Finite Element Approximation to the System of Shallow Water Equations, Part II: Discrete Time A Priori Error Estimates

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FINITE ELEMENT APPROXIMATIONS TO THE SYSTEM OF SHALLOW WATER EQUATIONS, PART II: DISCRETE TIME A PRIORI ERROR ESTIMATES *

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Abstract. Various sophisticated finite element models for surface water flow exist in the literature. Gray, Kolar, Luettich, Lynch and Westerink have developed a hydrodynamic model based on the generalized wave continuity equation (GWCE) formulation, and have formulated a Galerkin finite element procedure based on combining the GWCE with the nonconservative momentum equations. Numerical experiments suggest that this method is robust, accurate and suppresses spurious oscillations which plague other models. In this paper, we analyze a closely related Galerkin method which uses the conservative momentum equations (CME). For this GWCE-CME system of equations, we present, for discrete time, an *a priori* error estimate based on an $\mathcal{L}^2$ projection.

Key words. shallow water equations, surface flow, mass conservation, momentum conservation, finite element model, error estimate, stability

AMS subject classifications. 35Q35, 35L65 65N30, 65N15

1. Introduction. In this paper, we derive *a-priori* error estimates for a discrete-time finite element approximation to a model of shallow water flow. These estimates extend our continuous-time analysis given in [1]. We consider a shallow water model of Gray *et al* described in a series of papers beginning in [7]; see also [6].

We denote by $\xi(\boldsymbol{x},t)$ the free surface elevation over a reference plane and by $h_b(\boldsymbol{x})$ the bathymetric depth under that reference plane so that $H(\boldsymbol{x},t) = \xi + h_b$ is the total water column. Also, we denote by $\boldsymbol{u} = [U(\boldsymbol{x},t)\ V(\boldsymbol{x},t)]^T$ the depth-averaged horizontal velocities and let $\boldsymbol{q} = \boldsymbol{u}H$.

The formulation we consider consists of deriving a wave equation for the free surface elevation and combining that with the conservative momentum equations for velocities. The generalized wave continuity equation (GWCE) is given by

$$\begin{aligned}\xi_{tt} + \tau_0\xi_t - \nabla\cdot\Big[\nabla\cdot\left(\frac{1}{H}\boldsymbol{q}^2\right) - (\tau_0\boldsymbol{q} - \tau_{bf}\boldsymbol{q}) \\ + f_c\boldsymbol{k}\times\boldsymbol{q} + Hg\nabla\xi + \mu\nabla\xi_t + H\boldsymbol{\mathcal{F}}\Big] = 0.\end{aligned} \tag{1}$$

Here, τ_0 is a time-independent positive constant, $\tau_{bf}(\xi,\boldsymbol{u})$ is a bottom friction function, $\boldsymbol{k}$ is a unit vector in the vertical direction, f_c is a Coriolis function, g is acceleration due to gravity, μ is the horizontal eddy diffusion/dispersion coefficient, and $\boldsymbol{\mathcal{F}} = (-\boldsymbol{\tau}_{ws}/H + \nabla p_a - g\nabla\eta)$, where $\boldsymbol{\tau}_{ws}$ is the applied free surface wind stress relative to the reference density of water, $p_a(\boldsymbol{x},t)$ is the atmospheric pressure at the free surface relative to the reference density of water, and $\eta(\boldsymbol{x},t)$ is the Newtonian equilibrium tide potential relative to the effective Earth elasticity factor. Now, as defined in [6], the bottom friction function is given by

$$\tau_{bf}(\xi,\boldsymbol{u}) = c_f\frac{\sqrt{U^2+V^2}}{H} \equiv c_f\frac{||\boldsymbol{u}||_{\ell^2}}{H} \equiv c_f\frac{||\boldsymbol{q}/H||_{\ell^2}}{H},$$

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where c_f is a friction coefficient. Futhermore, the Coriolis function is given by

$$f_c(\boldsymbol{x}) \;=\; 2\omega \sin\alpha,$$

where ω is the angular velocity of the earth in its daily rotation and α is the degrees latitude. We will treat $f_c, \boldsymbol{\tau}_{ws}, \nabla p_a$, and $\nabla\eta$ as data.

The GWCE can be coupled to the conservative momentum equations (CME), given by

$$\boldsymbol{q}_t + \nabla\cdot\left(\frac{1}{H}\boldsymbol{q}^2\right) + \tau_{bf}\boldsymbol{q} + f_c\boldsymbol{k}\times\boldsymbol{q} + Hg\nabla\xi - \mu\Delta\boldsymbol{q} + H\boldsymbol{\mathcal{F}} \;=\; 0, \tag{2}$$

or to the non-conservative momentum equations (NCME). In this paper we will consider the GWCE-CME formulation. For technical reasons, we have found that this model lends itself more easily to error analysis.

A finite element simulator based on the GWCE-NCME formulation has been developed by Luettich, et al. In [6], it was demonstrated that the approximations generated by this simulator accurately matched tidal data taken from the English Channel and southern North Sea. The temporal discretization scheme we will consider adheres closely to the scheme implemented in this simulator.

The rest of this paper is outlined as follows. In section 2, we review the weak formulation associated with the GWCE-CME system of equations and also detail the assumptions we will need in our analysis. In section 3, we introduce the discrete-time finite element approximation to the weak solution. In section 4, we derive an *a priori* error estimate based on a discrete $\mathcal{L}^2$ projection. The proof of the error estimate relies on on an induction argument to obtain $\mathcal{L}^\infty$ boundedness of the Galerkin approximations.

2. Preliminaries.

2.1. Nondimensional Form. Because mathematical inequalities involving dimensional units may have no meaning without some basis for comparison, we will look at the non-dimensionalized form of the shallow water equations (1)-(2).

Suppose there exist reference or characterisitic quantities: velocity $\hat{U}$, length $\hat{L}$, and depth $\hat{H}$. Then, we make the following change of variables

$$\boldsymbol{u} = \hat{U}\boldsymbol{u}^\circ, \quad H = \hat{H}H^\circ, \quad \boldsymbol{x} = \hat{L}\boldsymbol{x}^\circ, \quad t = \tfrac{\hat{L}}{\hat{U}}t^\circ.$$

Thus, for given ς, we denote by ς° the non-dimensionalized form.

In non-dimensionalized form, (1)-(2) become

$$\begin{aligned}\frac{\partial^2\xi^\circ}{(\partial t^\circ)^2} + \tau_\circ^\circ\frac{\partial\xi^\circ}{\partial t^\circ} - \nabla^\circ\cdot\Bigg[\nabla^\circ\cdot\left(\frac{1}{H^\circ}(\boldsymbol{q}^\circ)^2\right) - (\tau_\circ^\circ - \lambda\tau_{bf}{}^\circ)\,\boldsymbol{q}^\circ \\ + f_c{}^\circ\boldsymbol{k}\times\boldsymbol{q}^\circ + \frac{1}{Fr^2}H^\circ\nabla^\circ\xi^\circ + \mu^\circ\nabla^\circ\frac{\partial\xi^\circ}{\partial t^\circ} + H^\circ\boldsymbol{\mathcal{F}}^\circ\Bigg] = 0;\end{aligned} \tag{3}$$

$$\begin{aligned}\frac{\partial\boldsymbol{q}^\circ}{\partial t^\circ} + \nabla^\circ\cdot\left(\frac{1}{H^\circ}(\boldsymbol{q}^\circ)^2\right) + \lambda\tau_{bf}{}^\circ\boldsymbol{q}^\circ \\ + f_c{}^\circ\boldsymbol{k}\times\boldsymbol{q}^\circ + \frac{1}{Fr^2}H^\circ\nabla^\circ\xi^\circ - \mu^\circ\Delta^\circ\boldsymbol{q}^\circ + H^\circ\boldsymbol{\mathcal{F}}^0 = 0.\end{aligned} \tag{4}$$

where $\tau_\circ^\circ = \frac{\hat{L}}{\hat{U}}\tau_\circ$, $\lambda = \frac{\hat{L}}{\hat{H}}$, $f_c{}^\circ = \frac{\hat{L}}{\hat{U}} f_c$, $\mu^\circ = \frac{1}{\hat{U}\hat{L}}\mu$, $\boldsymbol{\mathcal{F}}^\circ = \frac{\hat{L}}{(\hat{U})^2}\boldsymbol{\mathcal{F}}$, and $Fr = \frac{\hat{U}}{\sqrt{g\hat{H}}}$ is the Froude number.

Observe that (3)-(4) is nothing more than (1)-(2) with multiplicative biasing factors that faciliate understanding the importance of each term under limiting conditions. Moreover, under the SWE assumption that $\lambda \gg 1$, we observe that the bottom friction term is a significant term in the equations.

Letting $\hat{U} = \sqrt{g\hat{L}}$ so that $Fr = 1$, letting $\tau_{bf}{}^\circ = \lambda \tau_{bf}{}^\circ$, and dropping all extraneous symbols in the nondimensionalized equations above, we obtain equations of the form (1)-(2) except that gravity g doesn't appear explicitly. We will analyze these equations with the implicit understanding that all the terms are now nondimensionalized.

2.2. Notation. For the purposes of our analysis, we define some notation used throughout the rest of this paper.

For $\bar{J} = [0, T]$, $N\Delta t = T$, $\Delta t \geq 0$, let us define a temporal subdomain by

$$\bar{J}_m \quad = \quad \{t^k \in \bar{J} \mid t^k = k\Delta t, \;\; k = 0, \ldots, m\}$$

so that $\bar{J}_N = \bar{J}$, $\bar{J}_{N-1} = [0, T - \Delta t]$, and $\bar{J}_1 = [0, \Delta t]$. Denote $J_m = \bar{J}_m - \{0, m\Delta t\}$ and $J_m^0 = \bar{J}_m - \{0\}$.

For X, a normed linear space with norm $||\cdot||_X$, and $\varsigma : [0,T] \to X$, define $\varsigma^k = \varsigma(\cdot, t^k)$ and the following norms

$$\begin{aligned}
||\varsigma||^p_{\ell^p(J_m;X)} &= \sum_{k=0}^{m} ||\varsigma^k(\boldsymbol{x})||^p_X \;\Delta t, && 1 \leq p < \infty,\\
||\varsigma||_{\ell^\infty(J_m;X)} &= \sup_{0\leq k\leq m} ||\varsigma^k(\boldsymbol{x})||_X,\\
||\varsigma||^p_{\mathcal{L}^p((0,T);X)} &= \int_0^T ||\varsigma(\boldsymbol{x},t)||^p_X \;\mathrm{d}t, && 1 \leq p < \infty,\\
||\varsigma||^p_{\mathcal{L}^\infty((0,T);X)} &= \sup_{0\leq t\leq T} ||\varsigma(\boldsymbol{x},t)||^p_X \;\mathrm{d}t\;.
\end{aligned}$$

Furthermore, define

$$\begin{aligned}
&\partial_{t^f}(\varsigma^k) = (\Delta t)^{-1}(\varsigma^{k+1} - \varsigma^k); \qquad \partial_{t^c}(\varsigma^k) = (2\Delta t)^{-1}(\varsigma^{k+1} - \varsigma^{k-1});\\
&\partial_{t^b}(\varsigma^k) = (\Delta t)^{-1}(\varsigma^k - \varsigma^{k-1}); \qquad \partial^2_{t^c}(\varsigma^k) = (\Delta t)^{-2}(\varsigma^{k+1} - 2\varsigma^k + \varsigma^{k-1});
\end{aligned}$$

and $\varsigma^{k+\frac{1}{2}} = (\varsigma^{k+1} + \varsigma^k)/2$. Finally, we let $K, K_i, (i = 0, 1, 2, ..)$ and ϵ be generic constants not necessarily the same at every occurrence.

2.3. Variational Formulation. We will consider the coupled system given by the GWCE-CME described in Section 1, with the following homogeneous Dirichlet boundary conditions for simplicity

$$\left.\begin{aligned} \xi(\boldsymbol{x},t) &= 0,\\ \boldsymbol{u}(\boldsymbol{x},t) &= 0, \end{aligned}\right\} \;\boldsymbol{x} \in \partial\Omega,\; t > 0, \tag{5}$$

and with the compatible initial conditions

$$\left.\begin{aligned} \xi(\boldsymbol{x},0) &= \xi_0(\boldsymbol{x}),\\ \xi_t(\boldsymbol{x},0) &= \xi_1(\boldsymbol{x}),\\ \boldsymbol{u}(\boldsymbol{x},0) &= \boldsymbol{u}_0(\boldsymbol{x}), \end{aligned}\right\} \;\boldsymbol{x} \in \bar{\Omega}, \tag{6}$$

where $\partial\Omega$ is the boundary of $\Omega \subset \mathbb{R}^2$ and $\bar{\Omega} = \Omega \cup \partial\Omega$. By compatible, we mean $\xi_1 = -\nabla\cdot\boldsymbol{q}_0$, where $\boldsymbol{q}_0 = \boldsymbol{u}_0 H_0 = \boldsymbol{u}_0(h_b + \xi_0)$, see [5].

The weak form of this system in discrete time that we will consider is the following: For $t^k \in J_N^0$, find $\xi^k(\boldsymbol{x}) \in \mathcal{H}_0^1(\Omega)$ and $\boldsymbol{q}^k(\boldsymbol{x}) \in \boldsymbol{\mathcal{H}}_0^1(\Omega)$ such that

$$
\begin{aligned}
(7)\qquad & \left({\xi_{tt}}^0, v\right) + \tau_\circ\left({\xi_t}^0, v\right) + \left(\nabla\cdot\left\{\frac{1}{H^0}\left(\boldsymbol{q}^0\right)^2\right\}, \nabla v\right) - \left((\tau_\circ - {\tau_{bf}}^0)\boldsymbol{q}^0, \nabla v\right) \\
& + \left(f_c\boldsymbol{k}\times\boldsymbol{q}^0, \nabla v\right) + \left(h_b\nabla\xi^0, \nabla v\right) + \left(\xi^0\nabla\xi^0, \nabla v\right) \\
& + \mu\left(\nabla{\xi_t}^0, \nabla v\right) + \left(H^0\boldsymbol{\mathcal{F}}^0, \nabla v\right) = 0, \qquad \forall v \in \mathcal{H}_0^1(\Omega),
\end{aligned}
$$

$$
\begin{aligned}
(8)\quad & \left({\xi_{tt}}^k, v\right) + \tau_\circ\left({\xi_t}^k, v\right) + \left(\nabla\cdot\left\{\frac{1}{H^k}\left(\boldsymbol{q}^k\right)^2\right\}, \nabla v\right) - \left((\tau_\circ - {\tau_{bf}}^k)\boldsymbol{q}^k, \nabla v\right) \\
& + \left(f_c\boldsymbol{k}\times\boldsymbol{q}^k, \nabla v\right) + \left(h_b\nabla\xi^k, \nabla v\right) + \left(\xi^k\nabla\xi^k, \nabla v\right) \\
& + \mu\left(\nabla{\xi_t}^k, \nabla v\right) + \left(H^k\boldsymbol{\mathcal{F}}^k, \nabla v\right) = 0, \qquad \forall v \in \mathcal{H}_0^1(\Omega),\ k \geq 1,
\end{aligned}
$$

$$
\begin{aligned}
(9)\qquad & \left({\boldsymbol{q}_t}^k, \boldsymbol{w}\right) + \left(\nabla\cdot\left\{\frac{1}{H^k}\left(\boldsymbol{q}^k\right)^2\right\}, \boldsymbol{w}\right) + \left({\tau_{bf}}^k\boldsymbol{q}^k, \boldsymbol{w}\right) + \left(f_c\boldsymbol{k}\times\boldsymbol{q}^k, \boldsymbol{w}\right) \\
& + \left(H^k\nabla\xi^k, \boldsymbol{w}\right) + \mu\left(\nabla\boldsymbol{q}^k, \nabla\boldsymbol{w}\right) + \left(H^k\boldsymbol{\mathcal{F}}^k, \boldsymbol{w}\right) = 0, \quad \forall\boldsymbol{w} \in \boldsymbol{\mathcal{H}}_0^1(\Omega),\ k \geq 0,
\end{aligned}
$$

with initial conditions

$$
(10)\qquad
\begin{array}{rcll}
(\xi^0, v) & = & (\xi_0, v), & \forall v \in \mathcal{H}_0^1(\Omega), \\
(\xi^{-1}, v) & = & (\xi^1, v) - 2\Delta t(\xi_1, v), & \forall v \in \mathcal{H}_0^1(\Omega), \\
(\boldsymbol{q}^0, \boldsymbol{w}) & = & (\boldsymbol{q}_0, \boldsymbol{w}), & \forall \boldsymbol{w} \in \boldsymbol{\mathcal{H}}_0^1(\Omega).
\end{array}
$$

Observe that (7) is simply (8) at initial time using the initial conditions on elevation. The second initial condition, in (10), on elevation will allows us to handle the truncation error for the second-order time derivative approximation at time $t = 0$, in the spirit of [2].

2.4. Some Assumptions. Our analysis requires that we make certain physically reasonable assumptions about the solutions and the data. First, we assume for $(\boldsymbol{x}, t) \in \bar{\Omega} \times J_N^0$,

A1 the solution $(\xi,\, \boldsymbol{q})$ to (7)-(10) exist and are unique,
A2 $\exists$ positive constants H_* and H^* such that $H_* \leq H(\boldsymbol{x}, t) \leq H^*$,
A3 the velocities $U(\boldsymbol{x}, t), V(\boldsymbol{x}, t)$ are bounded,
A4 $\nabla h_b(\boldsymbol{x})$ is bounded,
A5 $\exists$ non-negative constants τ_* and τ^* such that $\tau_* \leq \tau_{bf}(\xi, \boldsymbol{u}) \leq \tau^*$,
A6 $f_c(\boldsymbol{x})$ is bounded,
A7 μ is a positive constant,
A8 $\nabla p_a(\boldsymbol{x}, t)$ and $\nabla\eta(\boldsymbol{x}, t)$ are bounded.

Finally, we make the following smoothness assumptions on the initial data and on the solutions

A9 $\xi_0(\boldsymbol{x}), \xi_1(\boldsymbol{x}) \in \mathcal{H}_0^1(\Omega) \cap \mathcal{H}^\ell(\Omega)$,
A10 $\boldsymbol{q}_0(\boldsymbol{x}) \in \boldsymbol{\mathcal{H}}_0^1(\Omega) \cap \boldsymbol{\mathcal{H}}^\ell(\Omega)$,
A11 $H(\boldsymbol{x}, t) \in \mathcal{H}_0^1(\Omega) \cap \mathcal{H}^\ell(\Omega) \cap \mathcal{W}_\infty^1(\Omega),\ \ t \in J_N$,
A12 $\boldsymbol{q}(\boldsymbol{x}, t) \in \boldsymbol{\mathcal{H}}_0^1(\Omega) \cap \boldsymbol{\mathcal{H}}^\ell(\Omega) \cap \boldsymbol{\mathcal{W}}_\infty^1(\Omega),\ \ t \in J_N$.

And, additionally, for $(\boldsymbol{x},t)\in\bar{\Omega}\times J_N$,

A13 $\xi(\boldsymbol{x},t)\in\mathcal{H}^2\left((0,T);\mathcal{H}^1(\Omega)\right)\cap\mathcal{H}^4\left((0,T);\mathcal{L}^2(\Omega)\right)$,

A14 $\xi_t(\boldsymbol{x},t),\xi_{tt}(\boldsymbol{x},t)\in\mathcal{L}^\infty\left((0,\Delta t);\mathcal{H}^1(\Omega)\right)$,

A15 $\xi_{ttt}(\boldsymbol{x},t)\in\mathcal{L}^\infty\left((0,\Delta t);\mathcal{L}^2(\Omega)\right)$,

A16 $\boldsymbol{q}(\boldsymbol{x},t)\in\mathcal{H}^2\left((0,T);\mathcal{H}^1(\Omega)\right)$.

where the non-negative integer ℓ is defined in the next section.

3. Finite Element Approximation.

3.1. The Discrete-Time Galerkin Approximation. Let $\mathcal{T}$ be a triangulation of Ω into elements $E_i,\ i=1,\ldots,m,$ with $\mathrm{diam}(E_i)=h_i$ and $h=\max_i h_i$. Let $\mathcal{S}^h$ denote a finite dimensional subspace of $\mathcal{H}_0^1(\Omega)$ defined on this triangulation consisting of piecewise polynomials of degree less than s_1. Define $\mathcal{H}(\Omega)=\mathcal{H}_0^1(\Omega)\cap\mathcal{H}^\ell(\Omega)$, and assume $\mathcal{S}^h$ satisfies the standard approximation property

$$\inf_{\varphi\in\mathcal{S}^h}||v-\varphi||_{\mathcal{H}^{s_0}(\Omega)}\ \le\ K_0h^{\ell-s_0}\,||v||_{\mathcal{H}^\ell(\Omega)},\quad v\in\mathcal{H}(\Omega),\tag{11}$$

and the inverse assumptions

$$\left.\begin{array}{lcl}||\varphi||_{\mathcal{H}^{\ell-s_0}(\Omega)} & \le & K_0\,||\varphi||_{\mathcal{L}^2(\Omega)}\,h^{-(\ell-s_0)},\\ ||\varphi||_{\mathcal{L}^\infty(\Omega)} & \le & K_0\,||\varphi||_{\mathcal{L}^2(\Omega)}\,h^{-1},\\ ||\nabla\varphi||_{\mathcal{L}^\infty(\Omega)} & \le & K_0\,||\nabla\varphi||_{\mathcal{L}^2(\Omega)}\,h^{-1},\end{array}\right\}\ \varphi\in\mathcal{S}^h(\Omega).\tag{12}$$

Here, s_0 and ℓ are integers, $0\le s_0\le k$ for any integer $k, 0\le k<s_1$, and ℓ satisfies $s_0\le\ell\le s_1$. Moreover, K_0 is a constant independent of h and v.

At time $t=t^k$, let $\Pi^k=h_b+\Xi^k$ and

$$\hat{\tau}_{bf}{}^k\ =\ c_f\frac{\left|\left|\mathcal{Q}^k/\Pi^k\right|\right|_{\ell^2}}{\Pi^k}.\tag{13}$$

We define the discrete-time Galerkin approximations to $\xi^k(\boldsymbol{x}),\boldsymbol{q}^k(\boldsymbol{x})$ to be the mappings $\Xi^k(\boldsymbol{x})\in\mathcal{S}^h,\mathcal{Q}^k(\boldsymbol{x})\in\boldsymbol{\mathcal{S}}^h$ satisfying

$$\begin{aligned}&\frac{2}{\Delta t}\left(\partial_{t^b}\Xi^1,v\right)+\tau_\circ\left(\partial_{t^b}\Xi^1,v\right)+\left(h_b\nabla\Xi^{\frac12},\nabla v\right)+\mu\left(\partial_{t^b}\nabla\Xi^1,\nabla v\right)\\&=\frac{2}{\Delta t}\left(\xi_1,v\right)-\left(\nabla\cdot\left\{\frac{1}{\Pi^0}\left(\mathcal{Q}^0\right)^2\right\},\nabla v\right)+\left((\tau_\circ-\hat{\tau}_{bf}{}^0)\mathcal{Q}^0,\nabla v\right)\\&-\left(f_c\boldsymbol{k}\times\mathcal{Q}^0,\nabla v\right)-\left(\Xi^0\nabla\Xi^0,\nabla v\right)-\left(\Pi^0\boldsymbol{\mathcal{F}}^0,\nabla v\right),\quad\forall v\in\mathcal{S}^h(\Omega);\end{aligned}\tag{14}$$

$$\begin{aligned}&\left(\partial^2_{t^c}\Xi^k,v\right)+\tau_\circ\left(\partial_{t^f}\Xi^k,v\right)+\left(\nabla\cdot\left\{\frac{1}{\Pi^k}\left(\mathcal{Q}^k\right)^2\right\},\nabla v\right)-\left((\tau_\circ-\hat{\tau}_{bf}{}^k)\mathcal{Q}^k,\nabla v\right)\\&+\left(f_c\boldsymbol{k}\times\mathcal{Q}^k,\nabla v\right)+\left(h_b\nabla\Xi^{k+\frac12},\nabla v\right)+\left(\Xi^k\nabla\Xi^k,\nabla v\right)\\&+\mu\left(\partial_{t^f}\nabla\Xi^k,\nabla v\right)+\left(\Pi^k\boldsymbol{\mathcal{F}}^k,\nabla v\right)=0,\qquad\forall v\in\mathcal{S}^h(\Omega),\ k\ge1;\end{aligned}\tag{15}$$

and with $\beta_1+\beta_2=1$,

$$\begin{aligned}&\left(\partial_{t^b}\mathcal{Q}^{k+1},\boldsymbol{w}\right)+\left(\nabla\cdot\left\{\frac{1}{\Pi^k}\left(\mathcal{Q}^k\right)^2\right\},\boldsymbol{w}\right)+\left(\hat{\tau}_{bf}{}^k\mathcal{Q}^{k+\frac12},\boldsymbol{w}\right)\\&+\left(f_c\boldsymbol{k}\times\mathcal{Q}^{k+\frac12},\boldsymbol{w}\right)+\left(\Pi^k\nabla\Xi^k,\boldsymbol{w}\right)+\mu\left(\beta_1\nabla\mathcal{Q}^{k+1}+\beta_2\nabla\mathcal{Q}^k,\nabla\boldsymbol{w}\right)\\&+\left(\Pi^k\boldsymbol{\mathcal{F}}^k,\boldsymbol{w}\right)=0,\qquad\forall\boldsymbol{w}\in\boldsymbol{\mathcal{S}}^h(\Omega),\ k\ge0,\end{aligned}\tag{16}$$

with boundary conditions

$$\Xi^k(\boldsymbol{x}) = 0, \qquad \mathcal{Q}^k(\boldsymbol{x}) = 0, \qquad \boldsymbol{x} \in \partial\Omega,\ k > 0; \tag{17}$$

and with initial conditions

$$\begin{aligned} (\Xi^0, v) &= (\xi_0, v), & \forall v \in \mathcal{S}^h(\Omega), \\ (\Xi^{-1}, v) &= (\Xi^1, v) - 2\Delta t(\xi_1, v), & \forall v \in \mathcal{S}^h(\Omega), \\ (\mathcal{Q}^0, \boldsymbol{w}) &= (\mathcal{Q}_0, \boldsymbol{w}), & \forall \boldsymbol{w} \in \boldsymbol{\mathcal{S}}^h(\Omega). \end{aligned} \tag{18}$$

As before, equation (14) arises from considering (15) at $k = 0$ and using the fictitious value Ξ^{-1} defined in (18).

REMARK: It should be noted that, in the simulator, a Crank-Nicolson approach is taken for solving the NCME. And thus, in the simulator, $\beta_1 = \beta_2 = \frac{1}{2}$ and the NCME is centered at $k + \frac{1}{2}$ except for advection terms which are treated explicitly.

However, it will be clear that because of the explicit treatement of advective terms (and other forcing terms), eliminating tight coupling between the GWCE and CME, the best that can be achieved is a first-order in time scheme. To that end, we only consider the cases when diffusion terms in the CME are treated explicitly ($\beta_1 = 0, \beta_2 = 1$) or implicitly ($\beta_1 = 1, \beta_2 = 0$).

4. ***A-Priori*** **Discrete-time Error Estimate.** Following the analysis in [1], we will compare our finite element approximations $(\Xi, \mathcal{Q})$ satisfying (15)–(18) to $\mathcal{L}^2$ projections $(\widetilde{\xi}, \widetilde{\boldsymbol{q}})$ satisfying

$$\begin{aligned} \left((\xi^k - \widetilde{\xi}^k), v\right) &= 0, & \forall v \in \mathcal{S}^h, k \geq -1, \\ \left((\boldsymbol{q}^k - \widetilde{\boldsymbol{q}}^k), \boldsymbol{w}\right) &= 0, & \forall \boldsymbol{w} \in \boldsymbol{\mathcal{S}}^h, k \geq 0. \end{aligned} \tag{19}$$

For the purpose of succinctness in the rest of the paper, we define

$$\left\{ \begin{array}{lllllll} \theta^k &=& \xi^k - \widetilde{\xi}^k, & \quad & \psi^k &=& \Xi^k - \widetilde{\xi}^k, \\ \boldsymbol{\phi}^k &=& \boldsymbol{q}^k - \widetilde{\boldsymbol{q}}^k, & & \boldsymbol{\chi}^k &=& \mathcal{Q}^k - \widetilde{\boldsymbol{q}}^k. \end{array} \right. \tag{20}$$

Clearly, $\xi^k - \Xi^k = \theta^k - \psi^k$ and $\boldsymbol{q}^k - \mathcal{Q}^k = \boldsymbol{\phi}^k - \boldsymbol{\chi}^k$. We shall call θ^k and $\boldsymbol{\phi}^k$ the *projection errors at time* t^k, and we shall call ψ^k and $\boldsymbol{\chi}^k$ the *affine errors at time* t^k.

Before proceeding, it will be necessary to make certain assumptions about the Galerkin approximations. We employ an inductive argument similar to that made in [3] to handle nonlinearities. In particular, we will assume that the Galerkin approximations are bounded by some constant (for $k = 0, \ldots, N-1$) in order to derive the *a priori* error estimate. Then we will show for sufficiently small mesh size h, in the case of polynomials of degree at least two ($s_1 \geq 3$), that we can remove the estimates' dependence on the assumed bound of the approximations (for $k = N$), being dependent instead on a smaller bound on the comparison projections.

Let us proceed with the inductive argument. Based on the continuous-time analog, Lemma 4.2 of [1], $\exists$ positive constants C_*, C^* and $C_{**} = C^*/2$, $C^{**} = 2C^*$ such that for $k = 0, \ldots, N$,

$$C_* \leq \left\{ h_b + \widetilde{\xi}^k,\ \nabla(h_b + \widetilde{\xi}^k) \right\} \leq C^*, \tag{21}$$

and

$$\left\{ \left|\widetilde{\boldsymbol{q}}^k\right|, \left|\nabla\widetilde{\boldsymbol{q}}^k\right| \right\} \leq C^*. \tag{22}$$

Next, we assume that for $k = 0, \ldots, (N-1)$

B1 $C_{**} < \Pi(\boldsymbol{x}, t^k) < C^{**}$,

B2 $\left|\mathcal{Q}^k\right| < C^{**}$.

We immediately have the base case for the inductive proof: $C_{**} < \Pi^0 < C^{**}$ and $\left|\mathcal{Q}^0\right| < C^{**}$. In the remainder of this paper, from the derivation of the *a priori* error estimate, we prove the hypothesis for time $t = t^N$.

Observe that **B1** and **B2**, in definition (13), imply

B3 $\exists$ non-negative constants $\hat{\tau}_*$ and $\hat{\tau}^*$ such that $\hat{\tau}_* \leq \hat{\tau}_{bf} \leq \hat{\tau}^*$.

Moreover, we have implicitly assumed that the Galerkin approximations exist and are unique.

4.1. Error Estimate. In order to obtain an error estimate for $(\xi - \Xi)$ and $(\boldsymbol{q} - \mathcal{Q})$, we must first obtain an estimate on the affine error terms $(\Xi - \widetilde{\xi})$ and $(\mathcal{Q} - \widetilde{\boldsymbol{q}})$. Then, with standard approximation results and with the estimate on the affine error to be obtained in the sequel, an application of the triangle inequality will yield an estimate for $(\xi - \Xi)$ and $(\boldsymbol{q} - \mathcal{Q})$.

It will be useful to employ the following expansion of the advective terms in (7)–(9), at time $t = t^k$,

$$\nabla\cdot\left\{\frac{1}{H}\boldsymbol{q}^2\right\} = \left(\frac{\boldsymbol{q}}{H}\cdot\nabla\boldsymbol{q}\right) + (\nabla\cdot\boldsymbol{q})\frac{\boldsymbol{q}}{H} - (\nabla H\cdot\boldsymbol{q})\frac{\boldsymbol{q}}{H^2}.$$

Similarly, the expansion of the advective terms in (14)–(16) gives, at time $t = t^k$,

$$\nabla\cdot\left\{\frac{1}{\Pi}\mathcal{Q}^2\right\} = \left(\frac{\mathcal{Q}}{\Pi}\cdot\nabla\mathcal{Q}\right) + (\nabla\cdot\mathcal{Q})\frac{\mathcal{Q}}{\Pi} - (\nabla\Pi\cdot\mathcal{Q})\frac{\mathcal{Q}}{\Pi^2}.$$

Subtract (7) from (14), (8) from (15), and (9) from (16), using the fact that we can write

$$\frac{\mathcal{Q}}{\Pi}\cdot\nabla(\mathcal{Q} - \widetilde{\boldsymbol{q}}) - \left(\frac{\boldsymbol{q}}{H}\cdot\nabla\boldsymbol{q}\right) + \left(\frac{\mathcal{Q}}{\Pi}\cdot\nabla\widetilde{\boldsymbol{q}}\right) = \left(\frac{\mathcal{Q}}{\Pi}\cdot\nabla\boldsymbol{\chi}\right) - \left(\frac{\boldsymbol{q}}{H}\cdot\nabla\boldsymbol{\phi}\right) - \left(\left[\frac{\boldsymbol{q}}{H} - \frac{\mathcal{Q}}{\Pi}\right]\cdot\nabla\widetilde{\boldsymbol{q}}\right);$$

$$\nabla\cdot(\mathcal{Q} - \widetilde{\boldsymbol{q}})\frac{\mathcal{Q}}{\Pi} - (\nabla\cdot\boldsymbol{q})\,\frac{\boldsymbol{q}}{H} + (\nabla\cdot\widetilde{\boldsymbol{q}})\,\frac{\mathcal{Q}}{\Pi} = (\nabla\cdot\boldsymbol{\chi})\frac{\mathcal{Q}}{\Pi} - (\nabla\cdot\boldsymbol{\phi})\frac{\boldsymbol{q}}{H} - (\nabla\cdot\widetilde{\boldsymbol{q}})\left[\frac{\boldsymbol{q}}{H} - \frac{\mathcal{Q}}{\Pi}\right];$$

$$\begin{aligned}\left(\nabla(\Xi - \widetilde{\xi})\cdot\mathcal{Q}\right)\frac{\mathcal{Q}}{\Pi^2} - (\nabla\xi\cdot\boldsymbol{q})\frac{\boldsymbol{q}}{H^2} + (\nabla\widetilde{\xi}\cdot\mathcal{Q})\frac{\mathcal{Q}}{\Pi^2}& \\ = (\nabla\psi\cdot\mathcal{Q})\,\frac{\mathcal{Q}}{\Pi^2} - (\nabla\theta\cdot\boldsymbol{q})\frac{\boldsymbol{q}}{H^2} - \nabla\widetilde{\xi}\cdot\left\{\left[\frac{\boldsymbol{q}}{H}\right]^2 - \left[\frac{\mathcal{Q}}{\Pi}\right]^2\right\};&\end{aligned}$$

and

$$\Pi\nabla(\Xi - \widetilde{\xi}) - H\nabla\xi + \Pi\nabla\widetilde{\xi} = \Pi\nabla\psi - H\nabla\theta - \theta\nabla\widetilde{\xi} + \psi\nabla\widetilde{\xi}.$$

Moreover, we obtain, using Taylor's Theorem with integral remainder, the following truncation-in-time terms:

$$\delta_0^0 = {\xi_{tt}}^0 + \frac{2}{\Delta t}\left(\xi_1 - \partial_{t^b}\xi^1\right) = -\frac{1}{(\Delta t)^2}\int_0^{\Delta t}(\Delta t - s)^2\xi_{ttt}\,\mathrm{d}s;$$

$$\begin{aligned}
\delta_0^{k\geq 1} &= {\xi_{tt}}^k - \partial_{t^c}^2 \xi^k = -\frac{1}{6(\Delta t)^2}\left\{\int_{t^{k-1}}^{t^k}(s-t^{k-1})^3\xi_{tttt}\ \mathrm{d}s + \int_{t^k}^{t^{k+1}}(t^{k+1}-s)^3\xi_{tttt}\ \mathrm{d}s\right\};\\
\delta_1^k &= {\xi_t}^k - \partial_{t^f}\xi^k = -\frac{1}{\Delta t}\int_{t^k}^{t^{k+1}}(t^{k+1}-s)\xi_{tt}\ \mathrm{d}s;\\
\delta_2^k &= -(\nabla\xi^{k+1}-\nabla\xi^k) = -\int_{t^k}^{t^{k+1}}\nabla\xi_t\ \mathrm{d}s;\\
\boldsymbol{\varepsilon}_1^k &= {\boldsymbol{q}_t}^k - \partial_{t^f}\boldsymbol{q}^k = -\frac{1}{\Delta t}\int_{t^k}^{t^{k+1}}(t^{k+1}-s)\boldsymbol{q}_{tt}\ \mathrm{d}s;\\
\boldsymbol{\varepsilon}_2^k &= -(\boldsymbol{q}^{k+1}-\boldsymbol{q}^k) = -\int_{t^k}^{t^{k+1}}\boldsymbol{q}_t\ \mathrm{d}s.
\end{aligned}$$

Consequently, we obtain the following GWCE-CME error equations:

(23)
$$\begin{aligned}
&\left(\partial_{t^c}^2\psi^k, v\right) + \tau_\circ\left(\partial_{t^f}\psi^k, v\right) + \left(\frac{\boldsymbol{\mathcal{Q}}^k}{\Pi^k}\cdot\nabla\boldsymbol{\chi}^k, \nabla v\right) + \left((\nabla\cdot\boldsymbol{\chi}^k)\frac{\boldsymbol{\mathcal{Q}}^k}{\Pi^k}, \nabla v\right)\\
&+\left((\nabla\psi^k\cdot\boldsymbol{\mathcal{Q}}^k)\frac{\boldsymbol{\mathcal{Q}}^k}{(\Pi^k)^2}, \nabla v\right) - \left((\tau_\circ - {\hat\tau_{bf}}^k)\boldsymbol{\chi}^k, \nabla v\right) + \left(f_c\boldsymbol{k}\times\boldsymbol{\chi}^k, \nabla v\right)\\
&+\left(h_b\nabla\psi^{k+\frac{1}{2}}, \nabla v\right) + \left(\Xi^k\nabla\psi^k, \nabla v\right) + \mu\left(\partial_{t^f}\nabla\psi^k, \nabla v\right) + \left(\psi^k\boldsymbol{\mathcal{F}}^k, \nabla v\right)\\
&= \left(\frac{\boldsymbol{q}^k}{H^k}\cdot\nabla\boldsymbol{\phi}^k, \nabla v\right) + \left(\left[\frac{\boldsymbol{q}^k}{H^k} - \frac{\boldsymbol{\mathcal{Q}}^k}{\Pi^k}\right]\cdot\nabla\widetilde{\boldsymbol{q}}^k, \nabla v\right) + \left((\nabla\cdot\boldsymbol{\phi}^k)\frac{\boldsymbol{q}^k}{H^k}, \nabla v\right)\\
&+\left(\left[\frac{\boldsymbol{q}^k}{H^k} - \frac{\boldsymbol{\mathcal{Q}}^k}{\Pi^k}\right](\nabla\cdot\widetilde{\boldsymbol{q}}^k), \nabla v\right) + \left(\nabla h_b\cdot\left\{\left(\frac{\boldsymbol{q}^k}{H^k}\right)^2 - \left(\frac{\boldsymbol{\mathcal{Q}}^k}{\Pi^k}\right)^2\right\}, \nabla v\right)\\
&+\left((\nabla\theta^k\cdot\boldsymbol{q}^k)\frac{\boldsymbol{q}^k}{(H^k)^2}, \nabla v\right) + \left(\nabla\widetilde{\xi}^k\cdot\left\{\left(\frac{\boldsymbol{q}^k}{H^k}\right)^2 - \left(\frac{\boldsymbol{\mathcal{Q}}^k}{\Pi^k}\right)^2\right\}, \nabla v\right)\\
&-\left((\tau_\circ - {\tau_{bf}}^k)\boldsymbol{\phi}^k, \nabla v\right) + \left(f_c\boldsymbol{k}\times\boldsymbol{\phi}^k, \nabla v\right) - \left(({\hat\tau_{bf}}^k - {\tau_{bf}}^k)\widetilde{\boldsymbol{q}}^k, \nabla v\right)\\
&+\left((H\nabla\theta)^{k+\frac{1}{2}}, \nabla v\right) + \left(\xi^k\nabla\theta^k, \nabla v\right) + \left(\theta^k\nabla\widetilde{\xi}^k, \nabla v\right) - \left(\psi^k\nabla\widetilde{\xi}^k, \nabla v\right)\\
&+\mu\left(\partial_{t^f}\nabla\theta^k, \nabla v\right) + \left(\theta^k\boldsymbol{\mathcal{F}}^k, \nabla v\right) + \left(\delta_0^k, v\right) + \tau_\circ\left(\delta_1^k, v\right)\\
&+\frac{1}{2}\left(gh_b\delta_2^k, \nabla v\right) + \mu\left(\nabla\delta_1^k, \nabla v\right), \qquad \forall v\in\mathcal{S}^h(\Omega),\ k\geq 1;
\end{aligned}$$

(24)
$$\begin{aligned}
&\left(\partial_{t^b}\boldsymbol{\chi}^{k+1}, \boldsymbol{w}\right) + \left(\frac{\boldsymbol{\mathcal{Q}}^k}{\Pi^k}\cdot\nabla\boldsymbol{\chi}^k, \boldsymbol{w}\right) + \left((\nabla\cdot\boldsymbol{\chi}^k)\frac{\boldsymbol{\mathcal{Q}}^k}{\Pi^k}, \boldsymbol{w}\right) + \left((\nabla\psi^k\cdot\boldsymbol{\mathcal{Q}}^k)\frac{\boldsymbol{\mathcal{Q}}^k}{(\Pi^k)^2}, \boldsymbol{w}\right)\\
&+\left({\hat\tau_{bf}}^k\boldsymbol{\chi}^{k+\frac{1}{2}}, \boldsymbol{w}\right) + \left(f_c\boldsymbol{k}\times\boldsymbol{\chi}^{k+\frac{1}{2}}, \boldsymbol{w}\right) + \left(\Pi^k\nabla\psi^k, \boldsymbol{w}\right)\\
&+\mu\left(\beta_1\nabla\boldsymbol{\chi}^{k+1} + \beta_2\nabla\boldsymbol{\chi}^k, \nabla\boldsymbol{w}\right) + \left(\psi^k\boldsymbol{\mathcal{F}}^k, \boldsymbol{w}\right)\\
&= \left(\frac{\boldsymbol{q}^k}{H^k}\cdot\nabla\boldsymbol{\phi}^k, \boldsymbol{w}\right) + \left(\left[\frac{\boldsymbol{q}^k}{H^k} - \frac{\boldsymbol{\mathcal{Q}}^k}{\Pi^k}\right]\cdot\nabla\widetilde{\boldsymbol{q}}^k, \boldsymbol{w}\right) + \left((\nabla\cdot\boldsymbol{\phi}^k)\frac{\boldsymbol{q}^k}{H^k}, \boldsymbol{w}\right)\\
&+\left(\left[\frac{\boldsymbol{q}^k}{H^k} - \frac{\boldsymbol{\mathcal{Q}}^k}{\Pi^k}\right](\nabla\cdot\widetilde{\boldsymbol{q}}^k), \boldsymbol{w}\right) + \left(\nabla h_b\cdot\left\{\left(\frac{\boldsymbol{q}^k}{H^k}\right)^2 - \left(\frac{\boldsymbol{\mathcal{Q}}^k}{\Pi^k}\right)^2\right\}, \boldsymbol{w}\right)
\end{aligned}$$

$$
\begin{aligned}
&+\left((\nabla\theta^k\cdot\boldsymbol{q}^k)\frac{\boldsymbol{q}^k}{(H^k)^2},\boldsymbol{w}\right)+\left(\nabla\widetilde{\xi}^k\cdot\left\{\left(\frac{\boldsymbol{q}^k}{H^k}\right)^2-\left(\frac{\mathcal{Q}^k}{\Pi^k}\right)^2\right\},\boldsymbol{w}\right)\\
&+\left({\tau_{bf}}^k\boldsymbol{\phi}^{k+\frac{1}{2}},\boldsymbol{w}\right)+\left(f_c\boldsymbol{k}\times\boldsymbol{\phi}^{k+\frac{1}{2}},\boldsymbol{w}\right)-\left(({\hat{\tau}_{bf}}^k-{\tau_{bf}}^k)\widetilde{\boldsymbol{q}}^{k+\frac{1}{2}},\boldsymbol{w}\right)\\
&+\left(H^k\nabla\theta^k,\boldsymbol{w}\right)+\left(\theta^k\nabla\widetilde{\xi}^k,\boldsymbol{w}\right)-\left(\psi^k\nabla\widetilde{\xi}^k,\boldsymbol{w}\right)\\
&+\mu\left(\beta_1\nabla\boldsymbol{\phi}^{k+1}+\beta_2\nabla\boldsymbol{\phi}^k,\nabla\boldsymbol{w}\right)+\left(\theta^k\boldsymbol{\mathcal{F}}^k,\boldsymbol{w}\right)+\left(\varepsilon_1^k,\boldsymbol{w}\right)\\
&+\frac{1}{2}\left({\tau_{bf}}^k\boldsymbol{\varepsilon}_2^k+f_c\boldsymbol{k}\times\boldsymbol{\varepsilon}_2^k,\boldsymbol{w}\right)+\mu\beta_1\left(\nabla\boldsymbol{\varepsilon}_2^k,\nabla\boldsymbol{w}\right),\quad\forall\boldsymbol{w}\in\boldsymbol{\mathcal{S}}^h(\Omega),\ k\geq 0.
\end{aligned}
$$

For reasons that will be clear later in the proof, we write (23) separately at $k=0$ as follows:

$$
\begin{aligned}
(25)&\left(\frac{2+\tau_\circ\Delta t}{\Delta t}\right)\left(\partial_{t^b}\psi^1,v\right)+\left(h_b\nabla\psi^{\frac{1}{2}},\nabla v\right)+\mu\left(\partial_{t^b}\nabla\psi^1,\nabla v\right)\\
&=\left(\frac{\boldsymbol{q}^0}{H^0}\cdot\nabla\boldsymbol{\phi}^0,\nabla v\right)+\left(\left[\frac{\boldsymbol{q}^0}{H^0}-\frac{\mathcal{Q}^0}{\Pi^0}\right]\cdot\nabla\widetilde{\boldsymbol{q}}^0,\nabla v\right)+\left((\nabla\cdot\boldsymbol{\phi}^0)\frac{\boldsymbol{q}^0}{H^0},\nabla v\right)\\
&+\left(\left[\frac{\boldsymbol{q}^0}{H^0}-\frac{\mathcal{Q}^0}{\Pi^0}\right](\nabla\cdot\widetilde{\boldsymbol{q}}^0),\nabla v\right)+\left(\nabla h_b\cdot\left\{\left(\frac{\boldsymbol{q}^0}{H^0}\right)^2-\left(\frac{\mathcal{Q}^0}{\Pi^0}\right)^2\right\},\nabla v\right)\\
&+\left((\nabla\theta^0\cdot\boldsymbol{q}^0)\frac{\boldsymbol{q}^0}{(H^0)^2},\nabla v\right)+\left(\nabla\widetilde{\xi}^0\cdot\left\{\left(\frac{\boldsymbol{q}^0}{H^0}\right)^2-\left(\frac{\mathcal{Q}^0}{\Pi^0}\right)^2\right\},\nabla v\right)\\
&-\left((\tau_\circ-{\hat{\tau}_{bf}}^0)\boldsymbol{\phi}^0,\nabla v\right)+\left(f_c\boldsymbol{k}\times\boldsymbol{\phi}^0,\nabla v\right)-\left(({\hat{\tau}_{bf}}^0-{\tau_{bf}}^0)\widetilde{\boldsymbol{q}}^0,\nabla v\right)+\left(H^0\nabla\theta^0,\nabla v\right)\\
&+\left(\xi^0\nabla\theta^0,\nabla v\right)+\left(\theta^0\nabla\widetilde{\xi}^0,\nabla v\right)+\mu\left(\partial_{tf}\nabla\theta^0,\nabla v\right)+\left(\theta^0\boldsymbol{\mathcal{F}}^0,\nabla v\right)\\
&+\left(\delta_0^0,v\right)+\tau_\circ\left(\delta_1^0,v\right)+\frac{1}{2}\left(gh_b\delta_2^0,\nabla v\right)+\mu\left(\nabla\delta_1^0,\nabla v\right),\quad\forall v\in\mathcal{S}^h(\Omega);
\end{aligned}
$$

We have used in (25) the fact that, from (10),(18), and (19), $\psi^0=0$, $\psi^{-1}=\psi^1$, $\boldsymbol{\chi}^0=0$.

4.2. Choice of Test Functions. We now choose the test functions employed to obtain the affine error estimate. Let r be a positive constant to be determined. Let $v=v_1^{k+1}=\sum_{j=k+1}^{N}e^{-rj\Delta t}\psi^j\Delta t$ and $v=v_2^{k+1}=\sum_{j=k+1}^{N}e^{-rj\Delta t}\,\partial_{t^b}\psi^j\Delta t$ be the test functions in (23). Let $v=\psi^1$ and $v=\partial_{t^b}\psi^1$ be the two test functions in (25). Let $\boldsymbol{w}=\boldsymbol{\chi}^{k+1}$ be the test function in (24).

First, multiply (23) and (24) by Δt and sum from $k=0$ to $k=(N-1)$ using the test functions above. Then manipulate (25) to derive upper bounds for $\left|\left|\psi^1\right|\right|_{\mathcal{H}^1(\Omega)}$ and for $\left|\left|\partial_{t^b}\psi^1\right|\right|_{\mathcal{H}^1(\Omega)}$.

Finally, after manipulating appropriate terms terms in (23) and (24), using the derived bounds on $\left|\left|\psi^1\right|\right|_{\mathcal{H}^1(\Omega)}$ and on $\left|\left|\partial_{t^b}\psi^1\right|\right|_{\mathcal{H}^1(\Omega)}$, adding the resulting inequalities, applying a generalized discrete Gronwall's inequality (GDGI), and taking bounds above and below, we will obtain a relation giving an estimate of the affine error.

REMARK: In order to complete our estimate, we will need to define a function

$$
\Lambda^k=e^{-rk\Delta t}\left|\left|\psi^k\right|\right|^2+\left|\left|\sum_{j=0}^{k}e^{-rj\Delta t}\psi^j\Delta t\right|\right|_{\mathcal{H}^1(\Omega)}^2+\left|\left|\boldsymbol{\chi}^k\right|\right|^2.
$$

Moreover, we will need to add $\left|\left|\sqrt{gh_b}\, v_1^0\right|\right|^2$ to both sides of the inequality obtained from (23), using $v = v_1^{k+1}$, after multiplying by Δt and summing over k. We will then, after some work, obtain Λ^N in which $\sum_{k=0}^{N} \Lambda^k \Delta t$ will be hidden in the spirit of the GDGI (see Heywood and Rannacher [4]).

4.3. Bounding the first time-step of the GWCE Error Equation. First, use $v = \psi^1$ in (25) to obtain

$$\left(\frac{2+\tau_\circ \Delta t}{2\Delta t^2}\right)\left|\left|\psi^1\right|\right|^2 + \frac{\gamma_*}{2}\left|\left|\nabla\psi^1\right|\right|^2 + \frac{\mu}{2\Delta t}\left|\left|\nabla\psi^1\right|\right|^2 \quad \leq \quad \mathcal{P}_1 + \cdots \mathcal{P}_{19}. \tag{26}$$

Here, $\mathcal{P}_i$ denotes a projection error term. Terms $\mathcal{P}_1 - \mathcal{P}_9$ and $\mathcal{P}_{11} - \mathcal{P}_{15}$ are treated similarly to the corresponding terms in continuous time (using test function v_1 with $r = 0$) investigated thoroughly in [1]. We now investigate the remaining terms.

In estimating $\mathcal{P}_{10}$, recall (2) and use Lemma 4.2 and Lemma 4.3 in [1] to obtain

$$\begin{aligned}
\mathcal{P}_{10} \;&=\; \left(({\hat{\tau}_{bf}}^{\,0} - {\tau_{bf}}^{0})\widetilde{\boldsymbol{q}}^0, \nabla\psi^1\right) \\
&=\; c_f\left(\left[\frac{\left|\left|\mathcal{Q}^0/\Pi^0\right|\right|_{\ell^2}}{\Pi^0} - \frac{\left|\left|\boldsymbol{q}^0/H^0\right|\right|_{\ell^2}}{H^0}\right]\widetilde{\boldsymbol{q}}^0, \nabla\psi^1\right) \\
&=\; c_f\left(\left[\frac{H^0\left|\left|\mathcal{Q}^0/\Pi^0\right|\right|_{\ell^2} - \Pi^0\left|\left|\boldsymbol{q}^0/H^0\right|\right|_{\ell^2}}{\Pi^0 H^0}\right]\widetilde{\boldsymbol{q}}^0, \nabla\psi^1\right) \\
&=\; c_f\left(\left[\frac{(H^0-\Pi^0)\left|\left|\mathcal{Q}^0/\Pi^0\right|\right|_{\ell^2}}{\Pi^0 H^0}\right.\right. \\
&\qquad\qquad \left.\left. + \frac{\Pi^0\left(\left|\left|\mathcal{Q}^0/\Pi^0\right|\right|_{\ell^2} - \left|\left|\boldsymbol{q}^0/H^0\right|\right|_{\ell^2}\right)}{\Pi^0 H^0}\right]\widetilde{\boldsymbol{q}}^0, \nabla\psi^1\right) \\
&\leq\; c_f\left(\left[\frac{(\psi^0-\theta^0)\left|\left|\mathcal{Q}^0\right|\right|_{\ell^2}}{(\Pi^0)^2 H^0}\right.\right. \\
&\qquad\qquad \left.\left. + \frac{(\psi^0-\theta^0)\left|\left|\boldsymbol{q}^0\right|\right|_{\ell^2}}{\Pi^0 (H^0)^2} + \frac{\left|\left|\boldsymbol{\phi}^0\right|\right|_{\ell^2} + \left|\left|\boldsymbol{\chi}^0\right|\right|_{\ell^2}}{\Pi^0 H^0}\right]\widetilde{\boldsymbol{q}}^0, \nabla\psi^1\right) \\
&\leq\; \epsilon\left|\left|\nabla\psi^1\right|\right|^2 + K\left|\left|\theta^0\right|\right|^2 + K\left|\left|\boldsymbol{\phi}^0\right|\right|^2.
\end{aligned}$$

In estimating $\mathcal{P}_{16}$, consider

$$\left|\left|\delta_0^0\right|\right|^2 \quad \leq \quad \frac{(\Delta t)^2}{5}\left|\left|\xi_{ttt}\right|\right|^2_{\mathcal{L}^\infty((0,\Delta t);\mathcal{L}^2(\Omega))}, \tag{27}$$

so that,

$$\mathcal{P}_{16} = \left(\delta_0^0, \psi^1\right) \quad \leq \quad \epsilon\left|\left|\psi^1\right|\right|^2 + K(\Delta t)^2\left|\left|\xi_{ttt}\right|\right|^2_{\mathcal{L}^\infty((0,\Delta t);\mathcal{L}^2(\Omega))}.$$

In estimating $\mathcal{P}_{17}$ and $\mathcal{P}_{19}$, consider, for instance,

$$\left|\left|\delta_1^0\right|\right|^2 \quad \leq \quad \frac{\Delta t^2}{3}\left|\left|\xi_{tt}\right|\right|^2_{\mathcal{L}^\infty((0,\Delta t);\mathcal{L}^2(\Omega))},$$

so that,

$$\begin{aligned}
\mathcal{P}_{17} \;&=\; \tau_\circ\left(\delta_1^0, \psi^1\right) \leq \epsilon\left|\left|\psi^1\right|\right|^2 + K(\Delta t)^2\left|\left|\xi_{tt}\right|\right|^2_{\mathcal{L}^\infty((0,\Delta t);\mathcal{L}^2(\Omega))}, \\
\mathcal{P}_{19} \;&=\; \mu\left(\nabla\delta_1^0, \nabla\psi^1\right) \leq \epsilon\left|\left|\nabla\psi^1\right|\right|^2 + K(\Delta t)^2\left|\left|\nabla\xi_{tt}\right|\right|^2_{\mathcal{L}^\infty((0,\Delta t);\mathcal{L}^2(\Omega))}.
\end{aligned}$$

Finally, in estimating $\mathcal{P}_{18}$, consider

$$\left|\left|\delta_2^0\right|\right|^2 \quad \leq \quad \Delta t^2 \left|\left|\nabla \xi_t\right|\right|^2_{\mathcal{L}^\infty((0,\Delta t);\mathcal{L}^2(\Omega))},$$

so that,

$$\mathcal{P}_{18} = \frac{1}{2}\left(h_b \delta_2^0, \psi^1\right) \quad \leq \quad \epsilon \left|\left|\psi^1\right|\right|^2 + K(\Delta t)^2 \left|\left|\nabla \xi_t\right|\right|^2_{\mathcal{L}^\infty((0,\Delta t);\mathcal{L}^2(\Omega))}.$$

Consequently, we observe that the right hand side of (26) is bounded above by

$$\begin{aligned}
&\epsilon_0 \left|\left|\psi^1\right|\right|^2 + \epsilon_1 \left|\left|\nabla\psi^1\right|\right|^2 + K\left|\left|\theta^0\right|\right|^2 + K\left|\left|\nabla\theta^0\right|\right|^2 + K\left|\left|\partial_{t^f}\nabla\theta^0\right|\right|^2 \\
&+K\left|\left|\phi^0\right|\right|^2 + K(\Delta t)^2 \left|\left|\xi_{ttt}\right|\right|^2_{\mathcal{L}^\infty((0,\Delta t);\mathcal{L}^2(\Omega))} + K(\Delta t)^2 \left|\left|\nabla\xi_t\right|\right|^2_{\mathcal{L}^\infty((0,\Delta t);\mathcal{L}^2(\Omega))} \\
&+K(\Delta t)^2 \left|\left|\xi_{tt}\right|\right|^2_{\mathcal{L}^\infty((0,\Delta t);\mathcal{L}^2(\Omega))} + K(\Delta t)^2 \left|\left|\nabla\xi_{tt}\right|\right|^2_{\mathcal{L}^\infty((0,\Delta t);\mathcal{L}^2(\Omega))}.
\end{aligned}$$

Let $\epsilon_0 = \frac{3}{16}\tau_\circ^2$ so that $\Delta t < \frac{8}{\tau_\circ} = \kappa_1$. Now, let $\sigma_0 = \left(\frac{2+\tau_\circ \Delta t}{2\Delta t^2}\right) - \epsilon_0 > 0$ and $\sigma_1 = \frac{\gamma_*}{2} - \epsilon_1 > 0$. Thus,

$$\begin{aligned}
&\sigma_0 \left|\left|\psi^1\right|\right|^2 + \sigma_1 \left|\left|\nabla\psi^1\right|\right|^2 + \frac{\mu}{\Delta t}\left|\left|\nabla\psi^1\right|\right|^2 \\
&\leq \quad K\left|\left|\theta^0\right|\right|^2 + K\left|\left|\nabla\theta^0\right|\right|^2 + K\left|\left|\partial_{t^f}\nabla\theta^0\right|\right|^2 + K\left|\left|\phi^0\right|\right|^2 \\
&\qquad +K(\Delta t)^2 \left|\left|\xi_{ttt}\right|\right|^2_{\mathcal{L}^\infty((0,\Delta t);\mathcal{L}^2(\Omega))} + K(\Delta t)^2 \left|\left|\nabla\xi_t\right|\right|^2_{\mathcal{L}^\infty((0,\Delta t);\mathcal{L}^2(\Omega))} \\
&\qquad +K(\Delta t)^2 \left|\left|\xi_{tt}\right|\right|^2_{\mathcal{L}^\infty((0,\Delta t);\mathcal{L}^2(\Omega))} + K(\Delta t)^2 \left|\left|\nabla\xi_{tt}\right|\right|^2_{\mathcal{L}^\infty((0,\Delta t);\mathcal{L}^2(\Omega))}.
\end{aligned}$$

Finally, we have

$$\left|\left|\psi^1\right|\right|^2_{\mathcal{H}^1(\Omega)} \quad \leq \quad K\left(h^{2(\ell-1)} + (\Delta t)^2\right). \tag{28}$$

Now, use $v = \partial_{t^b}\psi^1$ in (25) to obtain

$$\left(\frac{2+\tau_\circ\Delta t}{\Delta t}\right)\left|\left|\partial_{t^b}\psi^1\right|\right|^2 + \frac{\gamma_*}{2\Delta t}\left|\left|\nabla\psi^1\right|\right|^2 + \mu\left|\left|\partial_{t^b}\nabla\psi^1\right|\right|^2 \quad \leq \quad \mathcal{P}_1 + \cdots \mathcal{P}_{19}. \tag{29}$$

After some work similar to that above, we determine that the right hand side of (29) is bounded above by

$$\begin{aligned}
&\epsilon_2 \left|\left|\partial_{t^b}\psi^1\right|\right|^2 + \epsilon_3 \left|\left|\partial_{t^b}\nabla\psi^1\right|\right|^2 + K\left|\left|\theta^0\right|\right|^2 + K\left|\left|\nabla\theta^0\right|\right|^2 + K\left|\left|\partial_{t^f}\nabla\theta^0\right|\right|^2 \\
&+K\left|\left|\phi^0\right|\right|^2 + K(\Delta t)^2 \left|\left|\xi_{ttt}\right|\right|^2_{\mathcal{L}^\infty((0,\Delta t);\mathcal{L}^2(\Omega))} + K(\Delta t)^2 \left|\left|\nabla\xi_t\right|\right|^2_{\mathcal{L}^\infty((0,\Delta t);\mathcal{L}^2(\Omega))} \\
&+K(\Delta t)^2 \left|\left|\xi_{tt}\right|\right|^2_{\mathcal{L}^\infty((0,\Delta t);\mathcal{L}^2(\Omega))} + K(\Delta t)^2 \left|\left|\nabla\xi_{tt}\right|\right|^2_{\mathcal{L}^\infty((0,\Delta t);\mathcal{L}^2(\Omega))}.
\end{aligned}$$

Let $\sigma_2 = \tau_\circ - \epsilon_2 > 0$ and $\sigma_3 = \mu - \epsilon_3 > 0$. Then,

$$\begin{aligned}
&\left(\frac{2+\sigma_2\Delta t}{\Delta t}\right)\left|\left|\partial_{t^b}\psi^1\right|\right|^2 + \frac{\gamma_*}{2\Delta t}\left|\left|\nabla\psi^1\right|\right|^2 + \sigma_3\left|\left|\partial_{t^b}\nabla\psi^1\right|\right|^2 \\
&\leq \quad K\left|\left|\theta^0\right|\right|^2 + K\left|\left|\nabla\theta^0\right|\right|^2 + K\left|\left|\partial_{t^f}\nabla\theta^0\right|\right|^2 + K\left|\left|\phi^0\right|\right|^2 \\
&\qquad +K(\Delta t)^2 \left|\left|\xi_{ttt}\right|\right|^2_{\mathcal{L}^\infty((0,\Delta t);\mathcal{L}^2(\Omega))} + K(\Delta t)^2 \left|\left|\nabla\xi_t\right|\right|^2_{\mathcal{L}^\infty((0,\Delta t);\mathcal{L}^2(\Omega))} \\
&\qquad +K(\Delta t)^2 \left|\left|\xi_{tt}\right|\right|^2_{\mathcal{L}^\infty((0,\Delta t);\mathcal{L}^2(\Omega))} + K(\Delta t)^2 \left|\left|\nabla\xi_{tt}\right|\right|^2_{\mathcal{L}^\infty((0,\Delta t);\mathcal{L}^2(\Omega))}.
\end{aligned}$$

Finally, we have

$$\left|\left|\partial_{t^b}\psi^1\right|\right|^2_{\mathcal{H}^1(\Omega)} + \left|\left|\nabla\psi^1\right|\right|^2_{\mathcal{L}^2(\Omega)} \quad \leq \quad K\left(h^{2(\ell-1)} + (\Delta t)^2\right). \tag{30}$$

4.4. Bounding the GWCE-Error Equations. Recall that Π^0 and $|\mathcal{Q}^0|$ are bounded above by C^{**} and Π^0 is bounded below by C_{**}. And that we assumed that $C_{**} < \Pi^k < C^{**}$, and $|\mathcal{Q}^{k+1}| < C^{**}$ for $k = 0, \ldots, N-1$.

Using $v = v_1^{k+1}$ as the test function in (23), multiplying by Δt and summing from $k = 0$ to $k = (N-1)$, using the tools in Appendix A, and adding $\left|\left|\sqrt{gh_b}\; v_1^0\right|\right|^2$ to both sides of the resulting inequality yields

$$
\begin{aligned}
(31) \qquad & \frac{1}{2} e^{-rN\Delta t} ||\psi^N||^2 + \left(\tau_\circ + \frac{r}{2} e^{\theta_0}\right) \sum_{k=0}^{N} e^{-rk\Delta t} \left|\left|\psi^k\right|\right|^2 \Delta t \\
& + \frac{r\gamma_*}{2} e^{\theta} \sum_{k=0}^{N-1} e^{rk\Delta t} \left|\left|\nabla v_1^{k+1}\right|\right|^2 \Delta t + \mu \sum_{k=0}^{N} e^{-rk\Delta t} \left|\left|\nabla \psi^k\right|\right|^2 \Delta t \\
& + \frac{1}{2} \sum_{k=0}^{N} e^{-rk\Delta t} \left|\left|\partial_{t^b} \psi^k\right|\right|^2 (\Delta t)^2 + \frac{\gamma_*}{2} \left|\left|\nabla v_1^0\right|\right|^2 + \gamma_* \left|\left|v_1^0\right|\right|^2 \\
& \leq \frac{1}{2} \left|\left|\psi^1\right|\right|^2 - \left(\partial_{t^b} \psi^1, v_1^0\right) - \sum_{k=0}^{N-1} \left(\left(\frac{\mathcal{Q}^k}{\Pi^k} \cdot \nabla \boldsymbol{\chi}^k \right), \nabla v_1^{k+1} \right) \Delta t \\
& - \sum_{k=0}^{N-1} \left((\nabla \cdot \boldsymbol{\chi}^k) \frac{\mathcal{Q}^k}{\Pi^k}, \nabla v_1^{k+1} \right) \Delta t - \sum_{k=0}^{N-1} \left((\nabla \psi^k \cdot \mathcal{Q}^k) \frac{\mathcal{Q}^k}{(\Pi^k)^2}, \nabla v_1^{k+1} \right) \Delta t \\
& + \sum_{k=0}^{N-1} \left((\tau_\circ - \hat{\tau}_{bf}{}^k) \boldsymbol{\chi}^k, \nabla v_1^{k+1} \right) \Delta t - \sum_{k=0}^{N-1} \left(f_c \boldsymbol{k} \times \boldsymbol{\chi}^k, \nabla v_1^{k+1} \right) ddt \\
& - \sum_{k=0}^{N-1} \left(\Xi^k \nabla \psi^{k+\frac{1}{2}}, \nabla v_1^{k+1} \right) \Delta t - \sum_{k=0}^{N-1} \left(\psi^k \boldsymbol{\mathcal{F}}^k, \nabla v_1^{k+1} \right) \Delta t \\
& + \sum_{k=0}^{N-1} \left(\left(\frac{\boldsymbol{q}^k}{H^k} \cdot \nabla \boldsymbol{\phi}^k \right), \nabla v_1^{k+1} \right) \Delta t + \sum_{k=0}^{N-1} \left(\left[\frac{\boldsymbol{q}^k}{H^k} - \frac{\mathcal{Q}^k}{\Pi^k} \right] \cdot \nabla \widetilde{\boldsymbol{q}}^k, \nabla v_1^{k+1} \right) \Delta t \\
& + \sum_{k=0}^{N-1} \left((\nabla \cdot \boldsymbol{\phi}^k) \frac{\boldsymbol{q}^k}{H^k}, \nabla v_1^{k+1} \right) \Delta t + \sum_{k=0}^{N-1} \left((\nabla \cdot \widetilde{\boldsymbol{q}}^k) \left[\frac{\boldsymbol{q}^k}{H^k} - \frac{\mathcal{Q}^k}{\Pi^k} \right], \nabla v_1^{k+1} \right) \Delta t \\
& + \sum_{k=0}^{N-1} \left(\nabla h_b \cdot \left\{ \left[\frac{\boldsymbol{q}^k}{H^k} \right]^2 - \left[\frac{\mathcal{Q}^k}{\Pi^k} \right]^2 \right\}, \nabla v_1^{k+1} \right) \Delta t + \sum_{k=0}^{N-1} \left((\nabla \theta^k \cdot \boldsymbol{q}^k) \frac{\boldsymbol{q}^k}{(H^k)^2}, \nabla v_1^{k+1} \right) \Delta t \\
& + \sum_{k=0}^{N-1} \left(\nabla \widetilde{\xi}^k \cdot \left\{ \left[\frac{\boldsymbol{q}^k}{H^k} \right]^2 - \left[\frac{\mathcal{Q}^k}{\Pi^k} \right]^2 \right\}, \nabla v_1^{k+1} \right) \Delta t - \sum_{k=0}^{N-1} \left((\tau_\circ - \tau_{bf}{}^k) \boldsymbol{\phi}^k, \nabla v_1^{k+1} \right) \Delta t \\
& + \sum_{k=0}^{N-1} \left(f_c \boldsymbol{k} \times \boldsymbol{\phi}^k, \nabla v_1^{k+1} \right) \Delta t - \sum_{k=0}^{N-1} \left((\hat{\tau}_{bf}{}^k - \tau_{bf}{}^k) \widetilde{\boldsymbol{q}}^k, \nabla v_1^{k+1} \right) \Delta t \\
& + \sum_{k=0}^{N-1} \left(H^k \nabla \theta^{k+\frac{1}{2}}, \nabla v_1^{k+1} \right) \Delta t + \sum_{k=0}^{N-1} \left(\xi^k \nabla \theta^k, \nabla v_1^{k+1} \right) \Delta t + \sum_{k=0}^{N-1} \left(\theta^k \nabla \widetilde{\xi}^k, \nabla v_1^{k+1} \right) \Delta t \\
& - \sum_{k=0}^{N-1} \left(\psi^k \nabla \widetilde{\xi}^k, \nabla v_1^{k+1} \right) \Delta t + \mu \sum_{k=0}^{N-1} \left(\partial_{t^f} \nabla \theta^k, \nabla v_1^{k+1} \right) \Delta t + \sum_{k=0}^{N-1} \left(\theta^k \boldsymbol{\mathcal{F}}^k, \nabla v_1^{k+1} \right) \Delta t \\
& + \sum_{k=0}^{N-1} \left(\delta_0^k, v_1^{k+1} \right) \Delta t + \tau_\circ \sum_{k=0}^{N-1} \left(\delta_1^k, v_1^{k+1} \right) \Delta t + \frac{1}{2} \sum_{k=0}^{N-1} \left(g h_b \delta_2^k, \nabla v_1^{k+1} \right) \Delta t
\end{aligned}
$$

$$+\mu \sum_{k=0}^{N-1} \left(\nabla \delta_1^k, \nabla v_1^{k+1}\right) \Delta t + \gamma^* \left|\left|v_1^0\right|\right|^2$$
$$= \bar{S}_1 + \bar{S}_2 + \left(\bar{\mathcal{A}}_1 + \cdots + \bar{\mathcal{A}}_7\right) + \left(\bar{\mathcal{P}}_1 + \cdots + \bar{\mathcal{P}}_{21}\right).$$

Here, $\bar{\mathcal{A}}_i$ denotes an affine error term and $\bar{S}_i$ denotes a term resulting from summation by parts. Terms $\bar{\mathcal{A}}_1 - \bar{\mathcal{A}}_7, \bar{\mathcal{P}}_1 - \bar{\mathcal{P}}_9,$ and $\bar{\mathcal{P}}_{11} - \bar{\mathcal{P}}_{16}$ are treated similarly to their continuous-time analogs which are investigated thoroughly in [1]. Again, we investigate the remaining terms.

The treatment of terms $\bar{S}_1$ and $\bar{S}_2$ is straightforward. From (28) and (30),

$$\begin{aligned} \bar{S}_1 &= \frac{1}{2}\left|\left|\psi^1\right|\right|^2 \le K\left(h^{2(\ell-1)} + \Delta t^2\right); \\ \bar{S}_2 &= -\left(\partial_{t^b}\psi^1, v_1^0\right) \le K\left(h^{2(\ell-1)} + \Delta t^2\right) + \epsilon\left|\left|v_1^0\right|\right|^2. \end{aligned}$$

The treatment of $\bar{\mathcal{P}}_{10}$ is similar to that given in the previous section:

$$\begin{aligned} \bar{\mathcal{P}}_{10} &= \sum_{k=0}^{N-1}\left(\left(\hat{\tau}_{bf}{}^k - \tau_{bf}{}^k\right)\widetilde{\boldsymbol{q}}^k, \nabla v_1^{k+1}\right)\Delta t \\ &\le c_f \sum_{k=0}^{N-1}\left(\left[\frac{\left(\psi^k - \theta^k\right)\left|\left|\mathcal{Q}^k\right|\right|_{\ell^2}}{(\Pi^k)^2 H^k}\right.\right. \\ &\qquad \left.\left. + \frac{\left(\psi^k - \theta^k\right)\left|\left|\boldsymbol{q}^k\right|\right|_{\ell^2}}{\Pi^k (H^k)^2} + \frac{\left|\left|\boldsymbol{\phi}^k\right|\right|_{\ell^2} + \left|\left|\boldsymbol{\chi}^k\right|\right|_{\ell^2}}{\Pi^k H^k}\right]\widetilde{\boldsymbol{q}}^k, \nabla v_1^{k+1}\right)\Delta t \\ &\le \epsilon \sum_{k=0}^{N-1} e^{-rk\Delta t}\left|\left|\psi^k\right|\right|^2 \Delta t + K\sum_{k=0}^{N-1}\left|\left|\theta^k\right|\right|^2 \Delta t + K\sum_{k=0}^{N-1}\left|\left|\boldsymbol{\chi}^{k+1}\right|\right|^2 \Delta t \\ &\quad + K\sum_{k=0}^{N-1}\left|\left|\boldsymbol{\phi}^k\right|\right|^2 \Delta t + K\sum_{k=0}^{N-1} e^{rk\Delta t}\left|\left|\nabla v_1^{k+1}\right|\right|^2 \Delta t. \end{aligned}$$

In estimating $\bar{\mathcal{P}}_{17}$, consider

$$\left|\left|\delta_0^{k\ge 1}\right|\right|^2 \le \frac{(\Delta t)^3}{136}\left\{\int_{t^k}^{t^{k+1}} \left|\left|\xi_{tttt}\right|\right|^2 \,\mathrm{d}s + \int_{t^{k-1}}^{t^k}\left|\left|\xi_{tttt}\right|\right|^2 \,\mathrm{d}s\right\}.$$

Noting that $v_1^{k+1} = v_1^0 - \sum_{j=0}^k e^{-rt\Delta t}\psi^k \Delta t$ and using (27), we obtain

$$\begin{aligned} \bar{\mathcal{P}}_{17} &= \sum_{k=0}^{N-1}\left(\delta_0^k, v_1^{k+1}\right)\Delta t \le \epsilon\left|\left|v_1^0\right|\right|^2 + K\sum_{k=0}^{N-1}\left|\left|\sum_{j=0}^{k} e^{-rj\Delta t}\psi^k \Delta t\right|\right|^2 \Delta t, \\ &\quad + K(\Delta t)^4 \int_0^T \left|\left|\xi_{tttt}\right|\right|^2 \,\mathrm{d}s + K(\Delta t)^3 \left|\left|\xi_{ttt}\right|\right|^2_{\mathcal{L}^\infty((0,\Delta t);\mathcal{L}^2(\Omega))}. \end{aligned}$$

In estimating $\bar{\mathcal{P}}_{18}$ and $\bar{\mathcal{P}}_{20}$, consider, for instance,

$$\begin{aligned} \delta_1^k &\le \left|-\frac{1}{\Delta t}\right|\left(\int_{t^k}^{t^{k+1}} (t^{k+1} - s)^2 \,\mathrm{d}s\right)^{\frac{1}{2}}\left(\int_{t^k}^{t^{k+1}} (\xi_{tt})^2 \,\mathrm{d}s\right)^{\frac{1}{2}} \\ &= \left(\frac{\Delta t}{3}\right)^{1/2}\left(\int_{t^k}^{t^{k+1}} (\xi_{tt})^2 \,\mathrm{d}s\right)^{\frac{1}{2}}, \end{aligned}$$

so that

$$\left\|\delta_1^k\right\|^2 \quad \le \quad \frac{\Delta t}{3}\int_{t^k}^{t^{k+1}} \|\xi_{tt}\|^2 \, \mathrm{d}s \ .$$

Thus,

$$\begin{aligned}
\bar{\mathcal{P}}_{18} &= \tau_\circ \sum_{k=0}^{N-1} \left(\delta_1^k, v_1^{k+1}\right)\Delta t \\
&\le \epsilon \left\|v_1^0\right\|^2 + K\sum_{k=0}^{N-1}\left\|\sum_{j=0}^{k} e^{-rj\Delta t}\psi^k \Delta t\right\|^2 \Delta t + K(\Delta t)^2 \int_0^T \|\xi_{tt}\|^2 \, \mathrm{d}s; \\
\bar{\mathcal{P}}_{20} &= \mu \sum_{k=0}^{N-1} \left(\nabla\delta_1^k, \nabla v_1^{k+1}\right)\Delta t \\
&\le \epsilon \left\|\nabla v_1^0\right\|^2 + K\sum_{k=0}^{N-1}\left\|\sum_{j=0}^{k} e^{-rj\Delta t}\nabla\psi^k \Delta t\right\|^2 \Delta t + K(\Delta t)^2 \int_0^T \|\nabla\xi_{tt}\|^2 \, \mathrm{d}s.
\end{aligned}$$

The treatment of $\bar{\mathcal{P}}_{19}$ follows similarly

$$\begin{aligned}
\bar{\mathcal{P}}_{19} = \frac{1}{2}\sum_{k=0}^{N-1}\left(gh_b\delta_2^k, \nabla v_1^{k+1}\right)\Delta t \quad &\le \quad \epsilon\left\|\nabla v_1^0\right\|^2 + K\sum_{k=0}^{N-1}\left\|\sum_{j=0}^{k} e^{-rj\Delta t}\nabla\psi^k\Delta t\right\|^2 \Delta t \\
&\quad + K(\Delta t)^2\int_0^T \|\nabla\xi_t\|^2 \, \mathrm{d}s.
\end{aligned}$$

Finally, algebraic manipulation yields

$$\begin{aligned}
\bar{\mathcal{P}}_{21} = \gamma^*\left\|v_1^0\right\|^2 \quad &= \quad \gamma^*\left(\sum_{k=0}^{N} e^{-rk\Delta t}\psi^k\Delta t, \sum_{k=0}^{N} e^{-rk\Delta t}\psi^k\Delta t\right) \\
&\le \quad \epsilon\left\|\sum_{k=0}^{N} e^{-rk\Delta t}\psi^k\Delta t\right\|^2 + K\sum_{k=0}^{N}\left\|e^{-rk\Delta t}\psi^k\right\|^2\Delta t \\
&= \quad \epsilon\left\|v_1^0\right\|^2 + K\sum_{k=0}^{N}\left\|e^{-rk\Delta t}\psi^k\right\|^2\Delta t.
\end{aligned}$$

Using $v = v_2^{k+1}$ as the test function in (23), multiplying by Δt and summing from $k = 0$ to $k = (N-1)$, and using the relations above yields

$$\begin{aligned}
(32) \quad & \sum_{k=0}^{N} e^{-rk\Delta t}\left\|\partial_{t^b}\psi^k\right\|^2\Delta t + \frac{\tau_\circ}{2}\sum_{k=0}^{N} e^{-rk\Delta t}\left\|\partial_{t^b}\psi^k\right\|^2(\Delta t)^2 \\
& + \frac{\mu}{2}\sum_{k=0}^{N} e^{-rk\Delta t}\left\|\partial_{t^b}\nabla\psi^k\right\|^2(\Delta t)^2 + \frac{\tau_\circ}{2}\left\|v_2^0\right\|^2 + \frac{\mu}{2}\left\|\nabla v_2^0\right\|^2 \\
& + \frac{r\tau_\circ}{2}e^{\theta}\sum_{k=0}^{N} e^{rk\Delta t}\left\|v_2^{k+1}\right\|^2\Delta t + \frac{r\mu}{2}e^{\theta}\sum_{k=0}^{N} e^{rk\Delta t}\left\|\nabla v_2^{k+1}\right\|^2\Delta t \\
& \le -\left(\partial_{t^b}\psi^1, v_2^0\right) - \tau_\circ\left(\partial_{t^b}\psi^1, v_2^0\right)\Delta t - \mu\left(\partial_{t^b}\nabla\psi^1, \nabla v_2^0\right)\Delta t
\end{aligned}$$

$$
\begin{aligned}
&- \sum_{k=0}^{N-1} \left(\left(\frac{\mathcal{Q}^k}{\Pi^k} \cdot \nabla \boldsymbol{\chi}^k \right), \nabla v_2^{k+1} \right) \Delta t - \sum_{k=0}^{N-1} \left((\nabla \cdot \boldsymbol{\chi}^k) \frac{\mathcal{Q}^k}{\Pi^k}, \nabla v_2^{k+1} \right) \Delta t \\
&- \sum_{k=0}^{N-1} \left((\nabla \psi^k \cdot \mathcal{Q}^k) \frac{\mathcal{Q}^k}{(\Pi^k)^2}, \nabla v_2^{k+1} \right) \Delta t + \sum_{k=0}^{N-1} \left((\tau_\circ - \hat{\tau}_{bf}{}^k) \boldsymbol{\chi}^k, \nabla v_2^{k+1} \right) \Delta t \\
&- \sum_{k=0}^{N-1} \left(f_c \boldsymbol{k} \times \boldsymbol{\chi}^k, \nabla v_2^{k+1} \right) \Delta t - \sum_{k=0}^{N-1} \left(\Pi^k \nabla \psi^{k+\frac{1}{2}}, \nabla v_2^{k+1} \right) \Delta t - \sum_{k=0}^{N-1} \left(\psi^k \boldsymbol{\mathcal{F}}^k, \nabla v_2^{k+1} \right) \Delta t \\
&+ \sum_{k=0}^{N-1} \left(\left(\frac{\boldsymbol{q}^k}{H^k} \cdot \nabla \boldsymbol{\phi}^k \right), \nabla v_2^{k+1} \right) \Delta t + \sum_{k=0}^{N-1} \left(\left[\frac{\boldsymbol{q}^k}{H^k} - \frac{\mathcal{Q}^k}{\Pi^k} \right] \cdot \nabla \widetilde{\boldsymbol{q}}^k, \nabla v_2^{k+1} \right) \Delta t \\
&+ \sum_{k=0}^{N-1} \left((\nabla \cdot \boldsymbol{\phi}) \frac{\boldsymbol{q}^k}{H^k}, \nabla v_2^{k+1} \right) \Delta t + \sum_{k=0}^{N-1} \left((\nabla \cdot \widetilde{\boldsymbol{q}}^k) \left[\frac{\boldsymbol{q}^k}{H^k} - \frac{\mathcal{Q}^k}{\Pi^k} \right], \nabla v_2^{k+1} \right) \Delta t \\
&+ \sum_{k=0}^{N-1} \left(\nabla h_b \cdot \left\{ \left[\frac{\boldsymbol{q}^k}{H^k} \right]^2 - \left[\frac{\mathcal{Q}^k}{\Pi^k} \right]^2 \right\}, \nabla v_2^{k+1} \right) \Delta t + \sum_{k=0}^{N-1} \left((\nabla \theta^k \cdot \boldsymbol{q}^k) \frac{\boldsymbol{q}^k}{(H^k)^2}, \nabla v_2^{k+1} \right) \Delta t \\
&+ \sum_{k=0}^{N-1} \left(\nabla \widetilde{\xi}^k \cdot \left\{ \left[\frac{\boldsymbol{q}^k}{H^k} \right]^2 - \left[\frac{\mathcal{Q}^k}{\Pi^k} \right]^2 \right\}, \nabla v_2^{k+1} \right) \Delta t - \sum_{k=0}^{N-1} \left((\tau_\circ - \tau_{bf}{}^k) \boldsymbol{\phi}^k, \nabla v_2^{k+1} \right) \Delta t \\
&+ \sum_{k=0}^{N-1} \left(f_c \boldsymbol{k} \times \boldsymbol{\phi}^k, \nabla v_2^{k+1} \right) \Delta t - \sum_{k=0}^{N-1} \left((\hat{\tau}_{bf}{}^k - \tau_{bf}{}^k) \widetilde{\boldsymbol{q}}^k, \nabla v_2^{k+1} \right) \Delta t \\
&+ \sum_{k=0}^{N-1} \left(H^k \nabla \theta^{k+\frac{1}{2}}, \nabla v_2^{k+1} \right) \Delta t + \sum_{k=0}^{N-1} \left(\xi^k \nabla \theta^k, \nabla v_2^{k+1} \right) \Delta t + \sum_{k=0}^{N-1} \left(\theta^k \nabla \widetilde{\xi}^k, \nabla v_2^{k+1} \right) \Delta t \\
&- \sum_{k=0}^{N-1} \left(\psi^k \nabla \widetilde{\xi}^k, \nabla v_2^{k+1} \right) \Delta t + \mu \sum_{k=0}^{N-1} \left(\partial_{tf} \nabla \theta^k, \nabla v_2^{k+1} \right) \Delta t + \sum_{k=0}^{N-1} \left(\theta^k \boldsymbol{\mathcal{F}}^k, \nabla v_2^{k+1} \right) \Delta t \\
&+ \sum_{k=0}^{N-1} \left(\delta_0^k, v_2^{k+1} \right) \Delta t + \tau_\circ \sum_{k=0}^{N-1} \left(\delta_1^k, v_2^{k+1} \right) \Delta t + \mu \sum_{k=0}^{N-1} \left(\nabla \delta_1^k, \nabla v_2^{k+1} \right) \Delta t \\
&= \check{S}_1 + \check{S}_2 + \check{S}_3 + \left(\check{\mathcal{A}}_1 + \cdots + \check{\mathcal{A}}_7 \right) + \left(\check{\mathcal{P}}_1 + \cdots + \check{\mathcal{P}}_{19} \right).
\end{aligned}
$$

All of these terms are treated analogously to those in (32) with the exception of $\check{S}_2, \check{S}_3$ and $\check{\mathcal{P}}_{17} - \check{\mathcal{P}}_{19}$ which we detail now.

From (28) and (30),

$$
\begin{aligned}
\check{S}_2 &= -\tau_\circ \left(\partial_{t^b} \psi^1, v_2^0 \right) \Delta t \le \epsilon \left\| v_2^0 \right\| + K \Delta t^2 \left(h^{2(\ell-1)} + \Delta t^2 \right), \\
\check{S}_3 &= -\mu \left(\partial_{t^b} \nabla \psi^1, \nabla v_2^0 \right) \Delta t \le \epsilon \left\| \nabla v_2^0 \right\| + K \Delta t^2 \left(h^{2(\ell-1)} + \Delta t^2 \right).
\end{aligned}
$$

Observe that the treatment of $\check{\mathcal{P}}_{17} - \check{\mathcal{P}}_{19}$ differs slightly from the treatment of related terms in (31) as follows:

$$
\begin{aligned}
\check{\mathcal{P}}_{17} = \sum_{k=0}^{N-1} \left(\delta_0^k, v_2^{k+1} \right) \Delta t \quad &\le \quad K(\Delta t)^4 \int_0^T \|\xi_{tttt}\|^2 \, \mathrm{d}s + K \sum_{k=0}^{N-1} e^{rk\Delta t} \left\| v_2^{k+1} \right\|^2 \Delta t \\
&\quad + K(\Delta t)^3 \|\xi_{ttt}\|^2_{\mathcal{L}^\infty((0,\Delta t);\mathcal{L}^2(\Omega))};
\end{aligned}
$$

$$\check{\mathcal{P}}_{18} = \tau_\circ \sum_{k=0}^{N} \left(\delta_1^k, v_2^{k+1}\right)\Delta t \quad \le \quad K(\Delta t)^2 \int_0^{T+\Delta t} \|\xi_{tt}\|^2 \, \mathrm{d}s + K \sum_{k=0}^{N} e^{rk\Delta t} \left\|v_2^{k+1}\right\|^2 \Delta t;$$

$$\check{\mathcal{P}}_{19} = \mu \sum_{k=0}^{N} \left(\nabla\delta_1^k, \nabla v_2^{k+1}\right)\Delta t \quad \le \quad K(\Delta t)^2 \int_0^{T+\Delta t} \|\nabla\xi_{tt}\|^2 \, \mathrm{d}s + K \sum_{k=0}^{N} e^{rk\Delta t} \left\|\nabla v_2^{k+1}\right\|^2 \Delta t.$$

4.5. Bounding the CME-Error Equation. Using $\boldsymbol{w} = \boldsymbol{\chi}^{k+1}$ as the test function in (24) followed by summation in time, yields

$$
\begin{aligned}
(33)\ & \frac{1}{2}\left\|\boldsymbol{\chi}^N\right\|^2 + \frac{1}{2}\sum_{k=0}^{N-1}\left\|\partial_{t^b}\boldsymbol{\chi}^{k+1}\right\|^2(\Delta t)^2 + \frac{3}{4}\sum_{k=0}^{N-1}\left\|\sqrt{\hat{\tau}_{bf}{}^k}\ \boldsymbol{\chi}^{k+1}\right\|^2\Delta t \\
&+\mu\left(\beta_1 + \frac{\beta_2}{2}\right)\sum_{k=0}^{N-1}\left\|\nabla\boldsymbol{\chi}^{k+1}\right\|^2\Delta t \\
\le\ & \frac{\hat{\tau}^*}{4}\sum_{k=0}^{N-1}\left\|\partial_{t^b}\boldsymbol{\chi}^{k+1}\right\|^2(\Delta t)^3 + \frac{\mu\beta_2}{2}\sum_{k=0}^{N-1}\left\|\partial_{t^b}\nabla\boldsymbol{\chi}^{k+1}\right\|^2(\Delta t)^3 \\
&-\sum_{k=0}^{N-1}\left(\left(\frac{\mathcal{Q}^k}{\Pi^k}\cdot\nabla\boldsymbol{\chi}^k\right), \boldsymbol{\chi}^{k+1}\right)\Delta t - \sum_{k=0}^{N-1}\left((\nabla\cdot\boldsymbol{\chi}^k)\frac{\mathcal{Q}^k}{\Pi^k}, \boldsymbol{\chi}^{k+1}\right)\Delta t \\
&-\sum_{k=0}^{N-1}\left((\nabla\psi^k\cdot\mathcal{Q}^k)\frac{\mathcal{Q}^k}{(\Pi^k)^2}, \boldsymbol{\chi}^{k+1}\right)\Delta t - \frac{1}{2}\sum_{k=0}^{N-1}\left(f_c\boldsymbol{k}\times\boldsymbol{\chi}^k, \boldsymbol{\chi}^{k+1}\right)\Delta t \\
&-\sum_{k=0}^{N-1}\left(\Pi^k\nabla\psi^k, \boldsymbol{\chi}^{k+1}\right)\Delta t - \sum_{k=0}^{N-1}\left(\psi^k\boldsymbol{\mathcal{F}}^k, \boldsymbol{\chi}^{k+1}\right)\Delta t \\
&+\sum_{k=0}^{N-1}\left(\left(\frac{\boldsymbol{q}^k}{H^k}\cdot\nabla\boldsymbol{\phi}^k\right), \boldsymbol{\chi}^{k+1}\right)\Delta t + \sum_{k=0}^{N-1}\left(\left[\frac{\boldsymbol{q}^k}{H^k} - \frac{\mathcal{Q}^k}{\Pi^k}\right]\cdot\nabla\widetilde{\boldsymbol{q}}^k, \boldsymbol{\chi}^{k+1}\right)\Delta t \\
&+\sum_{k=0}^{N-1}\left((\nabla\cdot\boldsymbol{\phi}^k)\frac{\boldsymbol{q}^k}{H^k}, \boldsymbol{\chi}^{k+1}\right)\Delta t + \sum_{k=0}^{N-1}\left((\nabla\cdot\widetilde{\boldsymbol{q}}^k)\left[\frac{\boldsymbol{q}^k}{H^k} - \frac{\mathcal{Q}^k}{\Pi^k}\right], \boldsymbol{\chi}^{k+1}\right)\Delta t \\
&+\sum_{k=0}^{N-1}\left(\nabla h_b\cdot\left\{\left[\frac{\boldsymbol{q}^k}{H^k}\right]^2 - \left[\frac{\mathcal{Q}^k}{\Pi^k}\right]^2\right\}, \boldsymbol{\chi}^{k+1}\right)\Delta t + \sum_{k=0}^{N-1}\left((\nabla\theta^k\cdot\boldsymbol{q}^k)\frac{\boldsymbol{q}^k}{(H^k)^2}, \boldsymbol{\chi}^{k+1}\right)\Delta t \\
&+\sum_{k=0}^{N-1}\left(\nabla\widetilde{\xi}^k\cdot\left\{\left[\frac{\boldsymbol{q}^k}{H^k}\right]^2 - \left[\frac{\mathcal{Q}^k}{\Pi^k}\right]^2\right\}, \boldsymbol{\chi}^{k+1}\right)\Delta t + \sum_{k=0}^{N-1}\left(\tau_{bf}{}^k\boldsymbol{\phi}^{k+\frac{1}{2}}, \boldsymbol{\chi}^{k+1}\right)\Delta t \\
&+\sum_{k=0}^{N-1}\left(f_c\boldsymbol{k}\times\boldsymbol{\phi}^{k+\frac{1}{2}}, \boldsymbol{\chi}^{k+1}\right)\Delta t - \sum_{k=0}^{N-1}\left((\hat{\tau}_{bf}{}^k - \tau_{bf}{}^k)\widetilde{\boldsymbol{q}}^{k+\frac{1}{2}}, \boldsymbol{\chi}^{k+1}\right)\Delta t \\
&+\sum_{k=0}^{N-1}\left(H^k\nabla\theta^k, \boldsymbol{\chi}^{k+1}\right)\Delta t + \sum_{k=0}^{N-1}\left(\theta^k\nabla\widetilde{\xi}^k, \boldsymbol{\chi}^{k+1}\right)\Delta t - \sum_{k=0}^{N-1}\left(\psi^k\nabla\widetilde{\xi}^k, \boldsymbol{\chi}^{k+1}\right)\Delta t \\
&+\mu\sum_{k=0}^{N-1}\left(\beta_1\nabla\boldsymbol{\phi}^{k+1} + \beta_2\nabla\boldsymbol{\phi}^k, \nabla\boldsymbol{\chi}^{k+1}\right)\Delta t + \sum_{k=0}^{N-1}\left(\theta^k\boldsymbol{\mathcal{F}}^k, \boldsymbol{\chi}^{k+1}\right)\Delta t
\end{aligned}
$$

$$+\sum_{k=0}^{N-1}\left(\varepsilon_1^k,\boldsymbol{\chi}^{k+1}\right)\Delta t+\frac{1}{2}\sum_{k=0}^{N-1}\left(\tau_{bf}{}^k\varepsilon_2^k+f_c\boldsymbol{k}\times\varepsilon_2^k,\boldsymbol{\chi}^{k+1}\right)\Delta t+\mu\beta_1\sum_{k=0}^{N-1}\left(\nabla\varepsilon_2^k,\nabla\boldsymbol{\chi}^{k+1}\right)\Delta t$$
$$=\hat{S}_1+\hat{S}_2+\left(\hat{\mathcal{A}}_1+\cdots+\hat{\mathcal{A}}_6\right)+\left(\hat{\mathcal{P}}_1+\cdots+\hat{\mathcal{P}}_{18}\right).$$

Again, all the terms on the right-hand side are treated analogously to the terms in (32) with the exception of the following terms.

In estimating $\hat{S}_2$, we use the inverse assumption to obtain

$$\hat{S}_2=\frac{\mu}{2}\sum_{k=0}^{N-1}\left|\left|\partial_{t^b}\nabla\boldsymbol{\chi}^{k+1}\right|\right|^2(\Delta t)^3\quad\leq\quad\left(\frac{K_0\mu}{2h^2}\right)\sum_{k=0}^{N-1}\left|\left|\partial_{t^b}\boldsymbol{\chi}^{k+1}\right|\right|^2(\Delta t)^3.$$

And the truncation terms are estimated as follows

$$\hat{\mathcal{P}}_{16}=\sum_{k=0}^{N-1}\left(\varepsilon_1^k,\boldsymbol{\chi}^{k+1}\right)\Delta t\quad\leq\quad K(\Delta t)^2\int_0^T||\boldsymbol{q}_{tt}||^2\,\mathrm{d}s+K\sum_{k=0}^{N-1}\left|\left|\boldsymbol{\chi}^{k+1}\right|\right|^2\Delta t.$$

$$\hat{\mathcal{P}}_{17}=\frac{1}{2}\sum_{k=0}^{N-1}\left(\tau_{bf}{}^k\varepsilon_2^k+f_c\boldsymbol{k}\times\varepsilon_2^k,\boldsymbol{\chi}^{k+1}\right)\Delta t\ \leq\ K(\Delta t)^2\int_0^T||\boldsymbol{q}_t||^2\,\mathrm{d}s+K\sum_{k=0}^{N-1}\left|\left|\boldsymbol{\chi}^{k+1}\right|\right|^2\Delta t.$$

$$\hat{\mathcal{P}}_{18}=\sum_{k=0}^{N-1}\left(\nabla\varepsilon_2^k,\nabla\boldsymbol{\chi}^{k+1}\right)\Delta t\quad\leq\quad K(\Delta t)^2\int_0^T||\nabla\boldsymbol{q}_t||^2\,\mathrm{d}s+\epsilon\sum_{k=0}^{N-1}\left|\left|\nabla\boldsymbol{\chi}^{k+1}\right|\right|^2\Delta t.$$

4.6. Bounding the Sum of the Error Equations. We observe that the right-hand-side of (31) can be bounded by

$$\begin{aligned}(34)\qquad &\epsilon\sum_{k=0}^{N-1}e^{-rk\Delta t}\left|\left|\psi^k\right|\right|^2\Delta t+\epsilon\sum_{k=0}^{N-1}e^{-rk\Delta t}\left|\left|\nabla\psi^k\right|\right|^2\Delta t+\epsilon\sum_{k=0}^{N-1}\left|\left|\nabla\boldsymbol{\chi}^{k+1}\right|\right|^2\Delta t\\
&+\epsilon\left|\left|v_1^0\right|\right|^2+\epsilon\left|\left|\nabla v_1^0\right|\right|^2+K\sum_{k=0}^{N-1}\left(\left|\left|\theta^k\right|\right|^2+\left|\left|\nabla\theta^k\right|\right|^2+\left|\left|\partial_{t^f}\nabla\theta^k\right|\right|^2\right)\Delta t\\
&+K\sum_{k=0}^{N-1}\left|\left|\boldsymbol{\phi}^k\right|\right|^2\Delta t+K\sum_{k=0}^{N-1}\left|\left|\nabla\boldsymbol{\phi}^k\right|\right|^2\Delta t+K\left(h^{2(\ell-1)}+\Delta t^2\right)\\
&+K(\Delta t)^4\,||\xi_{tttt}||^2_{\mathcal{L}^2((0,T);\mathcal{L}^2(\Omega))}+K(\Delta t)^2\,||\xi_{tt}||^2_{\mathcal{L}^2((0,T);\mathcal{L}^2(\Omega))}\\
&+K(\Delta t)^2\,||\nabla\xi_{tt}||^2_{\mathcal{L}^2((0,T);\mathcal{L}^2(\Omega))}+K(\Delta t)^3\,||\xi_{ttt}||^2_{\mathcal{L}^\infty((0,\Delta t);\mathcal{L}^2(\Omega))}\\
&+K_1\sum_{k=0}^{N-1}e^{rk\Delta t}\left|\left|v_1^{k+1}\right|\right|^2\Delta t+K\sum_{k=0}^{N-1}\left|\left|\boldsymbol{\chi}^{k+1}\right|\right|^2\Delta t+K\sum_{k=0}^{N}\left|\left|e^{-rk\Delta t}\psi^k\right|\right|^2\Delta t\\
&+K\sum_{k=0}^{N-1}\left|\left|\sum_{j=0}^{k}e^{-rj\Delta t}\psi^j\Delta t\right|\right|^2\Delta t+K\sum_{k=0}^{N-1}\left|\left|\sum_{j=0}^{k}e^{-rj\Delta t}\nabla\psi^j\Delta t\right|\right|^2\Delta t.\end{aligned}$$

The right-hand-side of (32) can be bounded by

$$\begin{aligned}(35)\qquad &\epsilon\sum_{k=0}^{N-1} e^{-rk\Delta t}\left\|\psi^k\right\|^2\Delta t+\epsilon\sum_{k=0}^{N-1} e^{-rk\Delta t}\left\|\nabla\psi^k\right\|^2\Delta t+\epsilon\sum_{k=0}^{N-1}\left\|\nabla\boldsymbol{\chi}^{k+1}\right\|^2\Delta t\\
&+K\sum_{k=0}^{N-1}\left(\left\|\theta^k\right\|^2+\left\|\nabla\theta^k\right\|^2+\left\|\partial_{t^f}\nabla\theta^k\right\|^2\right)\Delta t+K\sum_{k=0}^{N-1}\left\|\boldsymbol{\phi}^k\right\|^2\Delta t\\
&+K\sum_{k=0}^{N-1}\left\|\nabla\boldsymbol{\phi}^k\right\|^2\Delta t+K\sum_{k=0}^{N-1}\left\|\boldsymbol{\chi}^{k+1}\right\|^2\Delta t+K\left(h^{2(\ell-1)}+\Delta t^2\right)\Delta t^2\\
&+K(\Delta t)^4\left\|\xi_{tttt}\right\|^2_{\mathcal{L}^2((0,T);\mathcal{L}^2(\Omega))}+K(\Delta t)^2\left\|\xi_{tt}\right\|^2_{\mathcal{L}^2((0,T);\mathcal{L}^2(\Omega))}\\
&+K(\Delta t)^2\left\|\nabla\xi_{tt}\right\|^2_{\mathcal{L}^2((0,T);\mathcal{L}^2(\Omega))}+K(\Delta t)^3\left\|\xi_{ttt}\right\|^2_{\mathcal{L}^\infty((0,\Delta t);\mathcal{L}^2(\Omega))}\\
&+K_2\sum_{k=0}^{N-1}e^{rk\Delta t}\left\|v_2^{k+1}\right\|^2\Delta t+K_3\sum_{k=0}^{N-1}e^{rk\Delta t}\left\|\nabla v_2^{k+1}\right\|^2\Delta t.\end{aligned}$$

Finally, the right-hand-side of (33) can be bounded by

$$\begin{aligned}(36)\qquad &\left(\frac{\hat{\tau}^* h^2+2K_0\mu\beta_2}{4h^2}\right)\sum_{k=0}^{N-1}\left\|\partial_{t^b}\boldsymbol{\chi}^{k+1}\right\|^2(\Delta t)^3+\epsilon\sum_{k=0}^{N-1} e^{-rk\Delta t}\left\|\psi^k\right\|^2\Delta t\\
&+\epsilon\sum_{k=0}^{N-1} e^{-rk\Delta t}\left\|\nabla\psi^k\right\|^2\Delta t+\epsilon\sum_{k=0}^{N-1}\left\|\nabla\boldsymbol{\chi}^{k+1}\right\|^2\Delta t\\
&+K\sum_{k=0}^{N-1}\left(\left\|\theta^k\right\|^2+\left\|\nabla\theta^k\right\|^2\right)\Delta t+K\sum_{k=0}^{N-1}\left(\left\|\boldsymbol{\phi}^k\right\|^2+\left\|\nabla\boldsymbol{\phi}^k\right\|^2\right)\Delta t\\
&+K\sum_{k=0}^{N-1}\left(\left\|\boldsymbol{\phi}^{k+1}\right\|^2+\left\|\nabla\boldsymbol{\phi}^{k+1}\right\|^2\right)\Delta t+K\sum_{k=0}^{N-1}\left\|\boldsymbol{\chi}^{k+1}\right\|^2\Delta t\\
&+K(\Delta t)^2\left\|\boldsymbol{q}_{tt}\right\|^2_{\mathcal{L}^2((0,T);\mathcal{L}^2(\Omega))}+K(\Delta t)^2\left\|\nabla\boldsymbol{q}_t\right\|^2_{\mathcal{L}^2((0,T);\mathcal{L}^2(\Omega))}\end{aligned}$$

Choose

$$r = \max\left\{\frac{2K_1}{\gamma_*},\frac{2K_2}{\tau_\circ},\frac{2K_3}{\mu}\right\},$$

such that,

$$\begin{aligned}r_1 &= r\frac{\gamma_*}{2}e^\theta-K_1\geq 0,\\ r_2 &= r\frac{\tau_\circ}{2}e^\theta-K_2\geq 0,\end{aligned}$$

and

$$r_3 = r\frac{\mu}{2}e^\theta-K_3\geq 0.$$

And, observe that

$$\begin{aligned}\Lambda^k &= e^{-rk\Delta t}\left\|\psi^k\right\|^2+\left\|\boldsymbol{\chi}^k\right\|^2+\left\|\sum_{j=0}^{k}e^{-erj\Delta t}\psi^j\Delta t\right\|^2_{\mathcal{H}^1(\Omega)}\\
&\geq \left\|e^{-rk\Delta t}\psi^k\right\|^2+\left\|\boldsymbol{\chi}^k\right\|^2+\left\|\sum_{j=0}^{k}e^{-erj\Delta t}\psi^j\Delta t\right\|^2_{\mathcal{H}^1(\Omega)}.\end{aligned}$$

Now, sum (31)–(33), using bounds (34)–(36) and the choice for r above to obtain

$$
\begin{aligned}
(37)\quad & \frac{1}{2}e^{-rN\Delta t}\left\|\psi^N\right\|^2 + \left(\tau_\circ + \frac{r}{2}e^{\theta_0}\right)\sum_{k=0}^{N} e^{-rk\Delta t}\left\|\psi^k\right\|^2\Delta t \\
& + \sum_{k=0}^{N} e^{-rk\Delta t}\left\|\partial_{t^b}\psi^k\right\|^2\Delta t + \underbrace{\left(\frac{\tau_\circ+1}{2}\right)\sum_{k=0}^{N} e^{-rk\Delta t}\left\|\partial_{t^b}\psi^k\right\|^2(\Delta t)^2}_{(a)} \\
& + \mu\sum_{k=0}^{N} e^{-rk\Delta t}\left\|\nabla\psi^k\right\|^2\Delta t + \underbrace{\frac{\mu}{2}\sum_{k=0}^{N} e^{-rk\Delta t}\left\|\partial_{t^b}\nabla\psi^k\right\|^2(\Delta t)^2}_{(b)} \\
& + \gamma_*\left\|v_1^0\right\|^2 + \frac{\gamma_*}{2}\left\|\nabla v_1^0\right\|^2 + \underbrace{\frac{\tau_\circ}{2}\left\|v_2^0\right\|^2}_{(c)} + \underbrace{\frac{\mu}{2}\left\|\nabla v_2^0\right\|^2}_{(d)} \\
& \underbrace{+\, r_1\sum_{k=0}^{N-1} e^{rk\Delta t}\left\|\nabla v_1^{k+1}\right\|^2\Delta t}_{(e)} + \underbrace{r_2\sum_{k=0}^{N-1} e^{rk\Delta t}\left\|v_2^{k+1}\right\|^2\Delta t}_{(f)} \\
& \underbrace{+\, r_3\sum_{k=0}^{N-1} e^{rk\Delta t}\left\|\nabla v_2^{k+1}\right\|^2\Delta t}_{(g)} + \frac{1}{2}\left\|\boldsymbol{\chi}^N\right\|^2 + \frac{1}{2}\sum_{k=0}^{N-1}\left\|\partial_{t^b}\boldsymbol{\chi}^{k+1}\right\|^2(\Delta t)^2 \\
& + \underbrace{\frac{1}{2}\sum_{k=0}^{N-1}\left\|\sqrt{\hat{\tau}_{bf}{}^k}\,(\boldsymbol{\chi}^{k+1})\right\|^2\Delta t}_{(h)} + \mu\left(\beta_1 + \frac{\beta_2}{2}\right)\sum_{k=0}^{N-1}\left\|\nabla\boldsymbol{\chi}^{k+1}\right\|^2\Delta t \\
\le\; & \left(\frac{\hat{\tau}^* h^2 + 2K_0\mu\beta_2}{4h^2}\right)\sum_{k=0}^{N-1}\left\|\partial_{t^b}\boldsymbol{\chi}^{k+1}\right\|(\Delta t)^3 + \epsilon\left\|v_1^0\right\|^2 + \epsilon\left\|\nabla v_1^0\right\|^2 \\
& + \epsilon\sum_{k=0}^{N-1} e^{-rk\Delta t}\left\|\psi^k\right\|^2\Delta t + \epsilon\sum_{k=0}^{N-1} e^{-rk\Delta t}\left\|\nabla\psi^k\right\|^2\Delta t + \epsilon\sum_{k=0}^{N-1}\left\|\nabla\boldsymbol{\chi}^{k+1}\right\|^2\Delta t \\
& + K_1\sum_{k=0}^{N-1} e^{rk\Delta t}\left\|\nabla v_1^{k+1}\right\|^2\Delta t + K_2\sum_{k=0}^{N-1} e^{rk\Delta t}\left\|v_2^{k+1}\right\|^2\Delta t \\
& + K_3\sum_{k=0}^{N-1} e^{rk\Delta t}\left\|\nabla v_2^{k+1}\right\|^2\Delta t + K\sum_{k=0}^{N-1}\left(\left\|\theta^k\right\|^2 + \left\|\nabla\theta^k\right\|^2 + \left\|\partial_{t^f}\nabla\theta^k\right\|^2\right)\Delta t \\
& + K\sum_{k=0}^{N}\left(\left\|\boldsymbol{\phi}^k\right\|^2 + \left\|\nabla\boldsymbol{\phi}^k\right\|^2\right)\Delta t + K\sum_{k=0}^{N}\left(\left\|\boldsymbol{\phi}^{k+1}\right\|^2 + \left\|\nabla\boldsymbol{\phi}^{k+1}\right\|^2\right)\Delta t \\
& + K(\Delta t)^4\left\|\xi_{tttt}\right\|^2_{\mathcal{L}^2((0,T);\mathcal{L}^2(\Omega))} + K(\Delta t)^2\left\|\xi_{tt}\right\|^2_{\mathcal{L}^2((0,T);\mathcal{L}^2(\Omega))} \\
& + K(\Delta t)^2\left\|\nabla\xi_{tt}\right\|^2_{\mathcal{L}^2((0,T);\mathcal{L}^2(\Omega))} + K(\Delta t)^3\left\|\xi_{ttt}\right\|^2_{\mathcal{L}^\infty((0,\Delta t);\mathcal{L}^2(\Omega))} \\
& + K(\Delta t)^2\left\|\boldsymbol{q}_{tt}\right\|^2_{\mathcal{L}^2((0,T);\mathcal{L}^2(\Omega))} + K(\Delta t)^2\left\|\nabla\boldsymbol{q}_t\right\|^2_{\mathcal{L}^2((0,T);\mathcal{L}^2(\Omega))}
\end{aligned}
$$

$$+\bar{K}\sum_{k=0}^{N}\Lambda^k\Delta t + K\left(h^{2(\ell-1)}+\Delta t^2\right) + K\left(h^{2(\ell-1)}+\Delta t^2\right)\Delta t^2\ .$$

Assume, in addition, that $\Delta t \le 2h^2\left(\hat{\tau}^* h^2 + 2K_0\mu\beta_2\right)^{-1} = \kappa_2$ so that

$$\sigma_4 \;=\; \frac{1}{2} - \left(\frac{\hat{\tau}^* h^2 + 2K_0\mu\beta_2}{4h^2}\right)\Delta t \;\ge\; 0.$$

Using the above choice for Δt, hiding all terms multiplied by ϵ, and observing that terms (a)-(h) are all non-negative, we can write (37) as follows

$$\begin{aligned}
(38)\qquad & \frac{1}{2}e^{-rN\Delta t}\left\|\psi^N\right\|^2 + \left(\frac{\tau_\circ}{2}+\frac{r}{2}e^{\theta_0}\right)\sum_{k=0}^{N} e^{-rk\Delta t}\left\|\psi^k\right\|^2\Delta t \\
& + \sum_{k=0}^{N} e^{-rk\Delta t}\left\|\partial_{t^b}\psi^k\right\|^2\Delta t + \frac{\mu}{2}\sum_{k=0}^{N} e^{-rk\Delta t}\left\|\nabla\psi^k\right\|^2\Delta t + \frac{\gamma_*}{4}\left\|v_1^0\right\|^2_{\mathcal{H}^1(\Omega)} \\
& + \frac{1}{2}\left\|\boldsymbol{\chi}^N\right\|^2 + \sigma_4\sum_{k=0}^{N-1}\left\|\partial_{t^b}\boldsymbol{\chi}^{k+1}\right\|^2(\Delta t)^2 + \frac{\mu}{2}\left(\beta_1+\frac{\beta_2}{2}\right)\sum_{k=0}^{N-1}\left\|\nabla\boldsymbol{\chi}^{k+1}\right\|^2\Delta t \\
& \le K\left\|\theta\right\|^2_{\ell^2(J_{N-1};\mathcal{L}^2(\Omega))} + K\left\|\nabla\theta\right\|^2_{\ell^2(J_{N-1};\mathcal{L}^2(\Omega))} + K\sum_{k=0}^{N-1}\left\|\partial_{t^f}\nabla\theta^k\right\|^2\Delta t \\
& + K\left\|\boldsymbol{\phi}\right\|^2_{\ell^2(J_{N-1};\mathcal{H}^1(\Omega))} + K\left\|\boldsymbol{\phi}\right\|^2_{\ell^2(J_N;\mathcal{H}^1(\Omega))} \\
& + K(\Delta t)^4\left\|\xi_{tttt}\right\|^2_{\mathcal{L}^2((0,T);\mathcal{L}^2(\Omega))} + K(\Delta t)^2\left\|\xi_{tt}\right\|^2_{\mathcal{L}^2((0,T);\mathcal{L}^2(\Omega))} \\
& + K(\Delta t)^2\left\|\nabla\xi_{tt}\right\|^2_{\mathcal{L}^2((0,T);\mathcal{L}^2(\Omega))} + K(\Delta t)^3\left\|\xi_{ttt}\right\|^2_{\mathcal{L}^\infty((0,\Delta t);\mathcal{L}^2(\Omega))} \\
& + K(\Delta t)^2\left\|\boldsymbol{q}_{tt}\right\|^2_{\mathcal{L}^2((0,T);\mathcal{L}^2(\Omega))} + K(\Delta t)^2\left\|\nabla\boldsymbol{q}_t\right\|^2_{\mathcal{L}^2((0,T);\mathcal{L}^2(\Omega))} \\
& + \bar{K}\sum_{k=0}^{N}\Lambda^k\Delta t + K\left(h^{2(\ell-1)}+\Delta t^2\right) + K\left(h^{2(\ell-1)}+\Delta t^2\right)\Delta t^2\ .
\end{aligned}$$

Recalling that

$$\Lambda^N = e^{-rN\Delta t}\left\|\psi^N\right\|^2 + \left\|\boldsymbol{\chi}^N\right\|^2 + \left\|v_1^0\right\|^2_{\mathcal{H}^1(\Omega)},$$

apply the GDGI to finally obtain

$$\begin{aligned}
(39)\qquad & e^{-rk\Delta t}\left\|\psi^k\right\|^2\Delta t + \frac{\tau_\circ}{2}\sum_{k=0}^{N} e^{-rk\Delta t}\left\|\psi^k\right\|^2\Delta t + \frac{r}{2}e^{\theta_0}\sum_{k=0}^{N-1} e^{-rk\Delta t}\left\|\psi^k\right\|^2\Delta t \\
& + \sum_{k=0}^{N} e^{-rk\Delta t}\left\|\partial_{t^b}\psi^k\right\|^2\Delta t + \frac{\mu}{2}\sum_{k=0}^{N} e^{-rk\Delta t}\left\|\nabla\psi^k\right\|^2\Delta t + \frac{\gamma_*}{4}\left\|v_1^0\right\|^2_{\mathcal{H}^1(\Omega)} \\
& + \frac{1}{2}\left\|\boldsymbol{\chi}^N\right\|^2 + \sigma_4\sum_{k=0}^{N-1}\left\|\partial_{t^b}\boldsymbol{\chi}^{k+1}\right\|^2(\Delta t)^2 + \frac{\mu}{2}\left(\beta_1+\frac{\beta_2}{2}\right)\sum_{k=0}^{N-1}\left\|\nabla\boldsymbol{\chi}^{k+1}\right\|^2\Delta t \\
& \le K\left\{\left\|\theta\right\|^2_{\ell^2(J_{N-1};\mathcal{H}^1(\Omega))} + \sum_{k=0}^{N-1}\left\|\partial_{t^f}\nabla\theta^k\right\|^2\Delta t + \left\|\boldsymbol{\phi}\right\|^2_{\ell^2(J_N;\mathcal{H}^1(\Omega))}\right. \\
& \qquad + (\Delta t)^4\left\|\xi_{tttt}\right\|^2_{\mathcal{L}^2((0,T);\mathcal{L}^2(\Omega))} + (\Delta t)^2\left\|\xi_{tt}\right\|^2_{\mathcal{L}^2((0,T);\mathcal{H}^1(\Omega))}
\end{aligned}$$

$$+(\Delta t)^3 \|\xi_{ttt}\|^2_{\mathcal{L}^\infty((0,\Delta t);\mathcal{L}^2(\Omega))} + (\Delta t)^2 \|\boldsymbol{q}_{tt}\|^2_{\mathcal{L}^2((0,T);\mathcal{L}^2(\Omega))}$$
$$+(\Delta t)^2 \|\nabla \boldsymbol{q}_t\|^2_{\mathcal{L}^2((0,T);\mathcal{L}^2(\Omega))} + \left(h^{2(\ell-1)} + \Delta t^2\right)\left(1+\Delta t^2\right)\Big\}$$

for Δt sufficiently small and $K = K \exp\left(\bar{K}T/(\sigma_5 - \bar{K}\Delta t)\right)$ where $\sigma_5 = \frac{1}{2}\min\{1, \gamma_*\}$.

Bounding above and below, we obtain the estimate on the affine errors.

$$\begin{aligned}
(40)\qquad & \left\|\psi^N\right\|^2 + \|\psi\|^2_{\ell^2(J_N;\mathcal{H}^1(\Omega))} + \left\|\partial_{t^b}\psi^k\right\|^2_{\ell^2(J_N;\mathcal{L}^2(\Omega))} \\
& + \left\|\boldsymbol{\chi}^N\right\|^2 + \|\nabla \boldsymbol{\chi}\|^2_{\ell^2(J_{N-1};\mathcal{L}^2(\Omega))} \\
& \leq K \Big\{ \|\theta\|^2_{\ell^2(J_{N-1};\mathcal{H}^1(\Omega))} + \left\|\partial_{t^f}\nabla\theta^k\right\|^2_{\ell^2(J_{N-1};\mathcal{L}^2(\Omega))} \\
& + \|\boldsymbol{\phi}\|^2_{\ell^2(J_N;\mathcal{H}^1(\Omega))} + (\Delta t)^4 \|\xi_{tttt}\|^2_{\mathcal{L}^2((0,T);\mathcal{L}^2(\Omega))} \\
& +(\Delta t)^2 \|\xi_{tt}\|^2_{\mathcal{L}^2((0,T);\mathcal{H}^1(\Omega))} + (\Delta t)^3 \|\xi_{ttt}\|^2_{\mathcal{L}^\infty((0,\Delta t);\mathcal{L}^2(\Omega))} \\
& +(\Delta t)^2 \|\boldsymbol{q}_{tt}\|^2_{\mathcal{L}^2((0,T);\mathcal{L}^2(\Omega))} + (\Delta t)^2 \|\nabla \boldsymbol{q}_t\|^2_{\mathcal{L}^2((0,T);\mathcal{L}^2(\Omega))} \\
& + \left(h^{2(\ell-1)} + \Delta t^2\right)\left(1+\Delta t^2\right)\Big\}\ .
\end{aligned}$$

Using the standard approximation results, we obtain

$$\begin{aligned}
(41)\qquad & \left\|\psi^N\right\| + \|\psi\|_{\ell^2(J_N;\mathcal{H}^1(\Omega))} + \left\|\partial_{t^b}\psi^k\right\|_{\ell^2(J_N;\mathcal{L}^2(\Omega))} \\
& + \left\|\boldsymbol{\chi}^N\right\| + \|\nabla\boldsymbol{\chi}\|_{\ell^2(J_{N-1};\mathcal{L}^2(\Omega))} \leq K\left\{h^{\ell-1} + \Delta t\right\}\ .
\end{aligned}$$

The result of the theorem now follows by an application of the triangle inequality to the projection error and to the affine error (41).

Consequently, we can prove the following result.

THEOREM 4.1. *Let $0 \leq s_0 \leq k,\ s_0 \leq \ell \leq s_1,\ 0 \leq k < s_1$. Let $(\xi^k, \boldsymbol{q}^k)$ be the solution to (8)-(10) and suppose that assumptions* **A2**–**A16** *hold. Let $(\Xi^k, \mathcal{Q}^k)$ be the Galerkin approximations to $(\xi^k, \boldsymbol{q}^k)$. If $\Xi^k \in \mathcal{S}^h(\Omega)$, $\mathcal{Q}^k \in \boldsymbol{\mathcal{S}}^h(\Omega)$ for each k; then, for h and Δt sufficiently small, $\exists$ a constant $\bar{K} = \bar{K}(T, s_1, r, C_*, C^*, C_{**}, C^{**})$ such that*

$$\begin{aligned}
& \left\|\ \xi^N - \Xi^N\ \right\| + \|\ \xi - \Xi\ \|_{\ell^2(J_N;\mathcal{H}^1(\Omega))} \\
& + \left\|\ \boldsymbol{q}^N - \mathcal{Q}^N\ \right\| + \|\ \nabla\boldsymbol{q} - \nabla\mathcal{Q}\ \|_{\ell^2(J_{N-1};\mathcal{L}^2(\Omega))} \leq \bar{K}\left(h^{\ell-1} + \Delta t\right).
\end{aligned}$$

In particular, $\Delta t \leq \min\{o(h), \kappa_1, \kappa_2\}$, where $\kappa_1 = 8/\tau_\circ,\ \kappa_2 = 2h^2\left(\hat{r}^* h^2 + 2K_0\mu\beta_2\right)^{-1}$.

In the case $s_1 \geq 3$ and $l > 2$, we can now complete our induction argument.

$$\begin{aligned}
|\mathcal{Q}^N| &\leq |\boldsymbol{\chi}^N| + |\widetilde{\boldsymbol{q}}^N| \\
&\leq Kh^{-1}(h^{l-1} + \Delta t) + C^* \\
&< C^{**}
\end{aligned}$$

for h and Δt sufficiently small (with $\Delta t = o(h)$). A similar argument give an upper bound for Π^N. For the lower bound on Π^N we have

$$\begin{aligned}
\Pi^N &= \psi^N + \widetilde{\xi}^N \\
&\geq -Kh^{-1}(h^{l-1} + \Delta t) + C_* \\
&> C_{**},
\end{aligned}$$

again for h and Δt sufficiently small (with $\Delta t = o(h)$).

A. Appendix: Summation by Parts and Other Tools. We now introduce tools used in the previous sections. First investigate the use of v_1^{k+1} and v_2^{k+1} as the test functions in (23) when we multiply by Δt and sum over k. For a generic $\varsigma \in \mathcal{H}^1(\Omega)$, consider the following relations:

$$\text{(A)} \quad \sum_{k=0}^{N-1} e^{\pm rk\Delta t} \left(\partial_{t^b}\varsigma^k, \varsigma^k\right) \Delta t = \frac{1}{2}\sum_{k=0}^{N-1}\left(\frac{e^{\pm rk\Delta t}\left\|\varsigma^k\right\|^2 - e^{\pm r(k-1)\Delta t}\left\|\varsigma^{k-1}\right\|^2}{\Delta t}\right)\Delta t$$
$$+\frac{1}{2}\sum_{k=0}^{N-1} e^{\pm rk\Delta t}\left\|\varsigma^k\right\|^2 + \frac{1}{2}\sum_{k=0}^{N-1} e^{\pm r(k-1)\Delta t}\left\|\varsigma^{k-1}\right\|^2 - \sum_{k=0}^{N-1} e^{\pm rk\Delta t}\left(\varsigma^{k-1}, \varsigma^k\right)$$
$$= \frac{1}{2}\sum_{k=0}^{N-1}\partial_{t^b}\left(e^{\pm rk\Delta t}\left\|\varsigma^k\right\|^2\right)\Delta t + \frac{1}{2}\sum_{k=0}^{N-1} e^{\pm r(k-1)\Delta t}(1-e^{\pm r\Delta t})\left\|\varsigma^{k-1}\right\|^2$$
$$+\frac{1}{2}\sum_{k=0}^{N-1} e^{\pm rk\Delta t}\left\|\partial_{t^b}\varsigma^k\right\|^2(\Delta t)^2$$
$$= \frac{1}{2}\sum_{k=0}^{N-1}\partial_{t^b}\left(e^{\pm rk\Delta t}\left\|\varsigma^k\right\|^2\right)\Delta t + \frac{1}{2}\sum_{k=0}^{N-1} e^{\pm rk\Delta t}(1-e^{\pm r\Delta t})\left\|\varsigma^{k}\right\|^2$$
$$+\frac{1}{2}e^{\pm r(-1)\Delta t}(1-e^{\pm r\Delta t})\left\|\varsigma^{-1}\right\|^2 - \frac{1}{2}e^{\pm r(N-1)\Delta t}(1-e^{\pm r\Delta t})\left\|\varsigma^{N-1}\right\|^2$$
$$+\frac{1}{2}\sum_{k=0}^{N-1} e^{\pm rk\Delta t}\left\|\partial_{t^b}\varsigma^k\right\|^2(\Delta t)^2$$
$$= \frac{1}{2}e^{\pm rN\Delta t}\left\|\varsigma^{N-1}\right\|^2 - \frac{1}{2}\left\|\varsigma^{-1}\right\|^2 \mp \frac{r}{2}e^{\theta}\sum_{k=0}^{N-1} e^{\pm rk\Delta t}\left\|\varsigma^k\right\|^2\Delta t$$
$$+\frac{1}{2}\sum_{k=0}^{N-1} e^{\pm rk\Delta t}\left\|\partial_{t^b}\varsigma^k\right\|^2(\Delta t)^2, \qquad \theta \in (0, \pm r\Delta t),$$

where the last equality results from an application of the Mean Value Theorem.

$$\text{(B)} \quad \sum_{k=0}^{N-1} e^{\pm rk\Delta t}\left(\partial_{t^f}\varsigma^k, \varsigma^{k+1}\right)\Delta t = \frac{1}{2}\sum_{k=0}^{N-1}\left(\frac{e^{\pm r(k+1)\Delta t}\left\|\varsigma^{k+1}\right\|^2 - e^{\pm rk\Delta t}\left\|\varsigma^k\right\|^2}{\Delta t}\right)\Delta t$$
$$-\frac{1}{2}\sum_{k=0}^{N-1} e^{\pm r(k+1)\Delta t}\left\|\varsigma^{k+1}\right\|^2 + \frac{1}{2}\sum_{k=0}^{N-1} e^{\pm rk\Delta t}\left\|\varsigma^k\right\|^2$$
$$+\sum_{k=0}^{N-1} e^{\pm rk\Delta t}\left\|\varsigma^{k+1}\right\|^2 - \sum_{k=0}^{N-1} e^{\pm rk\Delta t}\left(\varsigma^k, \varsigma^{k+1}\right)$$
$$= \frac{1}{2}\sum_{k=0}^{N-1}\partial_{t^f}\left(e^{\pm rk\Delta t}\left\|\varsigma^k\right\|^2\right)\Delta t + \frac{1}{2}\sum_{k=0}^{N-1} e^{\pm rk\Delta t}(1-e^{\pm r\Delta t})\left\|\varsigma^{k+1}\right\|^2\Delta t$$
$$+\frac{1}{2}\sum_{k=0}^{N-1} e^{\pm rk\Delta t}\left\|\partial_{t^f}\varsigma^k\right\|^2(\Delta t)^2$$
$$= \frac{1}{2}e^{\pm rN\Delta t}\left\|\varsigma^{N}\right\|^2 - \frac{1}{2}\left\|\varsigma^{0}\right\|^2 \mp \frac{r}{2}e^{\theta}\sum_{k=0}^{N-1} e^{\pm rk\Delta t}\left\|\varsigma^{k+1}\right\|^2\Delta t$$

$$+\frac{1}{2}\sum_{k=0}^{N-1} e^{\pm rk\Delta t}\left|\left|\partial_{tf}\varsigma^k\right|\right|^2(\Delta t)^2, \qquad \theta\in(0,\pm r\Delta t),$$

where the last equality follows from the Mean Value Theorem.

And finally,

$$\text{(C)}\quad \sum_{k=0}^{N-1} e^{\pm rk\Delta t}\left(\partial_{tf}\varsigma^k,\varsigma^k\right)\Delta t = \frac{1}{2}\sum_{k=0}^{N-1}\partial_{tf}\left(e^{\pm rk\Delta t}\left|\left|\varsigma^k\right|\right|^2\right)\Delta t \mp \frac{r}{2}e^{\theta}\sum_{k=0}^{N-1} e^{\pm rk\Delta t}\left|\left|\varsigma^{k+1}\right|\right|^2\Delta t$$
$$-\frac{1}{2}\sum_{k=0}^{N-1} e^{\pm rk\Delta t}\left|\left|\partial_{tf}\varsigma^k\right|\right|^2(\Delta t)^2, \qquad \theta\in(0,\pm r\Delta t)\,.$$

We understand the following to be true: $v_1^{k>N}=0$, $v_2^{k>N}=0$.

Now, consider the first two terms of (23). When $v=v_1^{k+1}$, we obtain

$$\begin{aligned}
\sum_{k=0}^{N-1}\left(\partial_{tc}^2\psi^k,v_1^{k+1}\right)\Delta t &= -\sum_{k=0}^{N-1}\left(\partial_{tb}\psi^k,\partial_{tf}v_1^k\right)\Delta t-\left(\partial_{tb}\psi^0,v_1^0\right)+\left(\partial_{tb}\psi^N,v_1^N\right)\\
&= \sum_{k=0}^{N} e^{-rk\Delta t}\left(\partial_{tb}\psi^k,\psi^k\right)\Delta t-\left(\partial_{tb}\psi^0,v_1^0\right)\\
&\overset{(A)}{=} \frac{1}{2}e^{-rN\Delta t}\left|\left|\psi^{N-1}\right|\right|^2-\frac{1}{2}\left|\left|\psi^{-1}\right|\right|^2+\frac{r}{2}e^{\theta_0}\sum_{k=0}^{N} e^{-rk\Delta t}\left|\left|\psi^k\right|\right|^2\Delta t\\
&\quad +\frac{1}{2}\sum_{k=0}^{N} e^{-rk\Delta t}\left|\left|\partial_{tb}\psi^k\right|\right|^2(\Delta t)^2-\left(\partial_{tb}\psi^0,v_1^0\right), \quad \theta_0\in(-r\Delta t,0);
\end{aligned}$$

$$\begin{aligned}
\tau_\circ\sum_{k=0}^{N-1}\left(\partial_{tf}\psi^k,v_1^{k+1}\right)\Delta t &= -\tau_\circ\sum_{k=0}^{N-1}\left(\psi^k,\partial_{tb}v_1^{k+1}\right)\Delta t-\tau_\circ\left(\psi^0,v_1^0\right)+\tau_\circ\left(\psi^N,v_1^N\right)\\
&\equiv \tau_\circ\sum_{k=0}^{N} e^{-rk\Delta t}\left|\left|\psi^k\right|\right|^2\Delta t\,.
\end{aligned}$$

The first equalities above result from temporal summation by parts. We also have from the diffusion term upon summing by parts in time:

$$\mu\sum_{k=0}^{N-1}\left(\partial_{tf}\nabla\psi^k,\nabla v_1^{k+1}\right)\Delta t = \mu\sum_{k=0}^{N} e^{-rk\Delta t}\left|\left|\nabla\psi^k\right|\right|^2\Delta t.$$

We are also able to manipulate the eighth term in (23) by using the definition of v_1 as follows:

$$\begin{aligned}
\frac{1}{2}\sum_{k=0}^{N-1}\left(gh_b\,\nabla\psi^k,\nabla v_1^{k+1}\right)\Delta t &\equiv -\frac{1}{2}\sum_{k=0}^{N-1} e^{rk\Delta t}\left(gh_b\,\partial_{tb}(\nabla v_1^{k+1}),\nabla v_1^{k+1}\right)\Delta t\\
&\overset{(B)}{=} -\frac{1}{4}\sum_{k=0}^{N-1}\partial_{tf}\left(e^{rk\Delta t}\left|\left|\sqrt{gh_b}\,\nabla v_1^k\right|\right|^2\right)\Delta t
\end{aligned}$$

$$
\begin{aligned}
& +\frac{r}{4}e^{\theta}\sum_{k=0}^{N-1} e^{rk\Delta t}\left\|\sqrt{gh_b}\,\nabla v_1^{k+1}\right\|^2\Delta t \\
& -\frac{1}{4}\sum_{k=0}^{N-1} e^{rk\Delta t}\left\|\partial_{t^f}(\sqrt{gh_b}\,\nabla v_1^{k})\right\|^2\Delta t^2, \quad \theta\in(0,r\Delta t) \\
\geq\ & \frac{\gamma_*}{4}\left\|\nabla v_1^0\right\|^2+\frac{r\gamma_*}{4}e^{\theta}\sum_{k=0}^{N-1} e^{rk\Delta t}\left\|\nabla v_1^{k+1}\right\|^2\Delta t \\
& -\frac{1}{4}\sum_{k=0}^{N} e^{-rk\Delta t}\left\|\sqrt{gh_b}\,\nabla \psi^{k}\right\|^2(\Delta t)^2, \quad \theta\in(0,r\Delta t)\,;
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{1}{2}\sum_{k=0}^{N-1}\left(gh_b\,\nabla\psi^{k+1},\nabla v_1^{k+1}\right)\Delta t \ &=\ \frac{1}{2}\sum_{k=0}^{N-1}\left(gh_b\,\nabla\psi^{k},\nabla v_1^{k}\right)\Delta t+\frac{1}{2}\left(gh_b\,\nabla\psi^{N},\nabla v_1^{N}\right)\Delta t \\
&\equiv\ -\frac{1}{2}\sum_{k=0}^{N} e^{rk\Delta t}\left(gh_b\,\partial_{t^b}(\nabla v_1^{k+1}),\nabla v_1^{k}\right)\Delta t \\
&\overset{(C)}{=}\ -\frac{1}{4}\sum_{k=0}^{N}\partial_{t^f}\left(e^{rk\Delta t}\left\|\sqrt{gh_b}\,\nabla v_1^{k}\right\|^2\right)\Delta t \\
&\quad +\frac{r}{4}e^{\theta}\sum_{k=0}^{N} e^{rk\Delta t}\left\|\sqrt{gh_b}\,\nabla v_1^{k+1}\right\|^2\Delta t \\
&\quad +\frac{1}{4}\sum_{k=0}^{N} e^{rk\Delta t}\left\|\partial_{t^f}(\sqrt{gh_b}\,\nabla v_1^{k})\right\|^2\Delta t^2, \quad \theta\in(0,r\Delta t) \\
&\geq\ \frac{\gamma_*}{4}\left\|\nabla v_1^0\right\|^2+\frac{r\gamma_*}{4}e^{\theta}\sum_{k=0}^{N} e^{rk\Delta t}\left\|\nabla v_1^{k+1}\right\|^2\Delta t \\
&\quad +\frac{1}{4}\sum_{k=0}^{N} e^{-rk\Delta t}\left\|\sqrt{gh_b}\,\nabla \psi^{k}\right\|^2(\Delta t)^2, \quad \theta\in(0,r\Delta t)\,.
\end{aligned}
$$

Similarly, using $v=v_2^{k+1}$ as the test function in (23), we obtain

$$
\begin{aligned}
\sum_{k=0}^{N-1}\left(\partial_{t^c}^2\psi^{k},v_2^{k+1}\right)\Delta t \ &=\ -\sum_{k=0}^{N-1}\left(\partial_{t^b}\psi^{k},\partial_{t^f}v_2^{k}\right)\Delta t-\left(\partial_{t^b}\psi^{0},v_2^{0}\right)+\left(\partial_{t^b}\psi^{N},v_2^{N}\right) \\
&\equiv\ \sum_{k=0}^{N} e^{-rk\Delta t}\left\|\partial_{t^b}\psi^{k}\right\|^2\Delta t+\left(\partial_{t^b}\psi^{1},v_2^{0}\right),
\end{aligned}
$$

where the first equality above results from summation by parts.

$$
\begin{aligned}
\tau_\circ\sum_{k=0}^{N-1}\left(\partial_{t^f}\psi^{k},v_2^{k+1}\right)\Delta t \ &\equiv\ \tau_\circ\sum_{k=0}^{N-1}\left(\partial_{t^b}\psi^{k},v_2^{k}\right)\Delta t+\tau_\circ\left(\partial_{t^b}\psi^{N},v_2^{N}\right)\Delta t-\tau_\circ\left(\partial_{t^b}\psi^{0},v_2^{0}\right)\Delta t \\
&\equiv\ -\tau_\circ\sum_{k=0}^{N} e^{rk\Delta t}\left(\partial_{t^b}v_2^{k+1},v_2^{k}\right)\Delta t-\tau_\circ\left(\partial_{t^b}\psi^{0},v_2^{0}\right)\Delta t
\end{aligned}
$$

$$\stackrel{(C)}{=} \frac{\tau_0}{2}\left\|v_2^0\right\|^2 + \frac{r\tau_0}{2}e^{\theta}\sum_{k=0}^{N} e^{rk\Delta t}\left\|v_2^{k+1}\right\|^2 \Delta t + \tau_0\left(\partial_{t^b}\psi^1, v_2^0\right)\Delta t$$
$$+\frac{\tau_0}{2}\sum_{k=0}^{N} e^{-rk\Delta t}\left\|\partial_{t^b}\psi^k\right\|^2 (\Delta t)^2, \quad \theta \in (0, r\Delta t);$$

$$\mu\sum_{k=0}^{N-1}\left(\partial_{t^f}(\nabla\psi^k), \nabla v_2^{k+1}\right)\Delta t = \frac{\mu}{2}\left\|\nabla v_2^0\right\|^2 + \frac{r\mu}{2}e^{\theta}\sum_{k=0}^{N} e^{rk\Delta t}\left\|\nabla v_2^{k+1}\right\|^2 \Delta t$$
$$+\mu\left(\partial_{t^b}\nabla\psi^1, \nabla v_2^0\right)\Delta t + \frac{\mu}{2}\sum_{k=0}^{N} e^{-rk\Delta t}\left\|\partial_{t^b}(\nabla\psi^k)\right\|^2 (\Delta t)^2,$$
$$\theta \in (0, r\Delta t).$$

The two terms above are manipulated using the definition of the test function followed by an application of (C).

Finally, algebraic manipulation of the following terms from (24) yields:

$$\sum_{k=0}^{N-1}\left(\partial_{t^b}\boldsymbol{\chi}^{k+1}, \boldsymbol{\chi}^{k+1}\right)\Delta t = \frac{1}{2}\sum_{k=0}^{N-1}\left(\left\|\boldsymbol{\chi}^{k+1}\right\|^2 - \left\|\boldsymbol{\chi}^{k}\right\|^2 + \left\|\partial_{t^b}\boldsymbol{\chi}^{k+1}\right\|^2 (\Delta t)^2\right)$$
$$= \frac{1}{2}\left\|\boldsymbol{\chi}^{N}\right\|^2 + \frac{1}{2}\sum_{k=0}^{N-1}\left\|\partial_{t^b}\boldsymbol{\chi}^{k+1}\right\|^2 (\Delta t)^2;$$

$$\sum_{k=0}^{N-1}\left(\hat{\tau}_{bf}{}^k\boldsymbol{\chi}^{k+\frac{1}{2}}, \boldsymbol{\chi}^{k+1}\right)\Delta t = \frac{1}{2}\sum_{k=0}^{N-1}\left\|\sqrt{\hat{\tau}_{bf}{}^k}\,\boldsymbol{\chi}^{k+1}\right\|^2 \Delta t$$
$$+\frac{1}{2}\sum_{k=0}^{N-1}\left(\sqrt{\hat{\tau}_{bf}{}^k}\,(\boldsymbol{\chi}^{k+1} - (\boldsymbol{\chi}^{k+1} - \boldsymbol{\chi}^{k})), \sqrt{\hat{\tau}_{bf}{}^k}\,\boldsymbol{\chi}^{k+1}\right)\Delta t$$
$$\geq \frac{3}{4}\sum_{k=0}^{N-1}\left\|\sqrt{\hat{\tau}_{bf}{}^k}\,\boldsymbol{\chi}^{k+1}\right\|^2 \Delta t - \frac{1}{4}\sum_{k=0}^{N-1}\left\|\sqrt{\hat{\tau}_{bf}{}^k}\,\partial_{t^b}\boldsymbol{\chi}^{k+1}\right\|^2 (\Delta t)^3;$$

$$\mu\sum_{k=0}^{N-1}\left(\beta_1\nabla\boldsymbol{\chi}^{k+1} + \beta_2\nabla\boldsymbol{\chi}^{k}, \nabla\boldsymbol{\chi}^{k+1}\right)\Delta t \geq \mu\left(\beta_1 + \frac{\beta_2}{2}\right)\sum_{k=0}^{N-1}\left\|\nabla\boldsymbol{\chi}^{k+1}\right\|^2 \Delta t$$
$$-\frac{\mu\beta_2}{2}\sum_{k=0}^{N-1}\left\|\partial_{t^b}(\nabla\boldsymbol{\chi}^{k+1})\right\|^2 (\Delta t)^3 .$$

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