume that a constant $K$ exists such that

\begin{equation}
||\xi_h||_{L^\infty(0,T;L^\infty)} + ||U_h||_{L^\infty(0,T;L^\infty)} \leq K,
\end{equation}

and that positive constants $H_{**}, H^{**}$ exist such that

\begin{equation}
A7. \quad H_{**} < H_h < H^{**}.
\end{equation}

Using an induction argument as in [3], one can show that $K$, $H_{**}$ and $H^{**}$ are independent of $h$ for $h$ sufficiently small, for polynomials of degree two and higher. We will also assume that
\( A9. \) \( \xi, h_0, U \in \mathcal{W}_{\infty}^1(\Omega), \Gamma \in \mathcal{H}^l(\Omega). \)

Define \( \psi_\xi, \psi_u, \theta_\xi \) and \( \theta_u \) as before. Integrate (45) in time and subtract the analogous equation obtained from (44) to find

\[
(\psi_\xi, v) = ((\psi_\xi)_t, 0, v) + \left( \int_0^t \pi(\Gamma - \Gamma_h)ds, \nabla v \right) + ((\theta_\xi)_t, v) + \mu \left( \int_0^t (\Delta_h U_h - \Delta U)ds, \nabla v \right) + \left( \int_0^t (\Gamma - \pi \Gamma)ds, \nabla v \right), \quad v \in \mathcal{S}_h.
\]

Integrate (46) and (47) in time and subtract the analogous equation obtained from (42) to find

\[
(\psi_u, w) = \left( \int_0^t \pi(\Gamma - \Gamma_h)ds, w \right) + \mu \left( \int_0^t (\Delta_h U_h - \Delta U)ds, w \right) + (\theta_u, w) - (\theta_u(\cdot, 0), w), \quad w \in \mathcal{S}_h.
\]

From (46) and (42) we obtain

\[
(\psi_u)_t = (\Gamma - \Gamma_h, w) + ((\psi_u)_t, w) = (\Gamma - \Gamma_h, w) + ((\psi_u)_t, w), \quad w \in \mathcal{S}_h^0.
\]

Setting \( v = \psi_\xi, \quad w = \pi(\nabla \psi_\xi) \) in (49) and \( w = \psi_u \) in (49). Subtracting (49) from (48) and adding (49) we obtain

\[
(\psi_\xi)_t + ((\psi_u)_t, \psi_u) + \mu(\nabla \psi_u, \nabla \psi_u)
= ((\psi_\xi)_t, 0, \psi_\xi) + (\psi_u, \nabla \psi_\xi) + \mu \left( \int_0^t (\Delta U - \pi(\Delta U))ds, \pi(\nabla \psi_\xi) - \nabla \psi_\xi \right)
+ \left( \int_0^t (\Gamma - \pi \Gamma)ds, \nabla \psi_\xi \right) + \theta_u(\pi(\nabla \psi_\xi) - (\theta_u(\cdot, 0), \pi \nabla \psi_\xi)
+ (\Gamma - \Gamma_h), \psi_u) + ((\theta_u)_t, \psi_u).
\]

The fourth and seventh terms on the right side of (50) are handled as follows.

\[
\left( \int_0^t (\Gamma - \pi \Gamma)ds, \nabla \psi_\xi \right) \leq Ch^{-2} \left( \int_0^t (\Gamma - \pi \Gamma)ds \right)^2 + Ch^2 \left( \nabla \psi_\xi \right)^2
\]

\[
\leq Ch^{2(l-1)} + C \left( \psi_\xi \right)^2.
\]

\[
(\Gamma - \Gamma_h, \psi_u) = \left( gH \nabla \xi - gH_h \nabla \xi_h, \psi_u \right) + \left( \nabla \cdot \left( \frac{U^2}{H} - \frac{U^2_h}{H_h} \right), \psi_u \right)
= \left( gH_h \nabla (\xi - \xi_h), \psi_u \right) + \frac{g}{2} \left( \nabla \xi^2 - \nabla \xi_h^2, \psi_u \right)
- \left( \frac{U^2}{H} - \frac{U^2_h}{H_h}, \nabla \psi_u \right)
= -g(\xi - \xi_h, \nabla \cdot (h_0 \psi_u)) - \frac{g}{2}(\xi^2 - \xi_h^2, \nabla \cdot \psi_u)
- \left( \frac{U^2 H_h - U^2_h H}{H H_h}, \nabla \psi_u \right).
\]
\[
= -g(\xi - \xi_h, \nabla \cdot (h_0 \psi u)) - \frac{g}{2} (\xi^2 - \xi_h^2, \nabla \cdot u) \\
- \left( \frac{U^2 (H_h - H) - H_h (U^2 - U_h^2)}{H H_h}, \nabla \psi u \right) \\
\leq C ||\psi||^2 + C ||\theta||^2 + C ||\psi u||^2 \\
+ C ||\theta u||^2 + \frac{\mu}{2} ||\nabla \psi u||^2.
\]

Combining (50), (51) and (53), choosing $\epsilon$ sufficiently small, using bounds previously derived for the remaining terms, and integrating in time, we obtain

\[
||\psi(\cdot, T)||^2 + ||\psi u(\cdot, T)||^2 + \mu \int_0^T ||\nabla \psi u||^2 dt \\
\leq C h^{2(l-1)} + C \int_0^T [||\psi||^2 + ||\psi u||^2] dt.
\]

Using Gronwall’s Lemma we obtain the following.

**Theorem 5.1.** Assume the finite element solutions $\xi_h$, $U_h$, $\Delta_h U_h$, and $\lambda_h$ to (15), (16), (45), (18), (46), (20) and (47) exist and are unique. Let the assumptions A1-A9 hold and assume $h$ is sufficiently small. Then, there exists a constant $C$ independent of $h$ such that

\[
||U - U_h||_{L^\infty(0,T;L^2)} + ||\xi - \xi_h||_{L^\infty(0,T;L^2)} \leq C h^{l-1}.
\]

6. A discrete time method. In this section, we return for simplicity to the linear model presented in section (1), with $g = \mathcal{F} = 0$, and formulate a discrete time method. We extend our continuous-time stability argument presented in section (4) and show that the discrete scheme satisfies the same stability bound. We leave the derivation of error estimates for this scheme to the reader.

Choose a time step $\Delta t > 0$ and set $t^k = k\Delta t$, $k = 0, 1, \ldots$. Denote $g(t^k)$ by $g^k$. A discrete time scheme based on (2) and (3)-(6) can be defined as follows. We define the initial approximations $\xi_h^0$ and $U_h^0$ as before, see (15), (18). We enforce the initial condition (6) by

\[
\left( \frac{\xi_h^1 - \xi_h^0}{\Delta t}, v \right) + (\nabla \cdot U_h^0, v) = 0, \quad v \in \mathcal{S}_h.
\]

Then, for $k = 1, 2, \ldots$,

\[
\left( \frac{U_h^k - U_h^{k-1}}{\Delta t}, w \right) + (G \nabla \xi_h^k, w) + \mu (\nabla U_h^k, \nabla w) = 0, \quad w \in \mathcal{S}_h^0.
\]

\[
\mu (\lambda_h^k, w_h) = \left( \frac{U_h^k - U_h^{k-1}}{\Delta t}, w_h \right) + (G \nabla \xi_h^k, w_h) \\
+ \mu (\nabla U_h^k, \nabla w_h), \quad w_h \in \mathcal{S}_h^{0\Omega}.
\]

\[
(\Delta_h U_h^k, w) = - (\nabla U_h^k, \nabla w) + (\lambda_h^k, w), \quad w \in \mathcal{S}_h.
\]
and

\[
\frac{\xi_{h}^{k+1} - 2\xi_{h}^{k} + \xi_{h}^{k-1}}{\Delta t^{2}}, \quad v + (G \pi(\nabla \xi_{h}^{k}), \nabla v) - \mu(\Delta_{h} U_{h}^{k}, \nabla v) = 0, \quad v \in S_{h}.
\]

Note that, at each step in the above procedure, the matrices which arise are symmetric and positive definite, and independent of time.

We now extend the stability argument given above for the continuous time scheme to this discrete scheme. This argument can also be used to show uniqueness (hence existence) for the solutions to the system given above.

Adding (56) and (58) and using the definition (58) of \( \Delta_{h} U_{h}^{k} \), we find

\[
(\frac{U_{h}^{k} - U_{h}^{k-1}}{\Delta t}, w) + (G \nabla \xi_{h}^{k}, w) - \mu(\Delta_{h} U_{h}^{k}, w) = 0, \quad w \in S_{h}.
\]

Multiplying this equation by \( \Delta t \) and summing on \( k, k = 1, \ldots, n \), for some integer \( n \geq 0 \), we find

\[
(U_{h}^{n}, w) + \left( \sum_{k=1}^{n} G \nabla \xi_{h}^{k} \Delta t, w \right)
- \mu(\sum_{k=1}^{n} \Delta_{h} U_{h}^{k} \Delta t, w) = (U_{h}^{0}, w), \quad w \in S_{h}.
\]

Multiplying (59) by \( \Delta t \) and summing on \( k \) we obtain

\[
(\frac{\xi_{h}^{n+1} - \xi_{h}^{n}}{\Delta t}, v) + \left( \sum_{k=1}^{n} G \pi(\nabla \xi_{h}^{k}) \Delta t, \nabla v \right)
- \mu(\sum_{k=1}^{n} \Delta_{h} U_{h}^{k} \Delta t, \nabla v) = (\frac{\xi_{h}^{1} - \xi_{h}^{0}}{\Delta t}, v), \quad v \in S_{h}.
\]

Setting \( v = \xi_{h}^{n+1} \) in (62) and \( w = \pi(\nabla \xi_{h}^{n+1}) \) in (61), subtracting (61) from (62) and substituting (55), we find

\[
(\frac{\xi_{h}^{n+1} - \xi_{h}^{n}}{\Delta t}, \xi_{h}^{n+1}) = -(U_{h}^{0}, \nabla \xi_{h}^{n+1}) - (\nabla \cdot U_{h}^{0}, \xi_{h}^{n+1}) + (U_{h}^{n}, \nabla \xi_{h}^{n+1}).
\]

Setting \( k = n \) in (56) and \( w = U_{h}^{n} \), we obtain

\[
(\frac{U_{h}^{n} - U_{h}^{n-1}}{\Delta t}, U_{h}^{n}) + \mu ||\nabla U_{h}^{n}||^{2} = -(G \nabla \xi_{h}^{n}, U_{h}^{n}).
\]

Adding (63) and (64), using the inequality \( a(b - a) \geq \tfrac{1}{2} a^{2} - \tfrac{3}{2} b^{2} \), and integrating by parts we find

\[
\frac{||\xi_{h}^{n+1}||^{2} - ||\xi_{h}^{n}||^{2} + ||U_{h}^{n}||^{2} - ||U_{h}^{n-1}||^{2} + \mu||\nabla U_{h}^{n}||^{2}}{\Delta t}
= (G \xi_{h}^{n}, \nabla \cdot U_{h}^{n}) - (\nabla \cdot U_{h}^{n}, \xi_{h}^{n+1}).
\]

Multiplying (65) by \( \Delta t \) and summing on \( n, n = 1, 2, \ldots, N \) where \( N \geq 1 \) is an integer, we find

\[
||\xi_{h}^{N+1}||^{2} + ||U_{h}^{N}||^{2} + \mu \sum_{n=1}^{N} ||\nabla U_{h}^{n}||^{2} dt
\]
\[\begin{align*}
\leq ||\xi_h^i||^2 + ||U_h^0||^2 + C \sum_{n=1}^{N+1} ||\xi_h^i||^2 \Delta t \\
+ \frac{\mu}{2} \sum_{n=1}^{N} ||\nabla U_h^i||^2 \Delta t.
\end{align*}\]

Finally, we note that, by (55), setting \(v = \xi_h^1\) we find

\[||\xi_h^1|| \leq ||\xi_h^0|| + \Delta t ||\nabla U_h^0||.\]

Combining (67) with (66) and applying the discrete version of Gronwall's inequality we obtain the following.

**Lemma 6.1.** For the case \(g = 0\) and \(\mathcal{F} = 0\), \(N\) a positive integer and \(\Delta t > 0\),

\[||\xi_h^N|| + ||U_h^N|| \leq C \left( ||\xi_h^0|| + ||U_h^0|| + \Delta t ||\nabla U_h^0|| \right).

**References**


