Space-Time Domain Decomposition for Parabolic Problems

Eldar Giladi and Herbert B. Keller

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Center for Research on Parallel Computation
Rice University
6100 South Main Street
CRPC - MS 41
Houston, TX 77005

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Eldar Giladi *
Applied Mathematics 217-50
Caltech, Pasadena CA 91125.

Herbert B. Keller †
Applied Mathematics 217-50
Caltech, Pasadena CA 91125.

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Abstract

We analyze a space-time domain decomposition iteration, for a model advection diffusion equation in one and two dimensions. The asymptotic convergence rate is superlinear, and it is governed by the diffusion of the error across the overlap between subdomains. Hence, it depends on both the size of this overlap and the diffusion coefficient in the equation. However, it is independent of the number of subdomains. The convergence rate for the heat equation in a large time window is initially linear and it deteriorates as the number of subdomains increases. The duration of the transient linear regime is proportional to the length of the time window. For advection dominated problems, the convergence rate is initially linear and it improves as the the ratio of advection to diffusion increases. Moreover, it is independent of the size of the time window and of the number of subdomains. In two space dimensions, the iteration possesses the smoothing property: high modes of the error are damped much faster then low modes. This is a result of the natural smoothing property of the heat equation. Numerical calculations illustrate our analysis.

1 Introduction

We analyze a space-time domain decomposition (DD) iteration for a model advection diffusion equation in one and two dimensions. Our study is motivated by several applications. The algorithm provides the foundation for an efficient solution of parabolic problems on parallel machines, because it minimizes communication. In the numerical simulation of

*Phone (o): (818)-395-4550, FAX: (818)-578-0124, e-mail: giladi@ama.caltech.edu
†Phone (o): (818)-395-4557, e-mail: hbk@ama.caltech.edu

problems with multiple time scales, and some solidification problems, an adaptive refinment in both space and time is required. This scheme allows the use of different time steps in different space-time subdomains. The method can also be used as a fast iterative solver for implicit schemes, on a serial machine.

The numerical implementation of this iteration is closely related to the waveform-relaxation method, for the solution of systems of ordinary differential equations (ODEs) [1]-[15]. Specifically, space-time DD is equivalent to a block waveform-relaxation method with overlapping splittings, applied to the semi-discretized parabolic equation.

Jeltsch and Pohl [16] extended the theory of waveform relaxation to the case of overlapping splittings. They showed that the rate of convergence is superlinear, in a finite time window. Our analysis of the continuous iteration shows that the asymptotic convergence rate is governed by the diffusion of the error, across the overlap between subdomains. This yields a much higher rate of convergence then the one implied by the waveform-relaxation theory, in view of the Gaussian nature of diffusion. Moreover, we find that the convergence rate depends on the size of the overlap between subdomains, the size of the time window and the diffusion coefficient in the equation. It is independent of the number of subdomains, provided the size of the overlap between subdomains remains fixed.

Gander and Stuart [17] analyzed domain decomposition as a form of waveform relaxation, for the one dimensional heat equation, on the infinite time interval. They found that convergence is linear and that the rate deteriorates as the number of subdomains increases. Our analysis for the heat equation shows that on large time intervals, convergence is initially linear with the the same convergence rate as for the infinite time interval. Moreover, the duration of the transient linear regime is proportional to the length of the time window.

In advection dominated problems, we find that convergence is initially linear with a convergence factor that is governed by the ratio of advection to diffusion. Specifically, it is an exponentially decaying function of magnitude of this ratio, and it is independent of the size of the time window. Hence, this iteration is particularly well suited for such problems.

The results obtained for the one dimensional model problem are also valid in two space dimensions, albeit in a different norm. In addition, in two (or more) space dimensions, we find that the iteration has the smoothing property: high modes of the error are damped much faster then low modes. This property is a result of the natural smoothing property of the heat equation, and it is independent of the length of the time window.

The analysis of this paper highlights the dependence of the convergence rate of the iteration on the parameters in the domain decomposition. This raises the question of which decomposition yields the minimum elapsed time for solving the problem to a given accuracy, on a specified parallel computer. We model this problem and discuss the means for it's solution.

In section 2 we present the algorithm and the model equations. Then in section 3 we present some preliminary error analysis. In section 4 we derive the asymptotic convergence rate of the iteration, for the one dimensional problem. We also analyze the transient linear rate, for advection dominated equations. In section 5 we analyze the transient linear rate

for the heat equation, in a large time window. In section 6 we extend the analysis to the two dimensional advection diffusion equation. We also derive the smoothing property of the iteration. In section 7 we discuss the optimal decomposition problem. Numerical calculations that demonstrate the accuracy of our results are presented in section 8. We end with some conclusions and future directions in section 9.

2 The model problem and the algorithm

We consider the one-dimensional advection-diffusion equation in $[x_l, x_h] \times [0, \infty)$

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} + b\frac{\partial u}{\partial x} - c^2 u + f(x,t), \quad t > 0, \tag{1}$$

$$u(x,0) = h(x), \tag{2}$$

$$u(x_l,t) = g_l(t), \quad u(x_h,t) = g_h(t), \tag{3}$$

and it's two dimensional counterpart in $[x_l, x_h] \times [y_l, y_h] \times [0, \infty)$

$$\frac{\partial u}{\partial t} = a^2 \Delta u + b[\sin(\theta), \cos(\theta)] \cdot \nabla u + f(x,y,t), \tag{4}$$

$$u(x,y,0) = h(x,y), \tag{5}$$

$$\begin{aligned} u(x_l,y,t) = g_{x_l}(y,t), \quad u(x_h,y,t) = g_{x_h}(y,t), \\ u(x,y_l,t) = g_{y_l}(y,t), \quad u(x,y_h,t) = g_{y_h}(y,t). \end{aligned} \tag{6}$$

In equations (1) and (4) the coefficients a, b and c are assumed constants.

We solve the problems (1)-(6) by a domain decomposition (DD) iteration in which $\Omega \times [0, \infty)$ is partitioned into space-time subdomains. Time is partitioned into windows of length T and we denote the resulting space-time windows by

$$W_n = \Omega \times [nT, (n+1)T], \quad n = 0, 1, \cdots. \tag{7}$$

Each space-time window is further decomposed into subdomains by partitioning space. In one dimension $\Omega = [x_l, x_h]$ is decomposed into sub-intervals of length S, and in two dimensions $\Omega = [x_l, x_h] \times [y_l, h_h]$ is partitioned into strips of width S. Other possible decompositions are described in section 6. Adjacent space-time subdomains overlap in space, and the width of the overlap is denoted by Δ. Figure 1 depicts the decomposition of $[x_l, x_h] \times [0, 2T]$ into two time windows which in turn are decomposed into three space-time subdomains.

The problem is solved successively in W_0, $W_1, \cdots$ to accuracy ϵ, by a subdomain iteration with red black ordering. For example, figure 2 indicates the decomposition of $W_0 = [x_0, x_2] \times [0, T]$ into two space-time subdomains. The red and black subdomains are $[x_0, x_1] \times [0, T]$ and $[x_1, x_2] \times [0, T]$, respectively. We denote the red and black iterates by

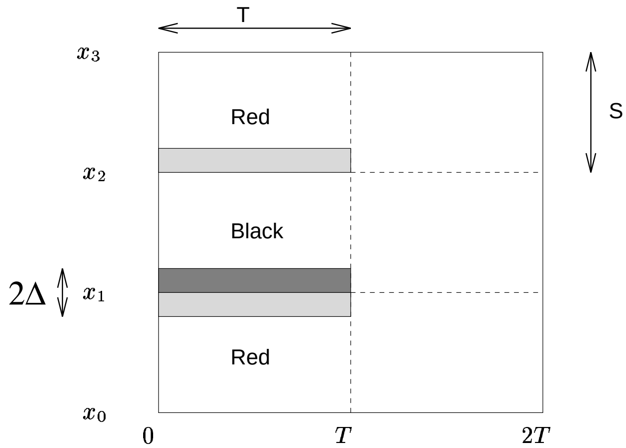

Figure 1: Decomposition of the domain $[x_0, x_3] \times [0, 2T]$ into two time windows and six space-time subdomains. The light grey areas are the overlap between the black subdomain and it's red neighbors. The overlap of the bottom red subdomain with it's black neighbor is shaded in dark grey.

$u_k^r(x,t)$ and $u_k^b(x,t)$ and we assume that $u_0^b(x,t)$ is given. At iteration $k \geq 1$, $u_k^r(x,t)$ is computed by solving equation (1) in $[x_0, x_1 + \Delta] \times [0, T]$, subject to the conditions

$$u_k^r(x,0) \;=\; h(x), \tag{8}$$

$$u_k^r(x_0,t) \;=\; g_l(t), \quad u_k^r(x_1+\Delta,t) \;=\; u_{k-1}^b(x_1+\Delta,t). \tag{9}$$

Then $u_k^b(x,t)$ is computed by solving equation (1) in $[x_1 - \Delta, x_2] \times [0, T]$, subject to

$$u_k^b(x,0) \;=\; h(x), \tag{10}$$

$$u_k^b(x_2,t) \;=\; g_h(t), \quad u_k^b(x_1-\Delta,t) \;=\; u_k^r(x_1-\Delta,t). \tag{11}$$

We denote the k'th iterate by $u_k(x,t)$,

$$u_k(x,t) \;=\; \begin{cases} u_k^r(x,t) & x \in [x_0, x_1] \\ u_k^b(x,t) & x \in [x_1, x_2], \end{cases} \tag{12}$$

for $k \geq 1$.

The initial value in W_n, $u(x, nT)$, is obtained from W_{n-1} for $n \geq 1$ and from the initial condition of the problem, for $n = 0$. Successive time windows do not overlap in time.

3 Error analysis for the 1-D model problem

3.1 The error equations

We now derive equations for the error in the DD iteration for problem (1)-(3). We first assume that $W_0 = [x_0, x_2] \times [0, T]$ is decomposed into two space-time subdomains, as

depicted in figure 2. The errors in the red and black iterates are defined by

$$e_k^b = u - u_k^b, \quad e_k^r = u - u_k^r, \quad e_k = u - u_k, \tag{13}$$

with u_k defined in equation (12).

In order to determine e_k^r we subtract equations (1), (8) and (9) for u_k^r from equations (1)-(3) for u to obtain

$$\frac{\partial e_k^r}{\partial t} = a^2 \frac{\partial^2 e_k^r}{\partial x^2} + b \frac{\partial e_k^r}{\partial x} - c^2 e_k^r, \tag{14}$$

$$e_k^r(x, 0) = 0, \tag{15}$$

$$e_k^r(x_0, t) = 0, \quad e_k^r(x_1 + \Delta, t) = e_{k-1}^b(x_1 + \Delta, t). \tag{16}$$

In a similar fashion we find that the error in $[x_1 - \Delta, x_2] \times [0, T]$ satisfies

$$\frac{\partial e_k^b}{\partial t} = a^2 \frac{\partial^2 e_k^b}{\partial x^2} + b \frac{\partial e_k^b}{\partial x} - c^2 e_k^b, \tag{17}$$

and the conditions

$$e_k^b(x, 0) = 0, \tag{18}$$

$$e_k^b(x_1 - \Delta, t) = e_k^r(x_1 - \Delta, t), \quad e_k^b(x_2, t) = 0. \tag{19}$$

The recursive system of equations (14)-(19) determines the evolution of the error in this iteration as a function of the iteration number k. When the number of subdomains $N \geq 3$ we derive a recursive system of N equations of the type (14) and (17), coupled through their boundary conditions, using the same method as for the two subdomain case.

In order to obtain an explicit expression for the error from the recursive system (14)-(19) we first derive the solution to the generic equation

$$\frac{\partial e}{\partial t} = a^2 \frac{\partial^2 e}{\partial x^2} + b \frac{\partial e}{\partial x} - c^2 e, \tag{20}$$

with $e(x, 0) = 0$ and Dirichlet conditions at $x = x_l$ and $x = x_h$, in Appendix A. We find

$$e(x,t) = e^{\alpha(x - x_l)} \mathcal{K}(t, x_h - x, a, b, c) * e(x_l, t) + e^{\alpha(x - x_h)} \mathcal{K}(t, x - x_l, a, b, c) * e(x_h, t), \tag{21}$$

where

$$\mathcal{K}(t, u, a, b, c) = \sum_{n=0}^{\infty} \left\{ F[t, \lambda_n(u), a, \gamma] - F[t, \mu_n(u), a, \gamma] \right\}, \tag{22}$$

$$\gamma^2 = \frac{b^2}{4a^2} + c^2, \quad \alpha = -\frac{b}{2a^2} \tag{23}$$

$$\lambda_n(u) = (2n+1)L - u \quad \text{and} \quad \mu_n(u) = (2n+1)L + u, \tag{24}$$

$$F(t, \eta, a) = \frac{\eta}{2\sqrt{\pi} a t^{3/2}} \exp\left(-\frac{\eta^2}{4a^2 t} - \gamma^2 t \right), \tag{25}$$

and

$$L = x_h - x_l. \tag{26}$$

3.2 Preliminary analysis

We begin by deriving simple bounds for expression (21) which we use in the convergence analysis of the iteration, in the next sections.

Lemma 1 *Let F be defined in equation (25). Suppose that $e(t) \geq 0$ is a non-decreasing function and that the scalars λ, μ satisfy $0 < \lambda < \mu$. Then for any γ*

$$\int_0^T \{F(\tau, \lambda, a, \gamma) - F(\tau, \mu, a, \gamma)\} e(T - \tau) d\tau \geq 0. \tag{27}$$

Proof: The ratio

$$\frac{F(t, \lambda, a, \gamma)}{F(t, \mu, a, \gamma)} = \frac{\lambda}{\mu} \exp\left(\frac{\mu^2 - \lambda^2}{4a^2 t}\right), \tag{28}$$

is a monotonically deceasing function of t that tends to infinity as $t \to 0$ and to λ/μ as $t \to \infty$. Hence, there exists a unique scalar t^* such that

$$F(t, \lambda, a, \gamma) \geq F(t, \mu, a, \gamma) \quad \text{for} \quad 0 < t \leq t^*, \tag{29}$$

and

$$F(t, \lambda, a, \gamma) \leq F(t, \mu, a, \gamma) \quad \text{for} \quad t^* \leq t. \tag{30}$$

We now rewrite the integral (27) as the sum

$$\begin{aligned} &\int_0^{\min(t^*,T)} [F(\tau, \lambda, a, \gamma) - F(\tau, \mu, a, \gamma)] e^{\gamma^2 \tau} e(T - \tau) e^{-\gamma^2 \tau} d\tau \; + \\ &\int_{\min(t^*,T)}^T [F(\tau, \lambda, a, \gamma) - F(\tau, \mu, a, \gamma)] \, e^{\gamma^2 \tau} e(T - \tau) e^{-\gamma^2 \tau} d\tau, \end{aligned} \tag{31}$$

and we note that

$$e(T - t^*) e^{-\gamma^2 t^*} = \min_{t \in [0, t^*]} e(T - t) e^{-\gamma^2 t} = \max_{t \in [t^*, T]} e(T - t) e^{-\gamma^2 t}, \tag{32}$$

in view of the monotonicity of $e(t)$. Combining equations (29)-(32) yields the following lower bound for the left size of (27)

$$\begin{aligned} &\int_0^T \{F(\tau, \lambda, a, \gamma) - F(\tau, \mu, a, \gamma)\} e(T - \tau) d\tau \geq \\ &e(T - t^*) e^{-\gamma^2 t^*} \left(\int_0^T \frac{\lambda}{2\sqrt{\pi} a t^{3/2}} \exp\left(-\frac{\lambda^2}{4a^2 t}\right) - \int_0^T \frac{\mu}{2\sqrt{\pi} a t^{3/2}} \exp\left(-\frac{\mu^2}{4a^2 t}\right) \right) dt. \end{aligned} \tag{33}$$

The change of variables $\tau = \frac{\lambda}{2a\sqrt{t}}$ and $\tau = \frac{\mu}{2a\sqrt{t}}$ in the first and second intergal of expression (33), respectively, yields the following bound and proves the Lemma

$$\int_0^T \{F(\tau, \lambda, a, \gamma) - F(\tau, \mu, a, \gamma)\} e(T - \tau) d\tau \geq \frac{2}{\sqrt{\pi}} e(T - t^*) e^{-\gamma^2 t^*} \int_{\frac{\lambda}{2a\sqrt{T}}}^{\frac{\mu}{2a\sqrt{T}}} e^{-\tau^2} d\tau. \tag{34}$$

$\square$

We now introduce the following truncated sums for the kernel (22)

$$\overline{\mathcal{K}_N}(t,u,a,b,c) \;=\; F[t,\lambda_0(u),a,\gamma] + \sum_{n=0}^{N-1} \left(F[t,\lambda_{n+1}(u),a,\gamma] - F[t,\mu_n(u),a,\gamma]\right), \qquad (35)$$

$$\underline{\mathcal{K}_N}(t,u,a,b,c) \;=\; \sum_{n=0}^{N} \left(F[t,\lambda_n(u),a,\gamma] - F[t,\mu_n(u),a,\gamma]\right), \qquad (36)$$

with γ defined in (23) and prove

Corollary 1 *Suppose that $e(t) \geq 0$ is a non-decreasing function and that $0 < u < L$*

$$\underline{\mathcal{K}_N}(t,u,a,b,c) * e(t) \;\leq\; \mathcal{K}(t,u,a,b,c) * e(t) \;\leq\; \overline{\mathcal{K}_N}(t,u,a,b,c) * e(t). \qquad (37)$$

Proof: We obtain both inequalities in (37) by determining the sign of the remainders $\mathcal{K}(t,u,a,b,c) * e(t) - \underline{\mathcal{K}_N}(t,u,a,b,c) * e(t)$ and $\mathcal{K}(t,u,a,b,c) * e(t) - \overline{\mathcal{K}_N}(t,u,a,b,c) * e(t)$, using Lemma 1 and the fact that

$$\lambda_n(u) < \mu_n(u) < \lambda_{n+1}(u). \qquad (38)$$

Here, λ_n and μ_n are defined in equation (24).

$\square$

The special case $N = 0$ of Corollary 1 is

Corollary 2 *Suppose that $e(t) \geq 0$ is a non-decreasing function and that $0 < u < L$*

$$\mathcal{K}(t,u,a,b,c) * e(t) \;\leq\; F[t,\lambda_0(u),a,\gamma] * e(t). \qquad (39)$$

We note that in this inequality, the expression on the left corresponds to the solution of equation (20) in a bounded domain, with a homogeneous condition on one of the boundaries. The formula on the right corresponds to the solution in the quarter plane, with the requirement that $|e| \to 0$ as $|x| \to \infty$. It is obtained by neglecting all corrections introduced by the images in the sum (22), for the derivative of the quarter plane green's function.

In view of the non-negativity of F and e in the right of inequality (39) we find that

$$F[t,\lambda_0(u),a,\gamma] * e(t) \;\leq\; \max_{\tau \leq t} e(\tau)\, \mathrm{I}(t,\eta,a,\gamma), \qquad (40)$$

where

$$\mathrm{I}(t,\eta,a,\gamma) \;=\; \int_0^t F(\tau,\eta,a,\gamma)d\tau. \tag{41}$$

In order to estimate I, we introduce the change of variables

$$\sigma \;=\; \frac{\eta}{2a\sqrt{\tau}} \tag{42}$$

in (41) and we find using [18]

$$\mathrm{I}(t,\eta,a,\gamma) \;=\; \frac{1}{2}\left\{e^{\frac{\gamma\eta}{a}}\mathrm{erfc}\left(\frac{\eta}{2a\sqrt{t}}+\gamma\sqrt{t}\right)+e^{-\frac{\gamma\eta}{a}}\mathrm{erfc}\left(\frac{\eta}{2a\sqrt{t}}-\gamma\sqrt{t}\right)\right\}. \tag{43}$$

Here $\mathrm{erfc}(z)$ is the complementary error function

$$\mathrm{erfc}(z) \;=\; \frac{2}{\sqrt{\pi}}\int_z^\infty e^{-t^2}dt, \tag{44}$$

which admits the expansion (see: [18])

$$\mathrm{erfc}(z) \;=\; \frac{e^{-z^2}}{\sqrt{\pi}z}(1+o(1)), \quad z\to\infty, \tag{45}$$

and the bound

$$\mathrm{erfc}(z) \;\le\; \frac{e^{-z^2}}{\sqrt{\pi}z}, \quad z>0. \tag{46}$$

We now derive bounds for I.

Lemma 2 *Let I be defined in equation (41) and suppose that $\eta, a > 0$.*

1. *If $\gamma = 0$ then for any $t > 0$*

$$I(t,\eta,a,\gamma) \;\le\; \sqrt{\frac{t}{\pi}}\frac{2a}{\eta}\exp\left(-\frac{\eta^2}{4a^2t}\right) \tag{47}$$

2. *If $\gamma > 0$ and $\eta \ge \frac{a}{\gamma\pi}$ then for any $t > 0$*

$$I(t,\eta,a,\gamma) \;\le\; \left(1+\frac{1}{2}\sqrt{\frac{a}{2\pi\eta\gamma}}\right)\exp\left(-\frac{\eta^2}{4a^2t}\right) \tag{48}$$

3. *If $\gamma > 0$ then for all $t \ge 0$*

$$I(t,\eta,a,\gamma) \;\le\; \left(1+\frac{1}{2}\sqrt{\frac{a}{2\pi\eta\gamma}}\right)\exp\left(-\frac{\eta\gamma}{a}\right) \tag{49}$$

Proof: The bound (46) yields

$$e^{\frac{\gamma\eta}{a}} \operatorname{erfc}\left(\frac{\eta}{2a\sqrt{t}} + \gamma\sqrt{t}\right) \;\le\; \frac{1}{\sqrt{\pi}\left(\frac{\eta}{2a\sqrt{t}} + \gamma\sqrt{t}\right)} \exp\left(-\frac{\eta^2}{4a^2 t} - \gamma^2 t\right). \tag{50}$$

The exponential and the pre-exponential factors in this expression are maximized at

$$t \;=\; \frac{\eta}{2a\gamma}, \tag{51}$$

and this yields the bound

$$e^{\frac{\gamma\eta}{a}} \operatorname{erfc}\left(\frac{\eta}{2a\sqrt{t}} + \gamma\sqrt{t}\right) \;\le\; \sqrt{\frac{a}{2\pi\eta\gamma}} \exp\left(-\frac{\eta\gamma}{a}\right). \tag{52}$$

We use this inequality and the fact that $\operatorname{erfc}(z) \le 2$ in expression (43) to obtain (49).

The substitution (51) for t in the pre-exponential factor of expression (50) yields

$$e^{\frac{\gamma\eta}{a}} \operatorname{erfc}\left(\frac{\eta}{2a\sqrt{t}} + \gamma\sqrt{t}\right) \;\le\; \sqrt{\frac{a}{2\pi\eta\gamma}} \exp\left(-\frac{\eta^2}{4a^2 t}\right). \tag{53}$$

The bound (46) implies that for $t \le \frac{\eta}{4a\gamma}$

$$e^{-\frac{\gamma\eta}{a}} \operatorname{erfc}\left(\frac{\eta}{2a\sqrt{t}} - \gamma\sqrt{t}\right) \;\le\; 2\sqrt{\frac{a}{\gamma\eta\pi}} \exp\left(-\frac{\eta^2}{4a^2 t}\right), \tag{54}$$

while inequality $\operatorname{erfc}(z) \le 2$ implies that for $t \ge \frac{\eta}{4a\gamma}$

$$e^{-\frac{\gamma\eta}{a}} \operatorname{erfc}\left(\frac{\eta}{2a\sqrt{t}} - \gamma\sqrt{t}\right) \;\le\; 2\exp\left(-\frac{\eta^2}{4a^2 t}\right). \tag{55}$$

We now use inequalities (53)-(55) in equation (43) to obtain (48).

Inequality (47) follows from identity (43) and the bound (46) for $\operatorname{erfc}(z)$.

□

Lemma 3 *Suppose that $e(x,t)$ is the solution to equation (20) in $[x_l, x_h] \times [0,T]$ with $e(x,0) = 0$ and Dirichlet conditions at $x = x_l$ and $x = x_h$. Suppose that $\bar{e}(x,t)$ is also a solution to equation (20) in $[x_l, x_h] \times [0,T]$ with $\bar{e}(x,0) = 0$ and that*

$$\bar{e}(x_l, t) \;\ge\; |e(x_l, t)|, \quad \bar{e}(x_h, t) \ge |e(x_h, t)|. \tag{56}$$

Then for all $(x,t) \in [x_l, x_h] \times [0,T]$

$$\bar{e}(x,t) \;\ge\; |e(x,t)|. \tag{57}$$

Proof: Both $u = \bar{e} - e$ and $w = \bar{e} + e$ satisfy equation (20) with non-negative boundary conditions. Hence u and w are non-negative, as shown in Appendix B for equation (20). This proves inequality (57).

$\square$

Lemma 4 *Suppose $e(t) \geq 0$ is a non-decreasing function and that $h(t) \geq 0$. The convolution*

$$H(t) \; = \; h(t) * e(t), \tag{58}$$

is a non-negative and non-decreasing function of t.

Proof: Suppose $t_2 \geq t_1 \geq 0$

$$H(t_2) - H(t_1) \; = \; \int_0^{t_1} h(\tau)\left[e(t_2-\tau) - e(t_1-\tau)\right] d\tau + \int_{t_1}^{t_2} h(\tau) e(t_2-\tau) d\tau. \tag{59}$$

In view of the monotonicity and non-negativity of e both integrands in equation (59) are non-negative. Hence, $H(t_2) \geq H(t_1)$ and this proves the lemma.

$\square$

Lemma 5 *Suppose $\theta > 0$ and let*

$$a_k(j) \; = \; \binom{2k-2}{j} \exp\left(-\frac{[(4(k-1)\Delta + j\theta]^2}{4a^2 t}\right). \tag{60}$$

Then

$$\sum_{j=0}^{2k-2} a_k(j) \; = \; \exp\left(-\frac{[(2(k-1)\Delta]^2}{a^2 t}\right)(1+o(1)), \quad k \to \infty. \tag{61}$$

Proof: We derive the asymptotic estimate (61) by essentially using Laplace's method for sums [19]. First we note that for all $j = 0, \ldots, 2k-3$

$$\frac{a_k(j+1)}{a_k(j)} \; \leq \; 2(k-1)\exp\left(-\frac{\theta 2(k-1)\Delta}{a^2 t}\right). \tag{62}$$

It follows that for sufficiently large k, say $k > k_0$

$$\frac{a_k(j+1)}{a_k(j)} \; < \; 1, \quad j = 0, \ldots, 2k-3, \tag{63}$$

and $a_k(j)$ is a monotonically decreasing function of j. Hence for all $k > k_0$

$$\sum_{j=0}^{2k-2} a_k(j) \; \leq \; \exp\left(-\frac{[(2(k-1)\Delta]^2}{a^2 t}\right)\left(1 + 2(k-1)\frac{a_k(1)}{a_k(0)}\right), \tag{64}$$

and this proves the Lemma, in view of inequality (62).

$\square$

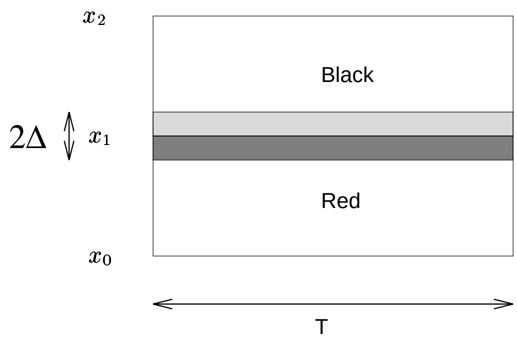

Figure 2: The decomposition of $W_0 = [x_0, x_2] \times [0, T]$ into two subdomains.

4 Convergence analysis for the 1-D model problem

We now consider the DD iteration of section 2 for the one dimensional model problem in $W_0 = [x_0, x_N] \times [0, T]$. We first derive a general bound, which we then use to obtain the asymptotic convergence rate of the iteration, and the transient linear regime for advection dominated equations.

4.1 General analysis

We now introduce the notation

$$|e(x,\tau)|_t = \max_{0\le\tau\le t} |e(x,\tau)|, \tag{65}$$

$$\| e(x,\tau) \|_t = \max_{x\in\Omega,\ 0\le\tau\le t} |e(x,\tau)|, \tag{66}$$

and we prove

Proposition 1 *Consider the space-time DD iteration for the one dimensional problem (1)-(3), and suppose $S > 2\Delta$. Then for any $0 \le t \le T$*

$$\frac{\| e_k \|_t}{\| e_0 \|_t} \le e^{|\alpha|(x_N - x_0)} \sum_{j=0}^{2k-2} \binom{2k-2}{j} I[t, 4(k-1)\Delta + j\theta, a, \gamma], \tag{67}$$

$$\theta = S - 2\Delta, \tag{68}$$

where α and γ are defined in equation (23).

Proof: We derive bounds for $|e_k^r(x,t)|$ and $|e_k^b(x,t)|$ which we denote by $\bar{e}_k^r(x,t)$ and $\bar{e}_k^b(x,t)$, respectively. We use an inductive construction which ensures that the bounds are non-decreasing functions of t. Initially we define

$$\bar{e}_0^b(x,t) = Ae^{\alpha x} \| e_0 \|_t, \tag{69}$$

$$A \;=\; \left\{ \begin{array}{ll} e^{-\alpha x_N} & \alpha < 0 \\ e^{-\alpha x_0} & \alpha \geq 0 \end{array} \right. , \tag{70}$$

and we note that $\bar{e}_0^b$ is a non-decreasing function of t. The derivation for $k \geq 1$ is presented in an incremental fashion. We first assume that $W_0 = [x_0, x_2] \times [0, T]$ is decomposed into the red and black subdomains $[x_0, x_1] \times [0, T]$ and $[x_1, x_2] \times [0, T]$, as indicated in figure 2. Then we present the general case.

Case I: Two subdomains

The error in the red subdomain at iteration $k \geq 1$, e_k^r, is determined by e_{k-1}^b, through equations (14)-(16). In order to define $\bar{e}_k^r$ we first substitute $\bar{e}_{k-1}^b$ for e_{k-1}^b in boundary condition (16), and solve the resulting problem using formula (21) to obtain

$$|e_k^r(x,t)| \;\leq\; e^{\alpha(x-x_1-\Delta)} \mathcal{K}(t, x-x_0, a, b, c) * \bar{e}_{k-1}^b(x_1+\Delta, t). \tag{71}$$

This inequality follows from Lemma 3 and the inequality $|e_{k-1}^b| \leq \bar{e}_{k-1}^b$. We now define $\bar{e}_k^r$ as the bound for $\mathcal{K} * \bar{e}_{k-1}^b$ obtained from Corollary 2

$$\bar{e}_k^r(x,t) \;=\; e^{\alpha(x-x_1-\Delta)} F[t, \lambda_0(x-x_0), a, \gamma] * \bar{e}_{k-1}^b(x_1+\Delta, t). \tag{72}$$

The monotonicity of $\bar{e}_k^r$ follows from the monotonicity of $\bar{e}_{k-1}^b$ and Lemma 4. Similar arguments yield

$$|e_k^b(x,t)| \;\leq\; e^{\alpha(x-x_1+\Delta)} \mathcal{K}(t, x_2-x, a, b, c) * \bar{e}_k^r(x_1-\Delta, t), \tag{73}$$

$$\bar{e}_k^b(x,t) \;=\; e^{\alpha(x-x_1+\Delta)} F[t, \lambda_0(x_2-x), a, \gamma] * \bar{e}_k^r(x_1-\Delta, t). \tag{74}$$

In equations (72) and (74) $\lambda_0(u) = S + \Delta - u$, in view of definition (24) for λ_0.

The maximum principle for equation (14) (see: Appendix B) and Lemma 3 imply

$$|e_k^r(x,t)| \;\leq\; \max_{0 \leq \tau \leq t} |e_{k-1}^b(x_1+\Delta, t)|, \tag{75}$$

$$|e_k^b(x,t)| \;\leq\; \max_{0 \leq \tau \leq t} |e_k^r(x_1-\Delta, t)|. \tag{76}$$

It follows from these inequalities and the monotonicity of $\bar{e}_{k-1}^b$ that

$$\| \, e_k \, \|_t \leq \; \bar{e}_{k-1}^b(x_1+\Delta, t). \tag{77}$$

We evaluate this bound by combining equations (72) and (74) repeatedly, using the identity

$$F(t, \eta_1, a, \gamma) * F(t, \eta_2, a, \gamma) \;=\; F(t, \eta_1+\eta_2, a, \gamma), \tag{78}$$

which follows from identity (187) in appendix A and the convolution Theorem for the Laplace transform, to obtain

$$\bar{e}_k^b(x_1+\Delta, t) \;=\; A e^{\alpha(x_1+\Delta)} F\left[t, 4(k-1)\Delta, a, \gamma\right] * \| \, e_0 \, \|_t \, . \tag{79}$$

Equations (77), (79) and the monotonicity of $\| \, e_0 \, \|_t$ yield

$$\| \, e_k \, \|_t \leq \; A e^{\alpha(x_1+\Delta)} \, \| \, e_0 \, \|_t \int_0^t F\left[\tau, 4(k-1)\Delta, a, \gamma\right] d\tau, \tag{80}$$

and it follows from definitions (41) for I and (70) for A that

$$\frac{\| e_k \|_t}{\| e_0 \|_t} \leq e^{|\alpha|(x_2-x_0)} \mathrm{I}(t, 4(k-1)\Delta, a, \gamma). \tag{81}$$

Case II: N subdomains

In general, $W_0 = [x_0, x_N] \times [0, T]$ is partitioned into N subdomains, $[x_j, x_{j+1}] \times [0, T]$, $j = 0, \ldots, N-1$, which we color red when $j = 2i$ and black otherwise. We assume, without loss of generality, that N is odd.

In all black subdomains $[x_{2i+1}, x_{2i+2}]$, $i = 0, \ldots, (N-1)/2 - 1$

$$\begin{aligned} \bar{e}_k^b(x,t) &= e^{\alpha(x-x_{2i+1}+\Delta)} F[t, \lambda_0(x_{2i+2}+\Delta-x), a, \gamma] * \bar{e}_k^r(x_{2i+1}-\Delta, t) \; + \\ & \quad e^{\alpha(x-x_{2i+2}-\Delta)} F[t, \lambda_0(x-x_{2i+1}+\Delta), a, \gamma] * \bar{e}_k^r(x_{2i+2}+\Delta, t). \end{aligned} \tag{82}$$

In interior red subdomains $[x_{2i}, x_{2i+1}]$, $i = 1, \ldots, (N-1)/2 - 1$

$$\begin{aligned} \bar{e}_k^r(x,t) &= e^{\alpha(x-x_{2i}+\Delta)} F[t, \lambda_0(x_{2i+1}+\Delta-x), a, \gamma] * \bar{e}_{k-1}^b(x_{2i}-\Delta, t) \; + \\ & \quad e^{\alpha(x-x_{2i+1}-\Delta)} F[t, \lambda_0(x-x_{2i}+\Delta), a, \gamma] * \bar{e}_{k-1}^b(x_{2i+1}+\Delta, t). \end{aligned} \tag{83}$$

In equations (83) and (82) $\lambda_0(u) = S + 2\Delta - u$, in view of definition (24) for λ_0.

In the red subdomains $[x_0, x_1] \times [0, T]$ and $[x_{N-1}, x_N] \times [0, T]$

$$\begin{aligned} |e_k^r(x,t)| &\leq e^{\alpha(x-x_1-\Delta)} F[t, \lambda_0(x-x_0), a, \gamma] * \bar{e}_{k-1}^b(x_1+\Delta, t), \\ |e_k^r(x,t)| &\leq e^{\alpha(x-x_{N-1}+\Delta)} F[t, \lambda_0(x_N-x), a, \gamma] * \bar{e}_{k-1}^b(x_{N-1}-\Delta, t), \end{aligned} \tag{84}$$

respectively, and in order to simplify the analysis we define $\bar{e}_k^r(x,t)$ as

$$e^{\alpha(x-x_1-\Delta)} \{F[t, \lambda_0(x-x_0), a, \gamma] + F[t, \lambda_0(x_1-x), a, \gamma]\} * \bar{e}_{k-1}^b(x_1+\Delta, t), \tag{85}$$

in $[x_0, x_1] \times [0, T]$ and as

$$e^{\alpha(x-x_{N-1}+\Delta)} \{F[t, \lambda_0(x_N-x), a, \gamma] + F[t, \lambda_0(x-x_{N-1}), a, \gamma]\} * \bar{e}_{k-1}^b(x_{N-1}-\Delta, t). \tag{86}$$

in $[x_{N-1}, x_N] \times [0, T]$. In these equations $\lambda_0(u) = S + \Delta - u$.

We denote by X_Δ^b the x coordinate of all points in the interior of black subdomains that are at distance Δ from the boundary

$$X_\Delta^b = \{x_{2i+1} + \Delta, \quad x_{2i+2} - \Delta \quad | \quad i = 0, \ldots, N/2 - 1\}. \tag{87}$$

The maximum principle for equation (14) and Lemma 3 yield

$$\| e_k \|_t \leq \max_{x_\Delta \in X_\Delta^b} \bar{e}_{k-1}^b(x_\Delta, t). \tag{88}$$

We evaluate this bound by induction using relations (82), (83), (85) and (86) to obtain

$$\bar{e}_k^r(x_\Delta, \tau) = A e^{\alpha x_\Delta} \left[F(t, 2\Delta, a, \gamma) + F(t, S, a, \gamma)\right]^{2(k-1)*} * \| e_0 \|_t, \tag{89}$$

for all $x_\Delta \in X_\Delta^b$. Here F^{k*} is defined by

$$F^{(k+1)*} = F * F^{k*}, \quad F^{1*} = F. \tag{90}$$

Identity (78), the monotonicity of $\| e_0 \|_t$ and definition (70) yield the bound (67).

□

4.2 Asymptotic convergence rate

We now show that the rate of convergence of the space-time DD iteration is super-linear.

Theorem 1 *Consider the space-time DD iteration for the one dimensional problem (1)-(3), and suppose $S > 2\Delta$. Then for any $0 \leq t \leq T$*

$$\frac{\| e_k \|_t}{\| e_0 \|_t} \leq e^{\frac{|b|(x_N - x_0)}{2a^2}} \exp\left(-\frac{[2(k-1)\Delta]^2}{a^2 t}\right)(1 + o(1)), \quad k \to \infty, \tag{91}$$

Proof: We use the bounds (47) and (48) in inequality (67) of Proposition 1 to obtain

$$\frac{\| e_k \|_t}{\| e_0 \|_t} \leq B(k) e^{\frac{|b|(x_N - x_0)}{2a^2}} \sum_{j=0}^{2k-2} \binom{2k-2}{j} \exp\left(-\frac{[4(k-1)\Delta + j\theta]^2}{4a^2 t}\right). \tag{92}$$

This inequality is valid for all k when $\gamma = 0$ with

$$B(k) = \frac{a\sqrt{t}}{\sqrt{\pi} 2(k-1)\Delta}, \tag{93}$$

and it is valid for $k \geq a/(4\Delta\gamma\pi) + 1$ when $\gamma > 0$ with

$$B(k) = 1 + \frac{1}{4}\sqrt{\frac{a}{2\pi(k-1)\Delta\gamma}}. \tag{94}$$

Inequality (91) follows from the bound (92), in view of Lemma 5.

□

The analysis of the semi-discrete iteration by the theory of waveform relaxation and multisplitting [16], yields the following bound for the decay of the error in the iteration

$$\frac{\| e_k \|_t}{\| e_0 \|_t} \leq O\left(\frac{(CT)^k}{k!}\right) = O\left(\exp\left[-k \log k + k(\log CT + 1) + \frac{1}{2}k\right]\right), \quad k \to \infty. \tag{95}$$

Here, the constant C deteriorates as the mesh parameter decreases to 0, in view of the stiff nature of the system of ODE's that results from the discretization of a parabolic PDE by the method of lines. Moreover, the dependence of the rate of convergence on Δ and the diffusion coefficient a^2 is not apparent in (95). The quadratic exponent in expression (91), which results from the Gaussian nature of diffusion, predicts a substantially faster rate of convergence. We also note that the bound (91) for the continuous iteration is independent of a mesh parameter.

In conclusion, the algorithm is particularly well suited to problems with small diffusion coefficients. Indeed, when the ratio

$$\frac{\Delta^2}{a^2 T}$$

is large the convergence rate is high. Hence, a small diffusion coefficient allows the use of a small overlap and a large time window, while maintaining a high rate of convergence. Moreover, it allows the use of a large number of subdomains, without adversely affecting the convergence rate, in view of the inequality $S > 2\Delta$.

4.3 Advection dominated equations

We now analyze the iteration when the ratio of advection to diffusion, $|b|/a^2$, is large. We first prove

Theorem 2 *Consider the space-time DD iteration for the one dimensional problem (1)-(3), and suppose $\gamma > 0$, with γ defined in equation (23). For all $t \geq 0$ and $k \geq a/(4\Delta\gamma\pi)+1$*

$$\frac{\| e_k \|_t}{\| e_0 \|_t} \leq B(k) e^{\frac{|b|(x_N - x_0)}{2a^2}} \left[\exp\left(-\frac{2\Delta\gamma}{a}\right) + \exp\left(-\frac{S\gamma}{a}\right) \right]^{2k-2}. \tag{96}$$

Here, $B(k)$ is defined in equation (94).

Proof: Proposition 1 and inequality (49) of Lemma 2 yield the bound

$$\frac{\| e_k \|_t}{\| e_0 \|_t} \leq B(k) e^{\frac{|b|(x_N - x_0)}{2a^2}} \sum_{j=0}^{2k-2} \binom{2k-2}{j} \exp\left(-[4(k-1)\Delta + j\theta]\gamma/a\right), \tag{97}$$

which is valid for $k \geq a/(4\Delta\gamma\pi) + 1$. This bound is equivalent to inequality (96), in view of the definition of θ in equation (68).

$\square$

The average convergence factor at iteration k, $\bar{\rho}_k$, is defined by

$$\bar{\rho}_k = \left(\frac{\| e_k \|_t}{\| e_0 \|_t} \right)^{1/k}, \tag{98}$$

and the asymptotic convergence factor, $\bar{\rho}$, is then

$$\bar{\rho} = \overline{\lim}_{k\to\infty} \bar{\rho}_k. \tag{99}$$

Theorem 2 yields a bound for the asymptotic convergence factor, in the linear regime

$$\bar{\rho} \leq \left[\exp\left(-\frac{2\Delta\gamma}{a}\right) + \exp\left(-\frac{S\gamma}{a}\right) \right]^2. \tag{100}$$

The improvement of this factor as the ratio $\frac{|b|}{a^2}$ increases, follows from the inequality

$$\frac{\gamma}{a} \geq \frac{|b|}{2a^2}, \tag{101}$$

with γ defined in equation (23). We also note that this factor is independent of T.

4.4 Sharper estimates for the heat equation

We now derive sharper estimates for the decay of the error in the iteration for the heat equation. First, we note that the magnitude of the error is maximized on the boundary of a red subdomain i.e. we replace inequality (88) by the sharper inequality

$$\| e_k \|_t \leq \bar{e}_k^r(x_j, t), \quad j = 1, \ldots, N-1. \tag{102}$$

This yields the bounds

$$\begin{aligned}
\frac{\| e_k \|_t}{\| e_0 \|_t} &\leq \operatorname{erfc}\left(\frac{(4k-3)\Delta}{2a\sqrt{t}}\right), \quad N = 2, && (103)\\
&\leq \sum_{j=0}^{k-1} \binom{k-1}{j} \operatorname{erfc}\left(\frac{(4k-3)\Delta + j\theta}{2a\sqrt{t}}\right), \quad N = 3, && (104)\\
&\leq \sum_{j=0}^{2k-2} \binom{2k-2}{j} \operatorname{erfc}\left(\frac{(4k-3)\Delta + j\theta}{2a\sqrt{t}}\right) + && (105)\\
&\quad \sum_{j=0}^{2k-2} \binom{2k-2}{j} \operatorname{erfc}\left(\frac{(4k-3)\Delta + S + j\theta}{2a\sqrt{t}}\right) \quad N \geq 4, && (106)
\end{aligned}$$

for two three and four subdomains, respectively. The arguments of Lemma 5 can be refined to show that each one of the sums above has the same order of magnitude as it's maximal term. In section 8, we compare the maximal term of the sum (104) with the error in numerical calculations with three subdomains.

5 The heat equation in a large time window

We now study the convergence of the DD iteration for the heat equation, in large time windows. The error initially decays at a linear rate before it transits to the superlinear regime.

5.1 The large time limit

We begin by proving a few preliminary Lemmas.

Lemma 6 *The function $\mathcal{K}(t, u, a, b, c)$ in equation (22) is continuous in t and non-negative for $t \geq 0$ and $0 \leq u \leq L$.*

Proof: Continuity follows from the continuity of F in (25) and the uniform convergence of the sum (22), which we show by noting that

$$\max_{0 \leq t} F(t, \lambda_n(u), a, \gamma) \leq \left(\frac{6}{e}\right)^{3/2} \frac{a^2}{8\sqrt{\pi} L^2} \frac{1}{n^2}. \tag{107}$$

In order to show that $\mathcal{K}(t,u,a,b,c) \geq 0$ we consider expression (21), which is the solution to equation (20) with Dirichlet data. This solution is non-negative when the data is non-negative, as shown in Appendix B. Hence $\mathcal{K} \geq 0$, in view of it's continuity.

$\square$

We now introduce the function

$$\phi(u,L,a,T,t) = \sum_{n=0}^{\infty} \chi_{[\frac{nL+\beta}{a\sqrt{T}},\frac{(n+1)L-\beta}{a\sqrt{T}}]}(t), \tag{108}$$

where

$$\beta = \frac{L-u}{2}, \tag{109}$$

and χ is the indicator function of an interval. In the following Lemmas we essentially show that for a continuous integrable function f

$$\lim_{T\to\infty} \int_0^\infty f\phi dt \ = \ \frac{u}{L}\int_0^\infty f dt.$$

Lemma 7 *Let $\phi(u,L,a,T,t)$ be defined in equation (108) and suppose that $0 \leq t_{j-1} < t_j \leq t$ and that $0 < u < L$ then*

$$\lim_{T\to\infty} \int_0^t \chi_{[t_{j-1},t_j]}\phi(u,L,a,T,\tau)d\tau = \frac{u}{L}\int_0^t \chi_{[t_{j-1},t_j]}. \tag{110}$$

Proof: For a given T, we define the integers n_1 and n_2 such that

$$\frac{(n_1-1)L}{a\sqrt{T}} \leq t_{j-1} \leq \frac{n_1 L}{a\sqrt{T}}, \qquad \frac{n_2 L}{a\sqrt{T}} \leq t_j \leq \frac{(n_2+1)L}{a\sqrt{T}}. \tag{111}$$

Hence,

$$\left|\int_0^t \chi_{[t_{j-1},t_j]}\phi(u,L,a,T,\tau) - \frac{u}{L}\int_0^t \chi_{[t_{j-1},t_j]}\right| \ \leq$$
$$\frac{2L}{a\sqrt{T}} \ + \ \left|\int_{\frac{n_1 L}{a\sqrt{T}}}^{\frac{n_2 L}{a\sqrt{T}}} \phi(u,L,a,T,\tau)d\tau - \frac{u}{L}(t_j - t_{j-1})\right|. \tag{112}$$

Now, we note that

$$\int_{\frac{n_1 L}{a\sqrt{T}}}^{\frac{n_2 L}{a\sqrt{T}}} \phi(u,L,a,T,\tau)d\tau \ = \ \frac{u}{L}\frac{(n_2-n_1)L}{a\sqrt{T}}, \tag{113}$$

and we use this expression and (111) in inequality (112) to obtain

$$\left|\int_0^t \chi_{[t_{j-1},t_j]}\phi(u,L,a,T,\tau) - \frac{u}{L}\int_0^t \chi_{[t_{j-1},t_j]}\right| \ \leq \ \frac{2}{a\sqrt{T}}(L+u). \tag{114}$$

Taking the limit $T \to \infty$ in (114) proves the Lemma.

$\square$

Lemma 8 *Let $\mathcal{K}(t,u,a,b,c)$ be defined in equation (22) and suppose that $0 < u < L$. Let*

$$g(T) = \int_0^T \mathcal{K}(\tau, u, a, 0, 0) d\tau, \tag{115}$$

then $g(T)$ is non-decreasing and

$$\lim_{T \to \infty} g(T) = \frac{u}{L}. \tag{116}$$

Proof: The monotonicity of g follows from the non-negativity of the kernel $\mathcal{K}(t,u,a,0,0)$ which was noted in Lemma 6. In order to prove the limit (116), we use the definition of $\mathcal{K}$ in (22) and evaluate the integral (115) using equations (43) and (44) to obtain

$$g(T) = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-t^2} \phi(u, L, a, T, t) dt, \tag{117}$$

where ϕ is defined in equation (108). Now, we use the representation (117) of $g(T)$ and the fact that $\operatorname{erfc}(0) = 1$ to note that for any $\epsilon > 0$ there exists a t_ϵ such that

$$\left| \frac{u}{L} - g(T) \right| \ \le \ \frac{\epsilon}{3} \ + \ \frac{2}{\sqrt{\pi}} \left| \frac{u}{L} \int_0^{t_\epsilon} e^{-\tau^2} d\tau - \int_0^{t_\epsilon} e^{-\tau^2} \phi(u, L, a, T, \tau) d\tau \right|. \tag{118}$$

Then, we define

$$dt = t_\epsilon / n, \quad t_j = j dt, \quad j = 0, 1, \ldots, n, \tag{119}$$

$$f_n(\tau) = \sum_{j=1}^n \chi_{[t_{j-1}, t_j]} e^{-t_{j-1}^2}, \tag{120}$$

which we use in inequality (118) and show that for sufficiently large n

$$\left| \frac{u}{L} - g(T) \right| \ \le \ \frac{2}{3}\epsilon \ + \ \frac{2}{\sqrt{\pi}} \left| \frac{u}{L} \int_0^{t_\epsilon} f_n(\tau) d\tau - \int_0^{t_\epsilon} \phi(u, L, a, T, \tau) f_n(\tau) d\tau \right|, \tag{121}$$

in view of the integrability of f_n and ϕf_n. The right of inequality (121) is bounded by ϵ, for sufficiently large T, in view of Lemma 7. This proves the Lemma, since ϵ was arbitrary.

$\square$

Corollary 3 *Let $\mathcal{K}$ be defined in equation (22) and $0 \le u \le L$ then*

1. *For any $t > 0$*

$$|e(t) * \mathcal{K}(t, u, a, 0, 0)| \le \frac{u}{L} |e|_t \tag{122}$$

2. *If in addition $e \geq 0$ and $e(t) \geq e(T_e)$ for all $t \geq T_e$. Then for any $\eta > 0$ there exists a number T_η such that for all $t \geq T_\eta + T_e$*

$$e(t) * \mathcal{K}(t, u, a, 0, 0) \geq \frac{u}{L}(1-\eta)e(T_e) \tag{123}$$

Proof: Inequality (122) follows from the mean value theorem for integrals and Lemma 8.

For a given η there exists T_η such that

$$\int_0^{T_\eta} \mathcal{K}(\tau, u, a, 0, 0)d\tau \ \geq \ \frac{u}{L}(1-\eta), \tag{124}$$

in view of Lemma 8. It follows that for $t \geq T_\eta + T_e$

$$e(t) * \mathcal{K}(t, u, a, 0, 0) \ \geq \ \int_0^{T_\eta} e(t-\tau)\mathcal{K}(\tau, u, a, 0, 0)d\tau, \tag{125}$$

and that $e(t-\tau) \geq e(T_e)$ for all τ in the range of integration. Combining this fact with inequalities (125) and (124) yields the bound (123).

□

Corollary 4 *Suppose $e(x,t)$ satisfies equation (20), with $b = c = 0$ in $[x_l, x_h] \times [0, T]$. Let $g_l(t) = e(x_l, t)$, $g_h(t) = e(x_h, t)$ and suppose that $e(x, 0) = 0$. Then*

1. *For any $t > 0$*

$$|e(x,\tau)|_t \ \leq \ \frac{x - x_l}{x_h - x_l}|g_h|_t + \frac{x_h - x}{x_h - x_l}|g_l|_t \tag{126}$$

2. *If in addition $g_l(t) \geq g_l(T_e) \geq 0$ and $g_h(t) \geq g_h(T_e) \geq 0$, for all $t \geq T_e$. Then for any $\eta > 0$ there exists a number T_η such that for all $t \geq T_\eta + T_e$*

$$e(x,t) \ \geq \ (1-\eta)\left[\frac{x - x_l}{x_h - x_l}g_h(T_e) + \frac{x_h - x}{x_h - x_l}g_l(T_e)\right]. \tag{127}$$

Proof: The proof follows from formula (21) and Corollary 3.

□

5.2 Convergence analysis

If T is sufficiently large, the error initially decays at a linear rate. Moreover, the number of iterations in the linear regime is a linear function of T. We now derive this result under the

assumption that the initial error $e_0(x,t) \geq 0$ and that there exists a small positive number ν and a time T_e such that

$$e_0(x,t) \;\geq\; (1-\nu) \parallel e_0 \parallel_t, \tag{128}$$

for all $x \in [x_0 + S/2, x_{N+1} - S/2]$ and $t \geq T_e$.

We first study the case of two and three subdomains. We derive lower and upper bounds for the error that decay at a linear rate. The lower bound is valid for a finite number of iterations, and both bounds become arbitrarily close as $T \to \infty$. The upper bound agrees with the result of Gander and Stewart [17] for the infinite time window.

Theorem 3 *Consider the space-time DD iteration for the one dimensional heat equation (20) in $[x_0, x_{N+1}] \times [0,T]$ with $N+1 = 2, 3$ subdomains. Suppose the initial error satisfies assumption (128). Then for any $\eta > 0$ there exists a time T_η such that if $k \leq (T-T_e)/(2T_\eta)$*

$$(1-\eta)^k \rho^{k-1}(1-\nu)C \;\leq\; \frac{\parallel e_k \parallel_t}{\parallel e_0 \parallel_t} \;\leq\; \rho^{k-1}C \tag{129}$$

where

$$C = \frac{1}{1+\Delta/S}, \qquad \rho = \left(\frac{1-\Delta/S}{1+\Delta/S}\right)^{3-N}. \tag{130}$$

Proof: We derive the upper bound for the error using an inductive construction analogous to that of section 4. In this construction we use inequality (122) instead of the right inequality in (37). This yields the right inequality in (129).

In order to derive a lower bound we define

$$\hat{\eta} = 1 - \sqrt{1-\eta}, \tag{131}$$

and T_η such that inequality (127) holds with $\hat{\eta}$ substituted for η. We first consider the case of two subdomains. We show by induction that for $t \geq T_e + (2k-1)T_\eta$

$$e_k^r(x,t) \;\geq\; \frac{x-x_0}{S+\Delta}(1-\hat{\eta})^{2k-1}\left(\frac{1-\Delta/S}{1+\Delta/S}\right)^{2(k-1)} (1-\nu) \parallel e_0 \parallel_t, \tag{132}$$

using Corollary 4 and assumption (128). Combining this result with definition (131) and the inequality

$$e_k^r(x_1,t) \;\leq \parallel e_k \parallel_t \tag{133}$$

yields the left inequality in (129) for $N+1 = 2$. The analysis for three subdomains is analogous.

□

We now analyze the case of an arbitrary number of subdomains. We derive a linear lower bound which is valid for a finite number of iterations. The convergence factor rapidly deteriorates as the number of subdomains increases.

Theorem 4 *Consider the space-time DD iteration for the one dimensional heat equation (20) with $N+1 \geq 4$ subdomains in $[x_0, x_{N+1}] \times [0,T]$. Suppose that the initial error satisfies assumption (128). Then for any $\eta > 0$ there exists a time T_η such that if $k \leq (T-T_e)/(2T_\eta)$*

$$\frac{\| e_k \|_t}{\| e_0 \|_t} \geq \left(1 - \frac{1+2\Delta/S}{N}\right)^k (1-\eta)^k (1-\nu), \tag{134}$$

for N even. For N odd inequality (134) holds with $N-1$ substituted for N.

Proof: We define the hat function

$$\phi_0(x,N) = \begin{cases} \frac{2x-S}{NS}(1-\nu) \| e_0 \|_t & S/2 \leq x \leq (N+1)S/2 \\ \frac{(2N+1)S-2x}{NS}(1-\nu) \| e_0 \|_t & (N+1)S/2 \leq x \leq (N+1)S - S/2 \\ 0 & \text{elsewhere.} \end{cases} \tag{135}$$

We also define $\hat{\eta}$ and T_η as in Theorem 3.

We first assume that $N+1$ is odd. In this case the subdomain $[x_{N/2}, x_{N/2+1}] \times [0,T]$ and the sub-domains near the boundaries $x = x_0$ and $x = x_{N+1}$ are red. Then we note that

$$e_0^b(x,t) \geq \phi_0(x-x_0,N), \quad t \geq T_e, \tag{136}$$

in view of assumption (128). Moreover $\phi_0(x-x_0,N)$ is convex in all subdomains, with the exception of the red subdomain $[x_{N/2}, x_{N/2+1}] \times [0,T]$. When combined with Corollary 4 the convexity of ϕ_0 yields

$$e_1^r(x,t) \geq (1-\hat{\eta})\phi_0(x-x_0,N), \quad t \geq T_e + T_\eta, \tag{137}$$

for $x \notin [x_{N/2}, x_{N/2+1}]$. An explicit calculation with Corollary 4 yields

$$e_1^r(x,t) \geq (1-\hat{\eta})\left(1 - \frac{1+2\Delta/S}{N}\right)\phi_0(x-x_0,N), \quad t \geq T_e + T_\eta, \tag{138}$$

for $x \in [x_{N/2}, x_{N/2+1}]$. This inequality is valid in all red subdomains, in view of inequality (137). Then, we use Corollary 4 and the convexity of ϕ_0 in all black subdomains to obtain

$$e_1^b(x,t) \geq (1-\hat{\eta})^2\left(1 - \frac{1+2\Delta/S}{N}\right)\phi_0(x-x_0,N), \quad t \geq T_e + 2T_\eta. \tag{139}$$

Repeated application of the above arguments and the use of definitions (135) and (131) yields inequality (134).

When $N+1$ is even we use the same arguments as above with $\phi(x-x_0, N-1)$ substituted for $\phi(x-x_0,N)$, thereby reducing the problem to the previous case.

□

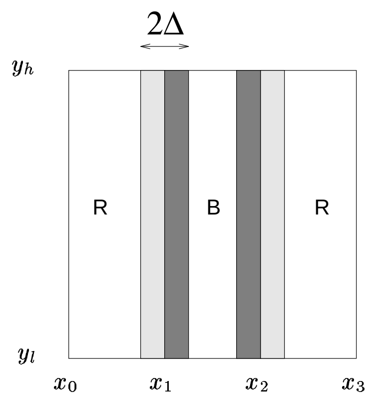

Figure 3: The decomposition of space into three vertical strips. The dark areas represent the overlap of a red subdomain over a black one. The light grey areas represent the overlap of the black subdomain over a red one.

Finally, we note that inequality (126) implies that the magnitude of the error in this iteration is bounded by $\| e_0 \|_t$. In conjunction with Theorem 4 this provides a linear lower an upper envelop with convergence factor that is arbitrarily close to 1 as $N \to \infty$.

In a practical application, it is desirable to choose T sufficiently small so as to avoid the transient linear regime. Indeed, the super-linear convergence rate, does not degrade with the number of subdomains, provided Δ is fixed. Moreover, the super-linear rate improves as diffusion decreases while the linear regime is independent of the diffusion coefficient.

6 The two dimensional advection diffusion equation

We now analyze the space-time domain decomposition iteration for equations (4)-(6) in $W_0 = [x_l, x_h] \times [y_l, y_h] \times [0, T]$. We consider a decomposition of space into strips of width S, as depicted in figure 3, for three subdomains. We note that each strip may be further decomposed into yellow and green cubes, as depicted in figure 4. In this case, we solve the problem in green subdomains of red strips followed by the yellow ones. Then we perform a green yellow sweep in black strips.

6.1 Error analysis for a decomposition into strips

We now derive equations for the error in the iteration, for the case of two subdomains. Hence, $\Omega = [x_0, x_2] \times [y_l, y_h]$ and the red and black subdomains are $[x_0, x_1] \times [y_l, y_h] \times [0, T]$ and $[x_1, x_2] \times [y_l, y_h] \times [0, T]$, respectively. We proceed as in section 3 to obtain an equation

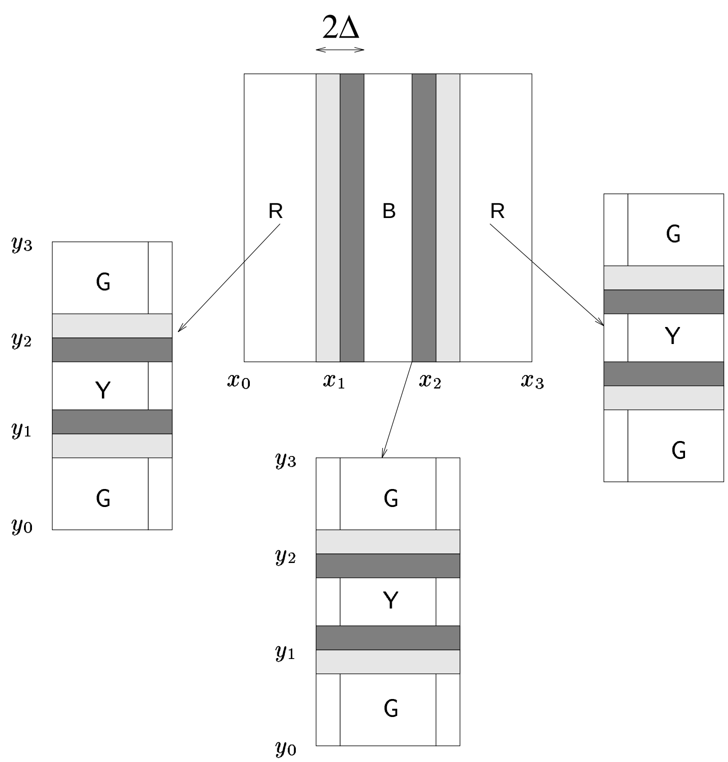

Figure 4: The decomposition of space into three vertical strips each of which is then decomposed into three cubes. The shaded areas represent the overlap between subdomains.

for the error in the red subdomain and it's overlap

$$\frac{\partial e_k^r}{\partial t} = a^2 \Delta e_k^r + b[\sin\theta, \cos\theta] \cdot \nabla e_k^r, \tag{140}$$

$$e_k^r(x, y, 0) = e_k^r(x_0, y, t) = e_k^r(x, y_l, t) = e_k^r(x, y_h, t) = 0, \tag{141}$$

$$e_k^r(x_1 + \Delta, y, t) = e_{k-1}^b(x_1 + \Delta, y, t). \tag{142}$$

In the black subdomain and it's overlap the error satisfies

$$\frac{\partial e_k^b}{\partial t} = a^2 \Delta e_k^b + b[\sin\theta, \cos\theta] \cdot \nabla e_k^b, \tag{143}$$

$$e_k^b(x, y, 0) = e_k^b(x_2, y, t) = e_k^b(x, y_l, t) = e_k^b(x, y_h, t) = 0, \tag{144}$$

$$e_k^b(x_1 - \Delta, y, t) = e_k^r(x_1 - \Delta, y, t). \tag{145}$$

We solve equations (140)-(145) by separation of variables [20] to obtain

$$e_k^r(x, y, t) = \sum_{n=1}^{\infty} M_{k,n}^r(x, t) N_n(y), \qquad e_k^b(x, y, t) = \sum_{n=1}^{\infty} M_{k,n}^b(x, t) N_n(y), \tag{146}$$

where

$$N_n(y) \; = \; \sqrt{\frac{2}{y_h - y_l}} e^{-\frac{b\cos\theta}{2a^2} y} \sin\left(\omega_n (y - y_l)\right), \quad \omega_n = \frac{\pi n}{y_h - y_l}. \tag{147}$$

The Fourier coefficients $M_{k,n}^r$ and $M_{k,n}^b$ satisfy

$$\frac{\partial M_{k,n}^r}{\partial t} = a^2 \frac{\partial^2 M_{k,n}^r}{\partial x^2} + b\sin\theta \frac{\partial M_{k,n}^r}{\partial x} - C_n^2 M_{k,n}^r, \quad n = 1, 2, \ldots \tag{148}$$

$$M_{k,n}^r(x_0, t) = M_{k,n}^r(x, 0) = 0, \qquad M_{k,n}^r(x_1 + \Delta, t) = M_{k-1,n}^b(x_1 + \Delta, t), \tag{149}$$

$$\frac{\partial M_{k,n}^b}{\partial t} = a^2 \frac{\partial^2 M_{k,n}^b}{\partial x^2} + b\sin\theta \frac{\partial M_{k,n}^b}{\partial x} - C_n^2 M_{k,n}^b, \quad n = 1, 2, \ldots \tag{150}$$

$$M_{k,n}^b(x_2, t) = M_{k,n}^b(x, 0) = 0, \qquad M_{k,n}^b(x_1 - \Delta, t) = M_{k,n}^r(x_1 - \Delta, t) \tag{151}$$

$$M_{0,n}^b(x, t) = M_{0,n}(x, t), \quad x_1 \le x \le x_2, \tag{152}$$

where

$$M_{0,n}(x, t) = \int_{y_l}^{y_h} p(y) e_0(x, y, t) N_n(y) dy, \tag{153}$$

$e_0(x, y, t)$ is the initial error,

$$C_n^2 \; = \; \left(\frac{b\cos\theta}{2a}\right)^2 \; + \; \left(\frac{\pi n a}{y_h - y_l}\right)^2, \tag{154}$$

and

$$p(y) = e^{\frac{b\cos\theta y}{a^2}}. \tag{155}$$

The original problem (140)-(145) has been transformed into a sequence of problems (148)-(151) that govern the evolution of each mode of the error in the iteration. We denote the coefficient of the n'th mode of the error at iteration k by $M_{k,n}(x,t)$

$$M_{k,n}(x,t) = \begin{cases} M^r_{k,n}(x,t) & x \in [x_0, x_1] \\ M^b_{k,n}(x,t) & x \in [x_1, x_2] \end{cases}, \tag{156}$$

and the error at iteration k by $e_k(x,y,t)$

$$e_k(x,y,t) = \sum_{n=1}^{\infty} M_{k,n}(x,t) N_n(y). \tag{157}$$

When the number of subdomains $N \geq 3$, we proceed in an analogous fashion to obtain for each mode of the error a recursive system of N differential equations, coupled through their boundary conditions.

6.2 Convergence analysis

The convergence behavior of the iteration for the two dimensional model problem is essentially the same as in the one-dimensional case. Indeed, let $\|\cdot\|_{2,t}$ denote the norm

$$\| e \|_{2,t} = \max_{0 \leq \tau \leq t} \left(\int_{x_l}^{x_h} \int_{y_l}^{y_h} p(y) e^2(x,y,t)\, dx dy \right)^{1/2} \tag{158}$$

with p defined in equation (155) then

Theorem 5 *Consider the space time DD iteration in strips for the advection diffusion equation, problem (4)-(6). Suppose that the initial iterate $u_0 \in C^1(W_0)$.*

1. *If $S > 2\Delta$ then for any $0 \leq t \leq T$*

$$\| e_k \|_{2,t} \leq C \exp\left(-\frac{4(k-1)^2 \Delta^2}{a^2 t} \right) (1 + o(1)), \quad k \to \infty \tag{159}$$

2. *If $|b| > 0$ then for all $t \geq 0$ and $k \geq 1 + [(2\Delta(b/a^2)\pi]^{-1}$*

$$\| e_k \|_{2,t} \leq C\, D(k) \left[\exp\left(-\frac{\Delta |b|}{a^2} \right) + \exp\left(-\frac{S|b|}{2a^2} \right) \right]^{2k-2} \tag{160}$$

In these equations

$$D(k) = 1 + \frac{1}{4\sqrt{\pi (k-1)\Delta |b| / a^2}} \tag{161}$$

and C is a constant which depends on the initial error.

Proof: The recursive system of equations for $M_{k,n}$, derived in section 6.1, is identical to the system of equations of section 3. Hence Theorems 1 and 2 apply to each mode separately, and they yield

$$\begin{aligned} \| M_{k,n} \|_t &\leq \| M_{0,n} \|_t \, C_1 \exp\left(-\frac{4(k-1)^2\Delta^2}{a^2 t}\right)(1+o(1)), && (162)\\ \| M_{k,n} \|_t &\leq \| M_{0,n} \|_t \, C_1 D(k) \left[\exp\left(-\frac{\Delta|b|}{a^2}\right) + \exp\left(-\frac{S|b|}{2a^2}\right)\right]^{2k-2}, && (163) \end{aligned}$$

where

$$\gamma^2 = \left(\frac{b}{2a}\right)^2 + \left(\frac{n\pi a}{y_h - y_l}\right)^2, \quad C_1 = e^{\frac{|b\sin\theta|(x_N - x_0)}{2a^2}}. \tag{164}$$

Inequality (163) follows from inequality (96) in view of the relation $\gamma/a \geq |b|/(2a^2)$, which is valid for all n. Moreover, the $o(1)$ term in inequality (162) is independent of n. This is shown by inspecting the proof of Theorem 1 and using the left bound for γ/a

$$\frac{\pi}{y_h - y_l} \leq \frac{n\pi}{y_h - y_l} \leq \frac{\gamma}{a}. \tag{165}$$

We now use the representation (157) for e_k, and the orthonormality of $\{N_n(y)\}_{n=1}^{\infty}$ with respect to the inner product

$$(f,g) = \int_{y_l}^{y_h} p(y) f(y) g(y) dy, \tag{166}$$

(see: [21]) to obtain

$$\| e_k \|_{2,t} \leq \sqrt{x_h - x_l}\left(\sum_{n=1}^{\infty} \| M_{k,n} \|_t^2\right)^{1/2}. \tag{167}$$

Then we note that

$$\left(\sum_{n=1}^{\infty} (\| M_{0,n} \|_t)^2\right)^{1/2} \leq \frac{\pi}{\sqrt{6}} C_2, \tag{168}$$

where

$$C_2 = (y_h - y_l)^{3/2} \frac{\sqrt{2}}{\pi} \max_{W_0}\left\{ e^{\frac{b\cos\theta y}{2a^2}} \left| \frac{\partial e_0}{\partial y} + \frac{b\cos\theta}{2a^2} e_0 \right| \right\}. \tag{169}$$

We obtain this inequality by integrating equation (153) for $M_{0,n}$ by parts to obtain

$$|M_{0,n}(x,t)| \leq \frac{C_2}{n} \tag{170}$$

This manipulation is justified since $e_0 \in C^1(W_0)$, in view of the fact that both u and u_0 are in $C^1(W_0)$. The right side of inequality (170) is independent of both x and t and this yields the bound (168). Combining the bounds (162) and (163) for $\| M_{k,n} \|_t$ with inequalities (167) and (168) proves the Theorem.

□

6.3 The smoothing property of space-time DD

Space-time DD iteration possesses the smoothing property [22]: high modes of the error are damped much faster then low modes. Indeed Theorem 2 and the relations (165) imply that for all $k \geq 1 + (y_h - y_l)/(4\Delta\pi^2)$

$$\frac{\| M_{k,n} \|_t}{\| M_{0,n} \|_t} \leq C_1 D(k) \left[\exp\left(-\frac{n\pi 2\Delta}{y_h - y_l}\right) + \exp\left(-\frac{n\pi S}{y_h - y_l}\right)\right]^{2(k-1)}, \qquad (171)$$

where

$$D(k) = 1 + \frac{1}{4\pi}\sqrt{\frac{y_h - y_l}{2(k-1)\Delta n}}, \qquad (172)$$

and C_1 is defined in equation (164).

The asymptotic convergence factor of the $n'th$ mode,

$$\bar{\rho}(n) = \left[\exp\left(-\frac{n\pi 2\Delta}{y_h - y_l}\right) + \exp\left(-\frac{n\pi S}{y_h - y_l}\right)\right]^2, \qquad (173)$$

is independent of T, because it results from two complementary phenomena. For short times, the error in the interior of each subdomain is small because it has to "diffuse" through the overlap. This is indicated by the term $-\frac{\eta^2}{4a^2 t}$ in the exponent of equation (50). As time increases the smoothing property of the heat equation takes over and damps the high modes of the error much faster then the low modes, as indicated by the term $-\gamma^2 t$, in that exponent. The maximum of this exponent decreases with increasing mode number, as indicated by the convergence factor (173).

7 The optimal partitioning problem

The analysis of the previous sections, shows that the convergence rate of the iteration, strongly depends on the decomposition parameters, Δ, T and S. We wish to choose these parameters in a way that will minimize the elapsed time on a given parallel machine. In this section we model this problem, and discuss means for it's solution.

The time required to solve the model problem in $\Omega \times [0, T_f]$ to accuracy ϵ, is given by the function E

$$\mathrm{E} = \frac{T_f}{T}\mathrm{N}(\epsilon, \Delta, S, a, \gamma, T)[\mathrm{P}(\Delta, S, h, \epsilon) + \mathrm{C}(T, h)]. \qquad (174)$$

Here, $\mathrm{N}(\epsilon, \Delta, S, a, \gamma, T)$ is the number of DD iteration required to solve the problem to accuracy ϵ. It can be estimated by the analysis of the previous sections. $\mathrm{P}(\Delta, S, h, \epsilon)$ denotes the computation time at each iteration, with h the discretization parameter. The communication cost at the end of each iteration is denoted by $\mathrm{C}(T, h)$. We wish to find Δ, S and T that minimize E.

The function P depends on the algorithm for solving the subproblem and on the computer we use. In a future report we shall develop a model for P based on complexity

estimates of the algorithm and on experimentation with several computers. The goal is to determine a small set of parameters that determine P for a given computer, via a simple functional relationship. The value of these parameters is determined by a small number of experiments.

Communication time is composed of two costs. Initiating communication has a fixed cost, which can be measured on a given computer. The time for transmitting a certain amount of data is usually assumed a linear function of the size of the data. The slope depends on the load on the computer at a certain point in time, and may be measured when the computation is initiated.

In the minimization of E the following tradeoffs are apparent. On a fixed time window T, a large overlap dramatically reduces N, as indicated by expressions (1), (159) and (171). However, it increases the size of each subproblem and reduces the maximum number of subdomains, thereby increasing P. A small time window T also reduces N but increases T_f/T. In the limit $T \to 0$ the number of communication requests converge to infinity. This yields an unbounded elapsed time as $T \to 0$, in view of the fixed cost for initiating communication.

8 Numerical calculations

We begin by comparing the error estimates of section 4.4 for the one dimensional problem, with the decay of the error in an actual computation. We solve the problem in $W_0 = [0,1] \times [0,.25]$, with 3 subdomains of width $S = 1/3$. The diffusion coefficient is $a^2 = 1/2$ and $b = 0$. In our computations we discretize the equation with the Crank-Nicholson method. The mesh parameters are $\Delta x = 1/210$ and $\Delta t = 1/50$. Figure 5 indicates

$$\log_{10} \frac{\| e_k \|_T}{\| e_0 \|_T}, \tag{175}$$

as a function of the iteration k, for three different values of the overlap Δ. The dashed lines indicate our estimate for the error. Our bound is in very good agreement with the computation and it's asymptotic nature is apparent.

We now compare the observed transient linear regime for the heat equation, in a large time interval, with the predictions of section 5.1. In this computation $\Delta = S/7$ and T varies, $T = 1,2,3,4$. Figure 6 indicates the error (175) as a function of the iteration number k. The transient linear bound of Theorem 3 is also indicated in this figure. It's validity for a finite number of iterations is apparent. Moreover, the duration of the linear regime is clearly proportional to T, as predicted by the analysis. In a second set of calculations we solve the problem in $W_0 = [0,17/3] \times T$, with 17 subdomains of width $S = 1/3$. The results are presented in figure 7. A comparison with figure 6 shows the deterioration of the linear bound as the number of subdomains increases, as predicted by Theorem 4. In contrast, the asymptotic convergence rate, depends only on the size of the overlap as indicated in Theorem 1.

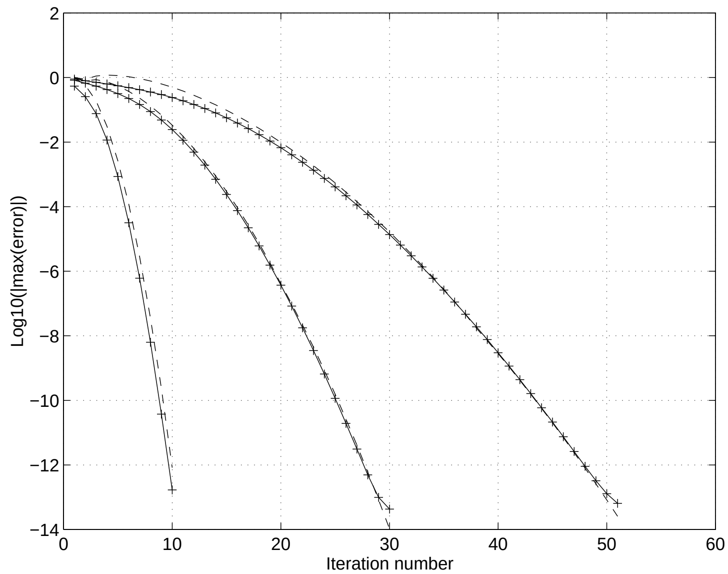

Figure 5: The magnitude of the error is indicated in solid and + for $\Delta = S/20$, $\Delta = S/10$, $\Delta = 3S/10$. The dashed line indicates the estimate of the bound.

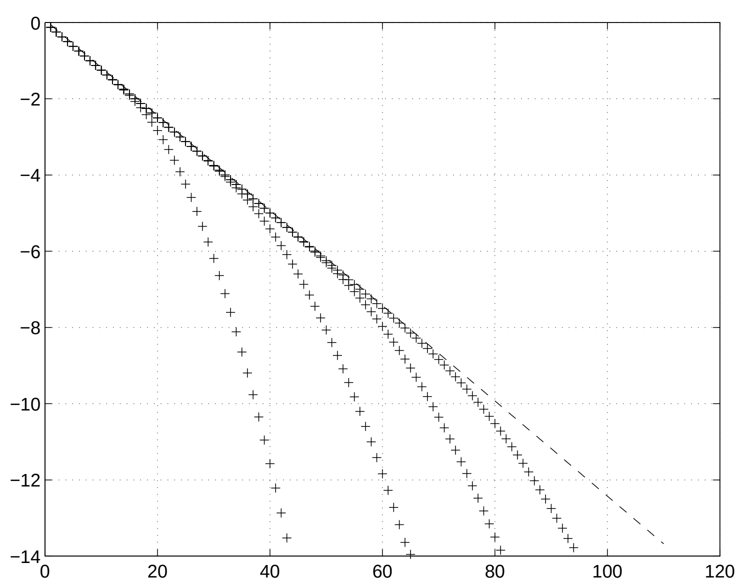

Figure 6: The $\log_{10}$ of the relative error for $T = 1$, $T = 2$, $T = 3$ and $T = 4$, is indicated in + from left to right, respectively. The dashed line indicates the transient linear bound. The number of subdomains is 3.

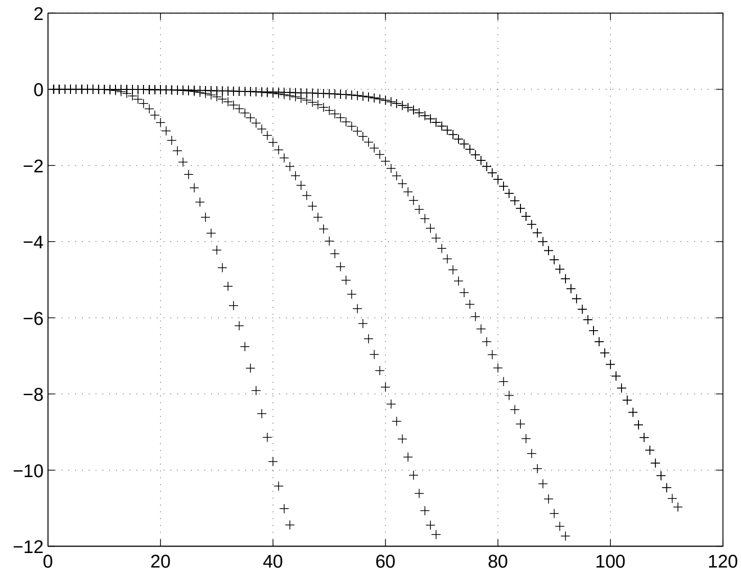

Figure 7: The $\log_{10}$ of the relative error for $T = 1$, $T = 2$, $T = 3$ and $T = 4$, is indicated in $+$ from left to right, respectively. The number of subdomains is 17.

b	Bound for $\bar{\rho}$	Measured $\bar{\rho}_k$
10	0.1777	0.1766
20	0.0225	0.0288
30	0.0033	0.0046

Table 1: The bound for the asymptotic convergence factor versus the measured average convergence factor.

We now compare the observed transient linear rate in advection dominated equations, with the analysis of section 4.3. Here, the diffusion coefficient is $a^2 = 1/2$ and the time window $T = 4$, is relatively large. The advection coefficient varies, $b = 30, 20, 10$. In Figure 8 we observe rapid convergence, in contrast to the case $b = 0$, indicated in figure 6. The improvement in the convergence rate, as the ratio b/a^2 increases, is apparent. In our computations, we measure the average convergence factor, defined in equation (98). Table 8 presents the measured convergence factor versus the bound for the asymptotic convergence factor of equation (100). There is good agreement between the two.

9 Conclusions and future directions

Space-time DD performs best on equations with a small diffusion coefficient and advection dominated equations. The smoothing property of the iteration, in two or more space dimensions, provides the foundation for a hybrid domain decomposition multigrid algorithm. We develop this algorithm in a subsequent paper [23] and compare it with the multigrid

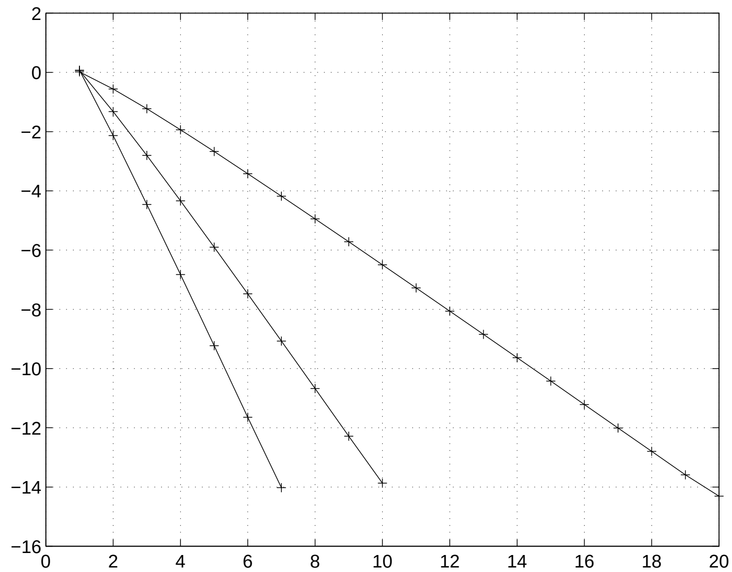

Figure 8: The $\log_{10}$ of the relative error for $b = 30$, $b = 20$ and $b = 10$, is indicated in solid and + from left to right, respectively. Here $a^2 = 1/2$ and $T = 4$.

waveform-relaxation method [6], [13].

The super-linear convergence rate, presented in Theorems 1 and 5 is strongly related to the Gaussian nature of diffusion. This leads us to believe that analogous results can be derived for equations with variable coefficients and possibly non-linear problems, in general domains.

10 Acknowledgements

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A Solution to error equation

In this appendix we derive the solution to equation

$$\frac{\partial e}{\partial t} = a^2 \frac{\partial^2 e}{\partial x^2} + b\frac{\partial e}{\partial x} - c^2 e \tag{176}$$

in $[x_l, x_h] \times [0, \infty)$ subject to the conditions

$$e(x,0) = 0, \quad e(x_l,t) = g_l(t), \quad e(x_h,t) = g_h(t). \tag{177}$$

We use the Laplace transform method (see: [20]) to obtain the equivalent problem

$$\Phi_{xx} + \frac{b}{a^2}\Phi_x - \frac{s+c^2}{a^2}\Phi = 0, \tag{178}$$

$$\Phi(x_l,s) = G_l(s), \quad \Phi(x_h,s) = G_h(s), \tag{179}$$

$$\Phi(x,s) = \int_0^t e^{-s\tau} e(\tau) d\tau, \quad G_l(s) = \int_0^t e^{-s\tau} g_l(\tau) d\tau, \quad G_h(s) = \int_0^t e^{-s\tau} g_h(\tau) d\tau. \tag{180}$$

The solution to equations (178)-(179) is

$$\Phi(x,s) = G_l(s) e^{\alpha(x-x_l)} \frac{\sinh(\beta(x_h - x))}{\sinh(\beta L)} + G_h(s) e^{\alpha(x-x_h)} \frac{\sinh(\beta(x - x_l))}{\sinh(\beta L)} \tag{181}$$

$$\alpha = -\frac{b}{2a^2}, \qquad \beta = \left(\frac{b^2}{4a^4} + \frac{s+c^2}{a^2}\right)^{1/2}. \tag{182}$$

We use the substitution

$$\frac{\sinh(\beta u)}{\sinh(\beta L)} = \frac{e^{-\beta L}\left(e^{\beta u} - e^{-\beta u}\right)}{1 - e^{-2\beta L}}, \tag{183}$$

in this expression and expand $1/(1 - \exp(-2\beta L))$ in Taylor series to obtain

$$\Phi(x,s) = G_l(s) e^{\alpha(x-x_l)} \sum_{n=0}^{\infty} \{\exp[-\beta\lambda_n(x_h - x)] - \exp[-\beta\mu_n(x_h - x)]\} +$$

$$G_h(s) e^{\alpha(x-x_h)} \sum_{n=0}^{\infty} \{\exp[-\beta\lambda_n(x - x_l)] - \exp[-\beta\mu_n(x - x_l)]\}, \tag{184}$$

where the functions λ_n and μ_n are defined in equation (24).

In order to determine e we now compute the inverse transform of each term in the sums (184) using the identities (see: [18])

$$\mathcal{L}^{-1}\{\exp(-k\sqrt{s})\} = \frac{k}{2\sqrt{\pi}t^{3/2}} \exp\left(-\frac{k^2}{4t^2}\right), \quad k > 0, \tag{185}$$

and

$$\mathcal{L}^{-1}\{G(s+h)\} = e^{-ht}\mathcal{L}^{-1}\{G(s)\} \tag{186}$$

to obtain with $\eta = \lambda_n$ or $\eta = \mu_n$

$$\mathcal{L}^{-1}\left\{\exp\left(-\beta\eta\right)\right\} = F(t, \eta, a, \gamma). \tag{187}$$

Here $\mathcal{L}^{-1}$ denotes the inverse Laplace transform and

$$\gamma^2 = c^2 + \frac{b^2}{4a^2}, \tag{188}$$

$$F(t, \eta, a, \gamma) = \frac{\eta}{2\sqrt{\pi}at^{3/2}} \exp\left(-\frac{\eta^2}{4a^2t} - \gamma^2 t\right). \tag{189}$$

We now use identity (187) and the convolution Theorem for the Laplace transform in expression (184) to obtain

$$e(x,t) = e^{\alpha(x-x_l)}\mathcal{K}(t, x_h - x, a, b, c) * g_l(t) + e^{\alpha(x-x_h)}\mathcal{K}(t, x - x_l, a, b, c) * g_h(t), \tag{190}$$

where

$$\mathcal{K}(t, u, a, b, c) = e^{-\gamma^2 t} \sum_{n=0}^{\infty} \left\{F[t, \lambda_n(u), a, \gamma] - F[t, \mu_n(u), a, \gamma]\right\} \tag{191}$$

and γ is defined in equation (188).

B A qualified maximum principle for the error equation

We now show that solutions to equation (192) below

$$\frac{\partial e}{\partial t} = a^2 \frac{\partial^2 e}{\partial x^2} + b\frac{\partial e}{\partial x} - c^2 e, \tag{192}$$

satisfy a maximum principle in $[x_l, x_h] \times [0, T]$ when the data is non-negative. Specifically we assume that $e \geq 0$ on

$$\hat{\partial} = \{x_l\} \times [0, T] \cup \{x_h\} \times [0, T] \cup [x_l, x_h] \times \{0\} \tag{193}$$

and we show that

1. The maximizer of $e \in \hat{\partial}$.
2. The function e is non-negative throughout $[x_l, x_h] \times [0, T]$.

We use similar arguments as in [20] for the heat equation. Let (x_0, t_0) be the maximizer of e and suppose by contradiction that $(x_0, t_0) \notin \hat{\partial}$. Then at (x_0, t_0)

$$\frac{\partial e}{\partial x} = 0, \quad \frac{\partial^2 e}{\partial x^2} \leq 0, \quad \frac{\partial e}{\partial t} \geq 0, \quad e \geq 0. \tag{194}$$

Let

$$\psi(x, t, \epsilon) = e(x, t) - \epsilon(t - t_0). \tag{195}$$

If ϵ is sufficiently small, the maximizer of ψ, (x_1, t_1), is sufficiently close to (x_0, t_0) such that $(x_1, t_1) \notin \hat{\partial}$ and $e(x_1, t_1) \geq 0$. Moreover at (x_1, t_1)

$$\frac{\partial^2 \psi}{\partial x^2} \leq 0, \quad \frac{\partial \psi}{\partial x} = 0, \quad \frac{\partial \psi}{\partial t} \geq 0 \tag{196}$$

and therefore at (x_1, t_1)

$$\frac{\partial^2 e}{\partial x^2} \leq 0, \quad \frac{\partial e}{\partial x} = 0, \quad \frac{\partial e}{\partial t} > 0, \quad e \geq 0. \tag{197}$$

This contradicts equation (192) and proves that $(x_1, t_1) \in \hat{\partial}$.

Let (x_0, t_0) denote the minimizer of e and suppose that $e(x_0, t_0) < 0$. Then $(x_0, t_0) \notin \hat{\partial}$ and therefore

$$\frac{\partial e}{\partial x} = 0, \quad \frac{\partial^2 e}{\partial x^2} \geq 0, \quad \frac{\partial e}{\partial t} \leq 0, \quad e < 0, \tag{198}$$

there. This contradicts equation (192) and proves that $e \geq 0$ throughout $[x_l, x_h] \times \Omega$.

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