Finite Element Approximation to the System of Shallow Water Equations, Part I: Continuous Time A Priori Error Estimates

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FINITE ELEMENT APPROXIMATIONS TO THE SYSTEM OF SHALLOW WATER EQUATIONS, PART I: CONTINUOUS TIME A PRIORI ERROR ESTIMATES *

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Abstract. Various sophisticated finite element models for surface water flow based on the shallow water equations exist in the literature. Gray, Kolar, Luettich, Lynch and Westerink have developed a hydrodynamic model based on the generalized wave continuity equation (GWCE) formulation, and have formulated a Galerkin finite element procedure based on combining the GWCE with the nonconservative momentum equations. Numerical experiments suggest that this method is robust, accurate and suppresses spurious oscillations which plague other models. We analyze a slightly modified Galerkin model which uses the conservative momentum equations (CME). For this GWCE-CME system of equations, we present a continous-time *a priori* error estimate based on an $\mathcal{L}^2$ projection.

Key words. shallow water equations, surface water flow, mass conservation, momentum conservation, finite element method, *a priori* error estimate

AMS subject classifications. 35Q35, 35L65 65N30, 65N15

1. Introduction. In recent years, there has been much interest in the numerical solutions of the shallow water equations. Simulation of shallow water systems can serve numerous purposes. First, it can serve as means for modeling tidal fluctuations for those interested in capturing tidal energy for commercial purposes. Second, these simulations can be used to compute tidal ranges and surges such as tsunamis and hurricanes caused by extreme earthquake and storm events. This information can be used in the development planning of coastal areas. Finally, the shallow water hydrodynamic model can be coupled to a transport model in considering flow and transport phenomenon, thus making it possible to study remediation options for polluted bays and estuaries, to predict the impact of commercial projects on fisheries, to model freshwater-saltwater interactions, and to study allocation of allowable discharges by municipalities and by industry in meeting water quality controls.

The 2-dimensional shallow water equations are obtained by depth (or vertical) averaging of the continuum mass and momentum balances given by the 3-dimensional incompressible Navier-Stokes equations. Shallow water equations can be used to study flow in fluid domains whose bathymetric depth is much smaller than the characteristic length scale in the horizontal direction. We denote by $\xi(\boldsymbol{x},t)$ the free surface elevation over a reference plane and by $h_b(\boldsymbol{x})$ the bathymetric depth under that reference plane so that $H(\boldsymbol{x},t) = \xi + h_b$ is the total water column (see Figure 1). Also, we denote by $\boldsymbol{u}(\boldsymbol{x},t) = [U(\boldsymbol{x},t)\ V(\boldsymbol{x},t)]^T$ the depth-averaged horizontal velocities. Letting $\boldsymbol{q} = \boldsymbol{u}H$, the 2-dimensional governing equations, in operator form [12], are the primitive continuity equation (CE)

$$\mathrm{L}(\xi,\boldsymbol{u};h_b) \quad \equiv \quad \frac{\partial \xi}{\partial t} + \nabla\cdot\boldsymbol{q} = 0,$$

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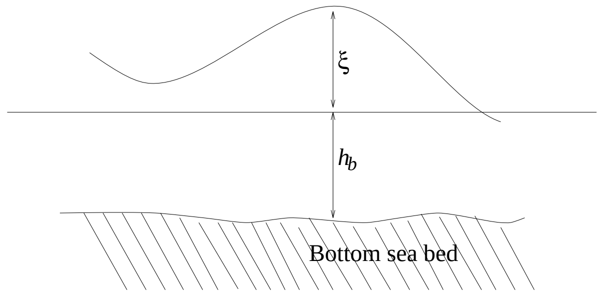
FIG. 1. *Definition of elevation and bathymetry*



and the primitive non-conservative momentum equations (NCME), as derived by Westerink et al [26],

$$\begin{aligned}\mathbf{M}(\xi, \boldsymbol{u}; \Phi) \quad \equiv \quad & \frac{\partial}{\partial t}\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\,\boldsymbol{u} + \tau_{bf}\boldsymbol{u} + \boldsymbol{k} \times f_c\boldsymbol{u} \\ & + g\nabla\xi - \frac{1}{H}E_h\Delta\boldsymbol{q} - \frac{1}{H}\boldsymbol{\tau}_{ws} + \boldsymbol{calF} = 0,\end{aligned}$$

where $\Phi = (h_b, \tau_{bf}, f_c, g, E_h, \boldsymbol{\tau}_{ws}, p_a, \eta)$. In particular, $\tau_{bf}(\xi, \boldsymbol{u})$ is a bottom friction function, $\boldsymbol{k}$ is a unit vector in the vertical direction, f_c is the Coriolis parameter, g is acceleration due to gravity, E_h is the horizontal eddy diffusion/dispersion coefficient, $\boldsymbol{\tau}_{ws}$ is the applied free surface wind stress relative to the reference density of water, and $\boldsymbol{calF} = (\nabla p_a - g\nabla\eta)$, where $p_a(\boldsymbol{x}, t)$ is the atmospheric pressure at the free surface relative to the reference density of water, and $\eta(\boldsymbol{x}, t)$ is the Newtonian equilibrium tide potential relative to the effective Earth elasticity factor. We will treat $\tau_{bf}, \boldsymbol{\tau}_{ws}, f_c, p_a$ and η as data. Moreover, we will treat the diffusion coefficient E_h as a constant.

The primitive conservative momentum equations (CME) are derived from the NCME as

$$\mathbf{M}^c \equiv H\mathbf{M} + \boldsymbol{u}\mathrm{L} = 0.$$

The numerical procedure used to solve the shallow water equations must resolve the physics of the problem without introducing spurious oscillations or excessive numerical diffusion. Westerink et al [26] note a need for greater grid refinement near land boundaries to resolve important processes and to prevent energy from aliasing. Permitting a high degree of grid flexibility, the finite element method is a good candidate.

There has been substantial effort over the past two decades in applying finite element methods to the CE coupled with either the NCME or the CME. Early finite element simulations of shallow water systems were plagued by spurious oscillations. Various methods were introduced to eliminate these oscillations through artificial diffusion [17, 22]. These methods were generally unsuccessful due to excessive damping of physical components of the solution. Recently, Agoshkov et al [2, 4, 3] have investigated a finite element approximation, where the velocity field is approximated by piecewise linear polynomials and the elevation is approximated by the same functions plus some additional ones. They have studied the effects of various boundary conditions, and proven stability of various time discretization schemes for a continuity equation-momentum equation system. In this paper, we will examine a finite element approximation to a modified shallow water model described below. Computational

and experimental evidence in the literature suggest that this formulation leads to approximate solutions with reduced oscillations. Moreover, these approximate solutions have accurately matched actual tidal data. This modified shallow water model is based on a reformulation of the CE, which we now describe.

1.1. Historical Development of the Wave Continuity and Generalized Wave Continuity Equations. In 1979, Lynch and Gray [14] derived the wave continuity equation (WCE) from the mass and momentum conservation equations,

$$\mathbf{W}(\xi, \boldsymbol{u}; \Phi) \equiv \frac{\partial \mathrm{L}}{\partial t} - \nabla\cdot\mathbf{M}^c + \tau \mathrm{L} = 0,$$

as a means to eliminate oscillations without resorting to numerical damping. Here, $\tau(\boldsymbol{x}, t)$ is a non-linear friction coefficient. In this shallow water formulation, the WCE is then coupled to either the CME or the NCME. The equivalence of this model to the more standard one based on the CE is discussed in [12].

This formulation has led to the development of robust finite element algorithms for depth-integrated coastal circulation models. The WCE approach has motivated a substantial computational and analytical effort [6, 8, 14, 18]. Using Fourier phase/space analysis of the linearized WCE-CME and of the WCE-NCME system of equations, Foreman [8] and Kinnmark [12] prove that the WCE formulation suppresses spurious oscillations of the numerical solution, and is capable of capturing "$2\Delta x$" waves. The WCE formulation has also motivated substantial field applications; see [9], [10], [11], [15], [16], [19], [20], [21], [24], [25]. These studies have demonstrated the advantage of the WCE formulation for finite element applications in terms of achieving both a high level of computational accuracy and efficiency.

The generalized wave continuity equation (GWCE) [12] is essentially the same as the WCE except that multiplication of the continuity equation by τ is replaced with multiplication by some general function that may be independent of time. Westerink and Luettich [13] chose to replace τ by a time-independent positive constant $\tau_\circ$. Their version of the GWCE is given by

$$(1) \qquad \frac{\partial^2 \xi}{\partial t^2} + \tau_\circ \frac{\partial \xi}{\partial t} - \nabla\cdot \left[\nabla\cdot \left(\frac{1}{H} \boldsymbol{q}^2 \right) - (\tau_\circ - \tau_{bf}) \boldsymbol{q} + (\boldsymbol{k} \times f_c \boldsymbol{q}) + H g \nabla \xi + E_h \nabla \frac{\partial \xi}{\partial t} - \boldsymbol{\tau}_{ws} + H \boldsymbol{calF} \right] = 0.$$

This choice of $\tau_\circ$ yields a system of time-independent matrices when the GWCE is discretized in time using a three-level implicit scheme for linear terms. (Here and in the equations below we have used tensor notation reviewed in Appendix A.)

The GWCE can be coupled to the CME, given by

$$(2) \quad \frac{\partial \boldsymbol{q}}{\partial t} + \nabla\cdot \left(\frac{1}{H} \boldsymbol{q}^2 \right) + \tau_{bf} \boldsymbol{q} + (\boldsymbol{k} \times f_c \boldsymbol{q}) + H g \nabla \xi - E_h \Delta \boldsymbol{q} - \boldsymbol{\tau}_{ws} + H \boldsymbol{calF} = 0$$

or to the NCME. A finite element simulator based on the GWCE-NCME, which uses same-order polynomials to approximate elevation and velocity unknowns, has been developed by Luettich, et al. In [13], it was demonstrated that the approximations generated by this simulator accurately matched tidal data taken from the English Channel and southern North Sea.

To date, no formal convergence analysis of finite element approximations to the WCE or GWCE combined with either the NCME or the CME exists in the literature. In this paper, we analyze the coupled GWCE-CME system of equations for a continuous time Galerkin finite element approximation.

The rest of this paper is outlined as follows. In section 2 we detail the assumptions we will need in our analysis. We also introduce the weak formulation associated with the GWCE-CME system of equations. In section 3, we introduce the finite element approximation to the weak solution.

In the derivation of an error estimate, we investigated various types of projections, such as the $\mathcal{L}^2$, elliptic, and parabolic projections. Because of the highly nonlinear nature of the coupled system of equations (1)-(2), we found these projections all led to suboptimal estimates. To that end, we present, in section 4, the simplest derivation of an *a priori* error estimate based on an $\mathcal{L}^2$ projection.

2. Preliminaries.

2.1. Notation and Definitions. For the purpose of our analysis, we define some notation used throughout the rest of this paper.

Let Ω be a bounded polygonal domain in $\mathbb{R}^2$ and $\boldsymbol{x}= (x_1, x_2) \in \mathbb{R}^2$.

The $\mathcal{L}^2$ inner product is denoted by

$$(\varphi, \omega) \quad = \quad \int_\Omega \varphi \diamond \omega \; \mathrm{d}x, \qquad \varphi, \omega \in [\mathcal{L}^2(\Omega)]^n,$$

where "$\diamond$" here refers to either multiplication, dot product, or double dot product as appropriate. We denote the $\mathcal{L}^2$ norm by $||\varphi|| = ||\varphi||_{\mathcal{L}^2(\Omega)} = (\varphi,\varphi)^{1/2}$. In $\mathbb{R}^n, \alpha = (\alpha_1, \ldots, \alpha_n)$ is an n-tuple with nonnegative integer components,

$$D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n} = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}$$

and $|\alpha| = \sum_{i=1}^n \alpha_i$.

For ℓ any nonnegative integer, let

$$\mathcal{H}^\ell \equiv \{\varphi \in \mathcal{L}^2(\Omega) \mid D^\alpha \varphi \in \mathcal{L}^2(\Omega) \text{ for } |\alpha| \leq \ell\}$$

be the Sobolev space with norm

$$||\varphi||_{\mathcal{H}^\ell(\Omega)} = \left(\sum_{|\alpha|\leq\ell} ||D^\alpha \cdot \, ||^2_{\mathcal{L}^2(\Omega)} \right)^{1/2} .$$

Additionally, $\mathcal{H}_0^1(\Omega)$ denotes the subspace of $\mathcal{H}^1(\Omega)$ obtained by completing $\mathcal{C}_0^\infty(\Omega)$ with respect to the norm $||\cdot||_{\mathcal{H}^1(\Omega)}$, where $\mathcal{C}_0^\infty(\Omega)$ is the set of infinitely differentiable functions with compact support in Ω.

Moreover, let

$$\mathcal{W}^\ell_\infty \equiv \{\varphi \in \mathcal{L}^\infty(\Omega) \mid D^\alpha \varphi \in \mathcal{L}^\infty(\Omega) \text{ for } |\alpha| \leq \ell\}$$

be the Sobolev space with norm

$$||\varphi||_{\mathcal{W}^\ell_\infty(\Omega)} \quad = \quad \max_{|\alpha|\leq\ell} ||D^\alpha \varphi \, ||_{\mathcal{L}^\infty(\Omega)} .$$

For relevant properties of these spaces, please refer to [1].

Observe, for instance, that $\mathcal{H}^\ell$ are spaces of $\mathbb{R}$-valued functions. Spaces of $\mathbb{R}^n$-valued functions will be denoted in boldface type, but their norms will not be distinguished. Thus, $\boldsymbol{\mathcal{L}}^2(\Omega) = [\mathcal{L}^2(\Omega)]^n$ has norm $||\boldsymbol{\varphi}||^2 = \sum_{i=1}^n ||\varphi_i||^2$; $\boldsymbol{\mathcal{H}}^1(\Omega) = [\mathcal{H}^1(\Omega)]^n$ has norm $||\boldsymbol{\varphi}||^2_{\mathcal{H}^1(\Omega)} = \sum_{i=1}^n \sum_{|\alpha|\leq 1} ||D^\alpha \varphi_i||^2$; etc.

For X, a normed space with norm $||\cdot||_X$ and a map f: $[0,T] \to X$, define

$$\begin{aligned} ||f||^2_{\mathcal{L}^2((0,T);X)} &= \int_0^T ||f(\cdot,t)||^2_X \,\mathrm{d}t, \\ ||f||_{\mathcal{L}^\infty((0,T);X)} &= \sup_{0\le t\le T} ||f(\cdot,t)||_X. \end{aligned}$$

Finally, we let $K, K_i, (i = 0, 1, 2, ..)$ and ϵ be generic constants not necessarily the same at every occurrence.

2.2. Variational Formulation. We will consider the coupled system given by the GWCE-CME described in Section 1, with the following homogeneous Dirichlet boundary conditions for simplicity

$$\left.\begin{aligned} \xi(\boldsymbol{x},t) &= 0, \\ \boldsymbol{u}(\boldsymbol{x},t) &= 0, \end{aligned}\right\} \boldsymbol{x} \in \partial\Omega,\ t>0, \tag{3}$$

and with the compatible initial conditions

$$\left.\begin{aligned} \xi(\boldsymbol{x},0) &= \xi_0(\boldsymbol{x}), \\ \tfrac{\partial\xi}{\partial t}(\boldsymbol{x},0) &= \xi_1(\boldsymbol{x}), \\ \boldsymbol{u}(\boldsymbol{x},0) &= \boldsymbol{u}_0(\boldsymbol{x}), \end{aligned}\right\} \boldsymbol{x} \in \bar{\Omega}, \tag{4}$$

where $\partial\Omega$ is the boundary of $\Omega \subset \mathbb{R}^2$ and $\bar{\Omega} = \Omega \cup \partial\Omega$. Extensions to more general land and sea boundary conditions will be treated in a later paper. As noted in Kinnmark [12], the condition necessary for the solution of the GWCE-CME system of equations to be the same as the solution of the primitive form is that

$$\xi_1(\boldsymbol{x}) = -\nabla\cdot\boldsymbol{q}(\boldsymbol{x},0).$$

The weak form of this system is the following: For $t \in (0,T]$, find $\xi(\boldsymbol{x},t) \in \mathcal{H}^1_0(\Omega)$ and $\boldsymbol{q}(\boldsymbol{x},t) \in \boldsymbol{\mathcal{H}}^1_0(\Omega)$ such that

$$\begin{aligned} &\left(\frac{\partial^2\xi}{\partial t^2}, v\right) + \tau_\circ\left(\frac{\partial\xi}{\partial t}, v\right) + \underbrace{\left(\nabla\cdot\left\{\frac{1}{H}\boldsymbol{q}^2\right\}, \nabla v\right)}_{(a)} + ((\tau_{bf}-\tau_\circ)\boldsymbol{q}, \nabla v) \\ &\quad + (\boldsymbol{k}\times f_c\boldsymbol{q}, \nabla v) + (Hg\nabla\xi, \nabla v) + E_h\left(\nabla\frac{\partial\xi}{\partial t}, \nabla v\right) \\ &\quad - (\boldsymbol{\tau}_{ws}, \nabla v) + (H\boldsymbol{calF}, \nabla v) = 0, \qquad \forall v \in \mathcal{H}^1_0(\Omega), t>0, \end{aligned} \tag{5}$$

$$\begin{aligned} &\left(\frac{\partial\boldsymbol{q}}{\partial t}, \boldsymbol{w}\right) + \underbrace{\left(\nabla\cdot\left\{\frac{1}{H}\boldsymbol{q}^2\right\}, \boldsymbol{w}\right)}_{(b)} + (\tau_{bf}\boldsymbol{q}, \boldsymbol{w}) + (\boldsymbol{k}\times f_c\boldsymbol{q}, \boldsymbol{w}) + (Hg\nabla\xi, \boldsymbol{w}) \\ &\quad + E_h(\nabla\boldsymbol{q}, \nabla\boldsymbol{w}) - (\boldsymbol{\tau}_{ws}, \boldsymbol{w}) + (H\boldsymbol{calF}, \boldsymbol{w}) = 0, \qquad \forall \boldsymbol{w} \in \boldsymbol{\mathcal{H}}^1_0(\Omega), t>0, \end{aligned} \tag{6}$$

with initial conditions

$$\begin{aligned} (\xi(\boldsymbol{x},0), v) &= (\xi_0(\boldsymbol{x}), v), & \forall v &\in \mathcal{H}^1_0(\Omega), \\ (\tfrac{\partial\xi}{\partial t}(\boldsymbol{x},0), v) &= (\xi_1(\boldsymbol{x}), v), & \forall v &\in \mathcal{H}^1_0(\Omega), \\ (\boldsymbol{q}(\boldsymbol{x},0), \boldsymbol{w}) &= (\boldsymbol{q}_0(\boldsymbol{x}), \boldsymbol{w}), & \forall \boldsymbol{w} &\in \boldsymbol{\mathcal{H}}^1_0(\Omega). \end{aligned} \tag{7}$$

Here, we have set $\boldsymbol{q}_0 = \boldsymbol{u}_0 H_0$.

2.3. Some Assumptions. Our analysis requires that we make certain physically reasonable assumptions about the solutions and the data. First, we assume for $(\boldsymbol{x}, t) \in \bar{\Omega} \times (0, T]$

A1 the solutions $(\xi,\, \boldsymbol{q})$ to (5)-(7) exist and are unique,
A2 $\exists$ positive constants H_* and H^* such that $H_* \leq H(\boldsymbol{x}, t) \leq H^*$,
A3 the velocities $U(\boldsymbol{x}, t), V(\boldsymbol{x}, t)$ are bounded,
A4 $\frac{\partial}{\partial x_i} h_b(\boldsymbol{x})$ is bounded.

Assumption **A2** is obvious from Figure 1. Dimensional analysis as explained in [23] accounts for assumption **A3**. Second, we assume that for $(\boldsymbol{x}, t) \in \bar{\Omega} \times (0, T]$

A5 $\exists$ positive constants γ_* and γ^* such that $\gamma_* \leq g h_b(\boldsymbol{x}) \leq \gamma^*$,
A6 $\exists$ non-negative constants τ_* and τ^* such that $\tau_* \leq \tau_{bf} \leq \tau^*$,
A7 $(\tau_{bf} - \tau_\circ)$ is bounded,
A8 $\exists$ non-negative constants f_* and f^* such that $f_* \leq f_c \leq f^*$,
A9 E_h is a positive constant,
A10 $\nabla p_a(\boldsymbol{x}, t)$ and $\nabla \eta(\boldsymbol{x}, t)$ are bounded.

Finally, we make the following smoothness assumptions on the initial data and on the solutions:

A11 $\xi_0(\boldsymbol{x}), \xi_1(\boldsymbol{x}) \in \mathcal{H}_0^1(\Omega)$,
A12 $\boldsymbol{q}_0(\boldsymbol{x}) \in \boldsymbol{\mathcal{H}}_0^1(\Omega)$,
A13 $H(\boldsymbol{x}, \cdot) \in \mathcal{H}_0^1(\Omega) \cap \mathcal{H}^\ell(\Omega) \cap \mathcal{W}_\infty^1(\Omega), \;\; t \in (0, T)$,
A14 $\boldsymbol{q}(\boldsymbol{x}, \cdot) \in \boldsymbol{\mathcal{H}}_0^1(\Omega) \cap \boldsymbol{\mathcal{H}}^\ell(\Omega) \cap \boldsymbol{\mathcal{W}}_\infty^1(\Omega), \;\; t \in (0, T)$,

where ℓ is a positive integer defined below.

3. Finite Element Approximation.

3.1. The Continuous-Time Galerkin Approximation. Let $\mathcal{T}$ be a quasi-uniform triangulation of Ω into elements $E_i,\ i = 1, \ldots, m,$ with $\mathrm{diam}(E_i) = h_i$ and $h = \max_i h_i$. Let $\mathcal{S}^h$ denote a finite dimensional subspace of $\mathcal{H}_0^1(\Omega)$ defined on this triangulation consisting of piecewise polynomials of degree at most $(s_1 - 1)$. Define $\mathcal{H}(\Omega) = \mathcal{H}_0^1(\Omega) \cap \mathcal{H}^\ell(\Omega)$, and assume $\mathcal{S}^h$ satisfies the standard approximation property

$$\inf_{\varphi \in \mathcal{S}^h} ||v - \varphi||_{\mathcal{H}^{s_0}(\Omega)} \;\; \leq \;\; K_0 h^{\ell - s_0} \, ||v||_{\mathcal{H}^\ell(\Omega)}, \quad v \in \mathcal{H}(\Omega), \tag{8}$$

and the inverse assumptions

$$\left.\begin{array}{lcl} ||\varphi||_{\mathcal{H}^{\ell - s_0}(\Omega)} & \leq & K_0 \, ||\varphi||_{\mathcal{L}^2(\Omega)} \, h^{-(\ell - s_0)}, \\ ||\varphi||_{\mathcal{L}^\infty(\Omega)} & \leq & K_0 \, ||\varphi||_{\mathcal{L}^2(\Omega)} \, h^{-1}, \\ ||\nabla \varphi||_{\mathcal{L}^\infty(\Omega)} & \leq & K_0 \, ||\nabla \varphi||_{\mathcal{L}^2(\Omega)} \, h^{-1}, \end{array}\right\} \;\; \varphi \in \mathcal{S}^h(\Omega), \tag{9}$$

for $0 \leq s_0 \leq k,\;\; s_0 \leq \ell \leq s_1$, with k defined from $0 \leq k \leq (s_1 - 1)$, and where K_0 is a constant independent of h and v.

We define the continuous-time Galerkin approximations to $\xi, \boldsymbol{q}$ to be the mappings $\Xi(\boldsymbol{x}, t) \in \mathcal{S}^h$, $\mathcal{Q}(\boldsymbol{x}, t) \in \boldsymbol{\mathcal{S}}^h$ for each $t > 0$ satisfying

$$\begin{aligned} &\left(\frac{\partial^2 \Xi}{\partial t^2}, v\right) + \tau_\circ \left(\frac{\partial \Xi}{\partial t}, v\right) + \underbrace{\left(\nabla \cdot \left\{\frac{1}{\Pi} \mathcal{Q}^2\right\}, \nabla v\right)}_{(a')} + ((\tau_{bf} - \tau_\circ) \mathcal{Q}, \nabla v) \\ &\quad + (\boldsymbol{k} \times f_c \mathcal{Q}, \nabla v) + (\Pi g \nabla \Xi, \nabla v) + E_h \left(\nabla \frac{\partial \Xi}{\partial t}, \nabla v\right) \\ &\quad - (\boldsymbol{\tau}_{ws}, \nabla v) + (\Pi \boldsymbol{calF}, \nabla v) = 0, \qquad\qquad \forall v \in \mathcal{S}^h(\Omega), \end{aligned} \tag{10}$$

$$(11)\qquad \left(\frac{\partial \mathcal{Q}}{\partial t}, \boldsymbol{w}\right) + \underbrace{\left(\nabla\cdot\left\{\frac{1}{\Pi}\mathcal{Q}^2\right\}, \boldsymbol{w}\right)}_{(b')} + (\tau_{bf}\mathcal{Q}, \boldsymbol{w}) + (\boldsymbol{k}\times f_c\mathcal{Q}, \boldsymbol{w}) + (\Pi g\nabla\Xi, \boldsymbol{w})$$
$$+E_h\left(\nabla\mathcal{Q}, \nabla\boldsymbol{w}\right) - (\boldsymbol{\tau}_{ws}, \boldsymbol{w}) + (\Pi \boldsymbol{calF}, \nabla v) = 0, \qquad \forall \boldsymbol{w}\in\boldsymbol{\mathcal{S}}^h(\Omega),$$

with boundary conditions

$$(12)\qquad \left.\begin{array}{rcl} \Xi(\boldsymbol{x},t) & = & 0, \\ \mathcal{Q}(\boldsymbol{x},t) & = & 0, \end{array}\right\} \boldsymbol{x}\in\partial\Omega,\ t>0,$$

and with initial conditions

$$(13)\qquad \begin{array}{rcll} (\Xi(\boldsymbol{x},0), v) & = & (\xi_0(\boldsymbol{x}), v), & \forall v\in\mathcal{S}^h(\Omega), \\ (\frac{\partial\Xi}{\partial t}(\boldsymbol{x},0), v) & = & (\xi_1(\boldsymbol{x}), v), & \forall v\in\mathcal{S}^h(\Omega), \\ (\mathcal{Q}(\boldsymbol{x},0), \boldsymbol{w}) & = & (\boldsymbol{q}_0(\boldsymbol{x}), \boldsymbol{w}), & \forall \boldsymbol{w}\in\boldsymbol{\mathcal{S}}^h(\Omega). \end{array}$$

Here, $\Pi(\boldsymbol{x},t) = h_b(\boldsymbol{x}) + \Xi(\boldsymbol{x},t)$.

4. *A Priori* **Error Estimate.** We will compare our finite element approximations Ξ and $\mathcal{Q}$, satisfying (10)–(13), to $\mathcal{L}^2$ projections $\widetilde{\xi}$ and $\widetilde{\boldsymbol{q}}$ satisfying

$$(14)\qquad \begin{array}{rcll} \left((\xi-\widetilde{\xi})(\cdot,t), v\right) & = & 0, & \forall v\in\mathcal{S}^h, t\geq 0, \\ ((\boldsymbol{q}-\widetilde{\boldsymbol{q}})(\cdot,t), \boldsymbol{w}) & = & 0, & \forall \boldsymbol{w}\in\boldsymbol{\mathcal{S}}^h, t\geq 0. \end{array}$$

For the purpose of succinctness in the rest of the paper, we define

$$(15)\qquad \left\{\begin{array}{rclcrcl} \theta & = & \xi-\widetilde{\xi}, & & \psi & = & \Xi-\widetilde{\xi}, \\ \boldsymbol{\phi} & = & \boldsymbol{q}-\widetilde{\boldsymbol{q}}, & & \boldsymbol{\chi} & = & \mathcal{Q}-\widetilde{\boldsymbol{q}}. \end{array}\right.$$

Clearly, $\xi - \Xi = \theta - \psi$ and $\boldsymbol{q} - \mathcal{Q} = \boldsymbol{\phi} - \boldsymbol{\chi}$. We shall call θ and $\boldsymbol{\phi}$ the *projection errors* and we shall call ψ and $\boldsymbol{\chi}$ the *affine errors.*

The following results are standard.

LEMMA 4.1. *Let $0\leq s_0\leq k$, $s_0\leq \ell\leq s_1$, $0\leq k\leq (s_1-1)$, and $\mathcal{H}(\Omega) = \mathcal{H}_0^1(\Omega)\cap\mathcal{H}^\ell(\Omega)$. Let $\xi\in\mathcal{L}^2((0,T),\mathcal{H}(\Omega))$ and $\boldsymbol{q}\in\mathcal{L}^2((0,T),\boldsymbol{\mathcal{H}}(\Omega))$ and let $(\widetilde{\xi},\widetilde{\boldsymbol{q}})$ be the corresponding $\mathcal{L}^2$ projections defined by (14). And let θ and $\boldsymbol{\phi}$ be defined as above. If for some integer $\jmath\geq 0$*

$$\frac{\partial^\jmath\xi}{\partial t^\jmath}\in\mathcal{L}^2\left((0,T);\mathcal{H}(\Omega)\right),\ \frac{\partial^\jmath\boldsymbol{q}}{\partial t^\jmath},\in\mathcal{L}^2\left((0,T);\boldsymbol{\mathcal{H}}(\Omega)\right),$$

then

$$\frac{\partial^\jmath\widetilde{\xi}}{\partial t^\jmath}\in\mathcal{L}^2\left((0,T);\mathcal{S}^h(\Omega)\right),\ \frac{\partial^\jmath\widetilde{\boldsymbol{q}}}{\partial t^\jmath}\in\mathcal{L}^2\left((0,T);\boldsymbol{\mathcal{S}}^h(\Omega)\right),$$

and

$$\begin{array}{rcl} \left\|\left(\frac{\partial}{\partial t}\right)^\jmath\theta\right\|_{\mathcal{L}^2((0,T);\mathcal{H}^s(\Omega))} & \leq & K_0h^{q-s}\left\|\left(\frac{\partial}{\partial t}\right)^\jmath\xi\right\|_{\mathcal{L}^2((0,T);\mathcal{H}^q(\Omega))}, \\ \left\|\left(\frac{\partial}{\partial t}\right)^\jmath\boldsymbol{\phi}\right\|_{\mathcal{L}^2((0,T);\mathcal{H}^s(\Omega))} & \leq & K_0h^{q-s}\left\|\left(\frac{\partial}{\partial t}\right)^\jmath q\right\|_{\mathcal{L}^2((0,T);\mathcal{H}^q(\Omega))} \end{array}$$

for some constant K_0 independent of ξ, q, h, ℓ, where $q = \min(\ell, s_1)$.

We will also need the following result.

LEMMA 4.2. *$\widetilde{\xi}, \widetilde{\boldsymbol{q}}$ and their first-order spatial derivatives are bounded above in $\mathcal{L}^\infty((0,T);\mathcal{L}^\infty(\Omega))$ by a positive constant K^*.*

Proof. See [5] Corollary 4.8.9. $\square$

Before proceeding, it will be necessary to make certain assumptions about the Galerkin approximations. We employ an argument similar to that made in [7] to handle nonlinearities. In particular, we will assume that the Galerkin approximations are bounded by some constant in order to derive the *a priori* error estimate. Then we will show for sufficiently small h, in the case of polynomials of degree at least two ($s_1 \geq 3$), that we can remove the estimates' dependence on the assumed bound of the approximations, being dependent instead on a smaller bound on the comparison projections.

To that end, we assume that, given K^* defined in Lemma 4.2, $\exists$ positive constants $C_* \leq \frac{H_*}{2}$ and $C^* \geq 2K^*$ such that

B1 $C_* \leq \Pi(\boldsymbol{x},t) \leq C^*$, and

B2 $||\mathcal{Q}||_{\mathcal{L}^\infty((0,T);\mathcal{L}^\infty(\Omega))} \leq C^*$.

The following lemma will be needed when we bound the right-hand sides of (18), (19), and (20).

LEMMA 4.3. *Let Assumptions* **A2, B1** *hold. There exists constants $K_1(C_*), K_2(C_*)$ such that*

$$\left|\left|\frac{\boldsymbol{q}}{H} - \frac{\mathcal{Q}}{\Pi}\right|\right| \leq K_1\left(||\theta|| + ||\psi||\right) + K_2\left(||\boldsymbol{\phi}|| + ||\boldsymbol{\chi}||\right).$$

Proof.

$$\begin{aligned}
\left|\left|\frac{\boldsymbol{q}}{H} - \frac{\mathcal{Q}}{\Pi}\right|\right| &= \left|\left|\frac{\boldsymbol{q}\,(\Pi - H) + (\boldsymbol{q} - \mathcal{Q})\,H}{H\Pi}\right|\right| \\
&\leq \left|\left|\frac{\boldsymbol{q}}{H\Pi}\right|\right|_{\mathcal{L}^\infty(\Omega)} ||\Pi - H|| + \left|\left|\frac{1}{\Pi}\right|\right|_{\mathcal{L}^\infty(\Omega)} ||\boldsymbol{q} - \mathcal{Q}|| \\
&= K_1\,||\Xi - \xi|| + K_2\,||\boldsymbol{q} - \mathcal{Q}||.
\end{aligned}$$

Assumptions **A2, B1** are used to get the first part of the inequality and assumption **B1** is used to get the second part of the inequality. $\square$

4.1. Error Estimate. In order to obtain an error estimate for $(\xi - \Xi)$ and $(\boldsymbol{q} - \mathcal{Q})$, we must first obtain an estimate on the affine error terms $(\Xi - \widetilde{\xi})$ and $(\mathcal{Q} - \widetilde{q})$. Then, with the approximation result stated in Lemma 4.1 and with the estimate on the affine error to be obtained in the proof of Theorem 4.4, an application of the triangle inequality will yield an estimate for $(\xi - \Xi)$ and $(\boldsymbol{q} - \mathcal{Q})$.

It will be useful to employ the following expansion of terms (a)–(b) in (5)–(6):

$$\nabla\cdot\left\{\frac{1}{H}\boldsymbol{q}^2\right\} = \left(\frac{\boldsymbol{q}}{H}\cdot\nabla\boldsymbol{q}\right) + (\nabla\cdot\boldsymbol{q})\frac{\boldsymbol{q}}{H} - (\nabla h_b\cdot\boldsymbol{q})\frac{\boldsymbol{q}}{H^2} - (\nabla\xi\cdot\boldsymbol{q})\frac{\boldsymbol{q}}{H^2}.$$

Similarly, the expansion of terms (a')–(b') in (10)–(11) gives

$$\nabla\cdot\left\{\frac{1}{\Pi}\mathcal{Q}^2\right\} = \left(\frac{\mathcal{Q}}{\Pi}\cdot\nabla\mathcal{Q}\right) + (\nabla\cdot\mathcal{Q})\frac{\mathcal{Q}}{\Pi} - (\nabla h_b\cdot\mathcal{Q})\frac{\mathcal{Q}}{\Pi^2} - (\nabla\Xi\cdot\mathcal{Q})\frac{\mathcal{Q}}{\Pi^2}.$$

Subtract (5) from (10) and (6) from (11), using the fact that we can write

$$
\begin{aligned}
&\left(\frac{\mathcal{Q}}{\Pi}\cdot\nabla(\mathcal{Q}-\widetilde{\boldsymbol{q}})\right)-\left(\frac{\boldsymbol{q}}{H}\cdot\nabla\boldsymbol{q}\right)+\left(\frac{\mathcal{Q}}{\Pi}\cdot\nabla\widetilde{\boldsymbol{q}}\right)\\
&\qquad= \left(\frac{\mathcal{Q}}{\Pi}\cdot\nabla\boldsymbol{\chi}\right)-\left(\frac{\boldsymbol{q}}{H}\cdot\nabla(\boldsymbol{q}-\widetilde{\boldsymbol{q}})\right)-\left(\left[\frac{\boldsymbol{q}}{H}-\frac{\mathcal{Q}}{\Pi}\right]\cdot\nabla\widetilde{\boldsymbol{q}}\right)\\
&\qquad= \left(\frac{\mathcal{Q}}{\Pi}\cdot\nabla\boldsymbol{\chi}\right)-\left(\frac{\boldsymbol{q}}{H}\cdot\nabla\boldsymbol{\phi}\right)-\left(\left[\frac{\boldsymbol{q}}{H}-\frac{\mathcal{Q}}{\Pi}\right]\cdot\nabla\widetilde{\boldsymbol{q}}\right);
\end{aligned}
$$

$$
\begin{aligned}
&\nabla\cdot(\mathcal{Q}-\widetilde{\boldsymbol{q}})\frac{\mathcal{Q}}{\Pi}-(\nabla\cdot\boldsymbol{q})\,\frac{\boldsymbol{q}}{H}+(\nabla\cdot\widetilde{\boldsymbol{q}})\,\frac{\mathcal{Q}}{\Pi}\\
&\qquad= (\nabla\cdot\boldsymbol{\chi})\frac{\mathcal{Q}}{\Pi}-(\nabla\cdot\boldsymbol{\phi})\frac{\boldsymbol{q}}{H}-(\nabla\cdot\widetilde{\boldsymbol{q}})\left[\frac{\boldsymbol{q}}{H}-\frac{\mathcal{Q}}{\Pi}\right];
\end{aligned}
$$

$$
\begin{aligned}
&\left(\nabla(\Xi-\widetilde{\xi})\cdot\mathcal{Q}\right)\frac{\mathcal{Q}}{\Pi^2}-(\nabla\xi\cdot\boldsymbol{q})\frac{\boldsymbol{q}}{H^2}+(\nabla\widetilde{\xi}\cdot\mathcal{Q})\frac{\mathcal{Q}}{\Pi^2}\\
&\qquad= (\nabla\psi\cdot\mathcal{Q})\,\frac{\mathcal{Q}}{\Pi^2}-(\nabla\theta\cdot\boldsymbol{q})\frac{\boldsymbol{q}}{H^2}-\nabla\widetilde{\xi}\cdot\left\{\left[\frac{\boldsymbol{q}}{H}\right]^2-\left[\frac{\mathcal{Q}}{\Pi}\right]^2\right\};
\end{aligned}
$$

and

$$
\begin{aligned}
\Pi g\nabla(\Xi-\widetilde{\xi})-Hg\nabla\xi+\Pi g\nabla\widetilde{\xi} &= \Pi g\nabla\psi-Hg\nabla(\xi-\widetilde{\xi})-(H-\Pi)g\nabla\widetilde{\xi}\\
&= \Pi g\nabla\psi-Hg\nabla\theta-\theta g\nabla\widetilde{\xi}+\psi g\nabla\widetilde{\xi}.
\end{aligned}
$$

Consequently, we obtain the following GWCE-CME error equations:

$$
\begin{aligned}
(16)\qquad &\left(\frac{\partial^2\psi}{\partial t^2},v\right)+\tau_\circ\left(\frac{\partial\psi}{\partial t},v\right)+\left(\left(\frac{\mathcal{Q}}{\Pi}\cdot\nabla\boldsymbol{\chi}\right),\nabla v\right)+\left((\nabla\cdot\boldsymbol{\chi})\,\frac{\mathcal{Q}}{\Pi},\nabla v\right)\\
&+\left((\nabla\psi\cdot\mathcal{Q})\,\frac{\mathcal{Q}}{\Pi^2},\nabla v\right)+((\tau_{bf}-\tau_\circ)\boldsymbol{\chi},\nabla v)+(\boldsymbol{k}\times f_c\boldsymbol{\chi},\nabla v)+(\Pi g\nabla\psi,\nabla v)\\
&+E_h\left(\nabla\frac{\partial\psi}{\partial t},\nabla v\right)+(\psi\boldsymbol{calF},\nabla v)\\
&=\left(\left(\frac{\boldsymbol{q}}{H}\cdot\nabla\boldsymbol{\phi}\right),\nabla v\right)+\left(\left[\frac{\boldsymbol{q}}{H}-\frac{\mathcal{Q}}{\Pi}\right]\cdot\nabla\widetilde{\boldsymbol{q}},\nabla v\right)+\left((\nabla\cdot\boldsymbol{\phi})\frac{\boldsymbol{q}}{H},\nabla v\right)\\
&+\left((\nabla\cdot\widetilde{\boldsymbol{q}})\left[\frac{\boldsymbol{q}}{H}-\frac{\mathcal{Q}}{\Pi}\right],\nabla v\right)+\left(\nabla h_b\cdot\left\{\left(\frac{\boldsymbol{q}}{H}\right)^2-\left(\frac{\mathcal{Q}}{\Pi}\right)^2\right\},\nabla v\right)\\
&+\left((\nabla\theta\cdot\boldsymbol{q})\,\frac{\boldsymbol{q}}{H^2},\nabla v\right)+\left(\nabla\widetilde{\xi}\cdot\left\{\left(\frac{\boldsymbol{q}}{H}\right)^2-\left(\frac{\mathcal{Q}}{\Pi}\right)^2\right\},\nabla v\right)+((\tau_{bf}-\tau_\circ)\boldsymbol{\phi},\nabla v)\\
&+(\boldsymbol{k}\times f_c\boldsymbol{\phi},\nabla v)+(Hg\nabla\theta,\nabla v)+\left(\theta g\nabla\widetilde{\xi},\nabla v\right)-\left(\psi g\nabla\widetilde{\xi},\nabla v\right)\\
&+E_h\left(\nabla\frac{\partial\theta}{\partial t},\nabla v\right)+(\theta\boldsymbol{calF},\nabla v),\qquad\qquad \forall v\in\mathcal{S}^h(\Omega),t>0.
\end{aligned}
$$

$$
(17)\qquad \left(\frac{\partial\boldsymbol{\chi}}{\partial t},\boldsymbol{w}\right)+\left(\left(\frac{\mathcal{Q}}{\Pi}\cdot\nabla\boldsymbol{\chi}\right),\boldsymbol{w}\right)+\left((\nabla\cdot\boldsymbol{\chi})\,\frac{\mathcal{Q}}{\Pi},\boldsymbol{w}\right)+\left((\nabla\psi\cdot\mathcal{Q})\,\frac{\mathcal{Q}}{\Pi^2},\boldsymbol{w}\right)
$$

$$
\begin{aligned}
&+(\tau_{bf}\boldsymbol{\chi},\boldsymbol{w})+(\boldsymbol{k}\times f_c\boldsymbol{\chi},\boldsymbol{w})+(\Pi g\nabla\psi,\boldsymbol{w})+E_h(\nabla\boldsymbol{\chi},\nabla\boldsymbol{w})+(\psi\boldsymbol{calF},\boldsymbol{w})\\
&=\left(\left(\frac{\boldsymbol{q}}{H}\cdot\nabla\boldsymbol{\phi}\right),\boldsymbol{w}\right)+\left(\left[\frac{\boldsymbol{q}}{H}-\frac{\boldsymbol{Q}}{\Pi}\right]\cdot\nabla\widetilde{\boldsymbol{q}},\boldsymbol{w}\right)+\left((\nabla\cdot\boldsymbol{\phi})\frac{\boldsymbol{q}}{H},\boldsymbol{w}\right)\\
&+\left((\nabla\cdot\widetilde{\boldsymbol{q}})\left[\frac{\boldsymbol{q}}{H}-\frac{\boldsymbol{Q}}{\Pi}\right],\boldsymbol{w}\right)+\left(\nabla h_b\cdot\left\{\left(\frac{\boldsymbol{q}}{H}\right)^2-\left(\frac{\boldsymbol{Q}}{\Pi}\right)^2\right\},\boldsymbol{w}\right)\\
&+\left((\nabla\theta\cdot\boldsymbol{q})\frac{\boldsymbol{q}}{H^2},\boldsymbol{w}\right)+\left(\nabla\widetilde{\xi}\cdot\left\{\left(\frac{\boldsymbol{q}}{H}\right)^2-\left(\frac{\boldsymbol{Q}}{\Pi}\right)^2\right\},\boldsymbol{w}\right)+(\tau_{bf}\boldsymbol{\phi},\boldsymbol{w})\\
&+(\boldsymbol{k}\times f_c\boldsymbol{\phi},\boldsymbol{w})+(Hg\nabla\theta,\boldsymbol{w})+\left(\theta g\nabla\widetilde{\xi},\boldsymbol{w}\right)-\left(\psi g\nabla\widetilde{\xi},\boldsymbol{w}\right)\\
&+E_h(\nabla\boldsymbol{\phi},\nabla\boldsymbol{w})+(\theta\boldsymbol{calF},\boldsymbol{w})\qquad\qquad \forall\boldsymbol{w}\in\boldsymbol{\mathcal{S}}^h(\Omega),t>0.
\end{aligned}
$$

4.1.1. Choice of Test Functions and their Manipulation. We now choose special test functions to obtain the affine error estimate. Let r be a positive constant to be chosen. We let $v_1(\cdot,t)=\int_t^T e^{-rs}\psi(\cdot,s)\,\mathrm{d}s$ and $v_2(\cdot,t)=\int_t^T e^{-rs}\frac{\partial}{\partial t}\psi(\cdot,s)\,\mathrm{d}s$ be the test functions in (16). And, we let $\boldsymbol{w}=\boldsymbol{\chi}$ be the test function in (17).

First, we will investigate the use of v_1 and v_2 as the test functions in (16) followed by temporally integrating over $(0,T]$. Note that $v_1(\cdot,T)=0$, $v_2(\cdot,T)=0$. Also recall that given ς, the following relations hold: $\left(\frac{\partial\varsigma}{\partial t},\varsigma\right)=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(||\varsigma||^2\right)$ and $\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-rt}\,||\varsigma||^2\right)=\frac{1}{2}e^{-rt}\frac{\mathrm{d}}{\mathrm{d}t}\left(||\varsigma||^2\right)-\frac{r}{2}e^{-rt}\,||\varsigma||^2$.

Now, consider the first two terms of (16). When $v=v_1$, we obtain, upon integrating by parts,

$$
\begin{aligned}
\int_0^T\left(\frac{\partial^2\psi}{\partial t^2},v_1\right)\mathrm{d}t &= -\int_0^T\left(\frac{\partial\psi}{\partial t},\frac{\partial v_1}{\partial t}\right)\mathrm{d}t+\left(\frac{\partial\psi}{\partial t},v_1\right)\Big|_0^T=\int_0^T e^{-rt}\left(\frac{\partial\psi}{\partial t},\psi\right)\mathrm{d}t\\
&= \frac{1}{2}\int_0^T\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-rt}\,||\psi||^2\right)\mathrm{d}t+\frac{r}{2}\int_0^T e^{-rt}\,||\psi||^2\mathrm{d}t\\
&= \frac{1}{2}e^{-rT}\,||\psi(\cdot,T)||^2+\frac{r}{2}\int_0^T e^{-rt}\,||\psi||^2\mathrm{d}t
\end{aligned}
$$

and

$$
\tau_\circ\int_0^T\left(\frac{\partial\psi}{\partial t},v_1\right)\mathrm{d}t \;=\; -\tau_\circ\int_0^T\left(\psi,\frac{\partial v_1}{\partial t}\right)\mathrm{d}t=\tau_\circ\int_0^T e^{-rt}\,||\psi||^2\mathrm{d}t.
$$

We also have from the diffusion term upon integrating by parts in time:

$$
E_h\int_0^T\left(\nabla\frac{\partial\psi}{\partial t},\nabla v_1\right)\mathrm{d}t \;=\; E_h\int_0^T e^{-rt}\,||\nabla\psi||^2\mathrm{d}t.
$$

We are also able to manipulate part of the eighth term in (16) by using the definition of v_1 as follows:

$$
\begin{aligned}
\int_0^T(gh_b\nabla\psi,\nabla v_1)\,\mathrm{d}t &= -\frac{1}{2}\int_0^T e^{rt}\frac{\mathrm{d}}{\mathrm{d}t}(gh_b\nabla v_1,\nabla v_1)\,\mathrm{d}t\\
&= -\frac{1}{2}\int_0^T\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{rt}\left|\left|\sqrt{gh_b}\nabla v_1\right|\right|^2\right)\mathrm{d}t+\frac{r}{2}\int_0^T e^{rt}\left|\left|\sqrt{gh_b}\nabla v_1\right|\right|^2\mathrm{d}t\\
&\geq \frac{\gamma_*}{2}\,||\nabla v_1(\boldsymbol{x},0)||^2+\frac{r\gamma_*}{2}\int_0^T e^{rt}\,||\nabla v_1||^2\mathrm{d}t.
\end{aligned}
$$

Similarly, using $v = v_2$ as the test function in (16) followed by temporally integrating over $(0, T]$ yields

$$\int_0^T \left(\frac{\partial^2 \psi}{\partial t^2}, v_2 \right) \mathrm{d}t = -\int_0^T \left(\frac{\partial \psi}{\partial t}, \frac{\partial v_2}{\partial t} \right) \mathrm{d}t = \int_0^T e^{-rt} \left|\left| \frac{\partial \psi}{\partial t} \right|\right|^2 \mathrm{d}t;$$

$$\begin{aligned} \tau_\circ \int_0^T \left(\frac{\partial \psi}{\partial t}, v_2 \right) \mathrm{d}t &= -\tau_\circ \int_0^T \left(\psi, \frac{\partial v_2}{\partial t} \right) \mathrm{d}t = \tau_\circ \int_0^T e^{-rt} \left(\psi, \frac{\partial \psi}{\partial t} \right) \mathrm{d}t \\ &= \frac{\tau_\circ}{2} e^{-rT} ||\psi(\cdot, T)||^2 + \tau_\circ \frac{r}{2} \int_0^T e^{-rt} ||\psi||^2 \mathrm{d}t. \end{aligned}$$

The first equality below follows from the definition of v_2:

$$\begin{aligned} E_h \int_0^T \left(\nabla \frac{\partial \psi}{\partial t}, \nabla v_2 \right) \mathrm{d}t &= -\frac{E_h}{2} \int_0^T e^{rt} \frac{\mathrm{d}}{\mathrm{d}t} (\nabla v_2, \nabla v_2)\, \mathrm{d}t \\ &= -\frac{E_h}{2} \int_0^T \frac{\mathrm{d}}{\mathrm{d}t} \left(e^{rt} ||\nabla v_2||^2 \right) \mathrm{d}t + \frac{rE_h}{2} \int_0^T e^{rt} ||\nabla v_2||^2 \mathrm{d}t \\ &= \frac{E_h}{2} ||\nabla v_2(\boldsymbol{x}, 0)||^2 + \frac{rE_h}{2} \int_0^T e^{rt} ||\nabla v_2||^2 \mathrm{d}t. \end{aligned}$$

The temporal integration of terms $\left(\frac{\partial \boldsymbol{\chi}}{\partial t}, \boldsymbol{\chi} \right), (\tau_{bf} \boldsymbol{\chi}, \boldsymbol{\chi}), E_h (\nabla \boldsymbol{\chi}, \nabla \boldsymbol{\chi})$ in (17) are straightforward.

4.1.2. Bounding the GWCE Error Equations. Using $v_1(\cdot, t) = \int_t^T e^{-rs} \psi(\cdot, s)\, \mathrm{d}s$ as the test function in (16), integrating in time over $(0, T]$, and using the relations above yields

$$\begin{aligned} (18)\quad & \frac{1}{2} e^{-rT} ||\psi(\cdot, T)||^2 + \frac{(r + 2\tau_\circ)}{2} \int_0^T e^{-rt} ||\psi||^2 \mathrm{d}t + \frac{\gamma_*}{2} ||\nabla v_1(\boldsymbol{x}, 0)||^2 \\ & + r \frac{\gamma_*}{2} \int_0^T e^{rt} ||\nabla v_1||^2 \mathrm{d}t + E_h \int_0^T e^{-rt} ||\nabla \psi||^2 \mathrm{d}t \\ & \leq -\int_0^T \left(\left(\frac{\mathcal{Q}}{\Pi} \cdot \nabla \boldsymbol{\chi} \right), \nabla v_1 \right) \mathrm{d}t - \int_0^T \left((\nabla \cdot \boldsymbol{\chi}) \frac{\mathcal{Q}}{\Pi}, \nabla v_1 \right) \mathrm{d}t \\ & + \int_0^T \left((\nabla \psi \cdot \mathcal{Q}) \frac{\mathcal{Q}}{\Pi^2}, \nabla v_1 \right) \mathrm{d}t - \int_0^T ((\tau_{bf} - \tau_\circ) \boldsymbol{\chi}, \nabla v_1)\, \mathrm{d}t \\ & - \int_0^T (\boldsymbol{k} \times f_c \boldsymbol{\chi}, \nabla v_1)\, \mathrm{d}t - \int_0^T (\Xi g \nabla \psi, \nabla v_1)\, \mathrm{d}t + \int_0^T (\psi \boldsymbol{calF}, \nabla v_1)\, \mathrm{d}t \\ & + \int_0^T \left(\left(\frac{\boldsymbol{q}}{H} \cdot \nabla \boldsymbol{\phi} \right), \nabla v_1 \right) \mathrm{d}t + \int_0^T \left(\left[\frac{\boldsymbol{q}}{H} - \frac{\mathcal{Q}}{\Pi} \right] \cdot \nabla \widetilde{\boldsymbol{q}}, \nabla v_1 \right) \mathrm{d}t \\ & + \int_0^T \left((\nabla \cdot \boldsymbol{\phi}) \frac{\boldsymbol{q}}{H}, \nabla v_1 \right) \mathrm{d}t + \int_0^T \left((\nabla \cdot \widetilde{\boldsymbol{q}}) \left[\frac{\boldsymbol{q}}{H} - \frac{\mathcal{Q}}{\Pi} \right], \nabla v_1 \right) \mathrm{d}t \\ & + \int_0^T \left(\nabla h_b \cdot \left\{ \left[\frac{\boldsymbol{q}}{H} \right]^2 - \left[\frac{\mathcal{Q}}{\Pi} \right]^2 \right\}, \nabla v_1 \right) \mathrm{d}t + \int_0^T \left((\nabla \theta \cdot \boldsymbol{q}) \frac{\boldsymbol{q}}{H^2}, \nabla v_1 \right) \mathrm{d}t \\ & + \int_0^T \left(\nabla \widetilde{\xi} \cdot \left\{ \left[\frac{\boldsymbol{q}}{H} \right]^2 - \left[\frac{\mathcal{Q}}{\Pi} \right]^2 \right\}, \nabla v_1 \right) \mathrm{d}t + \int_0^T ((\tau_{bf} - \tau_\circ) \boldsymbol{\phi}, \nabla v_1)\, \mathrm{d}t \end{aligned}$$

$$
\begin{aligned}
&+\int_0^T (\boldsymbol{k}\times f_c\boldsymbol{\phi},\nabla v_1)\,\mathrm{d}t+\int_0^T (Hg\nabla\theta,\nabla v_1)\,\mathrm{d}t+\int_0^T \left(\theta g\nabla\widetilde{\xi},\nabla v_1\right)\,\mathrm{d}t\\
&-\int_0^T \left(\psi g\nabla\widetilde{\xi},\nabla v_1\right)\,\mathrm{d}t+E_h\int_0^T \left(\nabla\frac{\partial\theta}{\partial t},\nabla v_1\right)\,\mathrm{d}t+\int_0^T (\theta\boldsymbol{calF},\nabla v_1)\,\mathrm{d}t\\
&=(\mathcal{A}_1+\cdots+\mathcal{A}_7)+(\mathcal{P}_1+\cdots+\mathcal{P}_{14})\,.
\end{aligned}
$$

Here, $\mathcal{A}$ denotes an affine error term and $\mathcal{P}$ denotes a projection error term.

We will explore the nonlinear terms appearing in the right-hand side of (18) in more detail. It will be implicitly understood that we use either the the Hölder Inequality or the Arithmetic Geometric Mean Inequality or both in the treatment of the affine and projection error terms. We will explicitly mention additional justifications in the derivation of bounds for these terms.

From Assumptions **A9, B1 B2**, there exists $K_3=K_3\left(E_h,C_*,C^*\right)$ and $K_4=K_4\left(E_h,C_*,C^*\right)$ with which we obtain the following bounds on $\mathcal{A}_1$, $\mathcal{A}_2$ and on $\mathcal{A}_3$:

$$
\begin{aligned}
\mathcal{A}_1 &= -\int_0^T \left(\left(\frac{\mathcal{Q}}{\Pi}\cdot\nabla\boldsymbol{\chi}\right),\nabla v_1\right)\,\mathrm{d}t\le\epsilon\int_0^T e^{-rt}\,||\nabla\boldsymbol{\chi}||^2\mathrm{d}t+K_3\int_0^T e^{rt}\,||\nabla v_1||^2\mathrm{d}t,\\
\mathcal{A}_2 &= -\int_0^T \left((\nabla\cdot\boldsymbol{\chi})\frac{\mathcal{Q}}{\Pi},\nabla v_1\right)\,\mathrm{d}t\le\epsilon\int_0^T e^{-rt}\,||\nabla\boldsymbol{\chi}||^2\mathrm{d}t+K_3\int_0^T e^{rt}\,||\nabla v_1||^2\mathrm{d}t,\\
\mathcal{A}_3 &= \int_0^T \left((\nabla\psi\cdot\mathcal{Q})\frac{\mathcal{Q}}{\Pi^2},\nabla v_1\right)\,\mathrm{d}t\le\epsilon\int_0^T e^{-rt}\,||\nabla\psi||^2\mathrm{d}t+K_4\int_0^T e^{rt}\,||\nabla v_1||^2\mathrm{d}t.
\end{aligned}
$$

From Assumption **A7**, there exists $K_5=K_5\left(||\tau_{bf}-\tau_\circ||_{\mathcal{L}^\infty((0,T);\mathcal{L}^\infty(\Omega))}\right)$ such that we obtain a bound on $\mathcal{A}_4$:

$$
\mathcal{A}_4 = -\int_0^T ((\tau_{bf}-\tau_\circ)\boldsymbol{\chi},\nabla v_1)\,\mathrm{d}t\le K\int_0^T e^{-rt}\,||\boldsymbol{\chi}||^2\mathrm{d}t+K_5\int_0^T e^{rt}\,||\nabla v_1||^2\mathrm{d}t.
$$

From Assumption **A8**, there exists $K_6=K_6\left(f^*\right)$ such that we obtain the following bound on $\mathcal{A}_5$:

$$
\mathcal{A}_5 = -\int_0^T (\boldsymbol{k}\times f_c\boldsymbol{\chi},\nabla v_1)\,\mathrm{d}t\le K\int_0^T e^{-rt}\,||\boldsymbol{\chi}||^2\mathrm{d}t+K_6\int_0^T e^{rt}\,||\nabla v_1||^2\mathrm{d}t.
$$

From Assumptions **A9, B1**, there exists $K_7=K_7\left(E_h,C^*\right)$ such that bound for $\mathcal{A}_6$ is as follows:

$$
\mathcal{A}_6 = -\int_0^T (\Xi g\nabla\psi,\nabla v_1)\,\mathrm{d}t\le\epsilon\int_0^T e^{-rt}\,||\nabla\psi||^2\mathrm{d}t+K_7\int_0^T e^{rt}\,||\nabla v_1||^2\mathrm{d}t.
$$

From Assumption **A10**, there exists $K_8=K_8\left(\tau_\circ,||\boldsymbol{calF}||_{\mathcal{L}^\infty((0,T);\mathcal{L}^\infty(\Omega))}\right)$ such that the bound for $\mathcal{A}_7$ is as follows:

$$
\mathcal{A}_7 = \int_0^T (\psi\boldsymbol{calF},\nabla v_1)\,\mathrm{d}t\le\epsilon\int_0^T e^{-rt}\,||\psi||^2\mathrm{d}t+K_8\int_0^T e^{rt}\,||\nabla v_1||^2\mathrm{d}t.
$$

From Assumptions **A2, A3**, there exists $K_9=K_9\left(\left|\left|\frac{\boldsymbol{q}}{H}\right|\right|_{\mathcal{L}^\infty((0,T);\mathcal{L}^\infty(\Omega))}\right)$ so that we obtain the following upper bound on the projection error terms $\mathcal{P}_1$ and $\mathcal{P}_3$:

$$\mathcal{P}_1 = \int_0^T \left(\left(\frac{\boldsymbol{q}}{H} \cdot \nabla \boldsymbol{\phi} \right), \nabla v_1 \right) \mathrm{d}t \leq K_9 \, ||\nabla \boldsymbol{\phi}||^2_{\mathcal{L}^2((0,T);\mathcal{L}^2(\Omega))} + K \int_0^T e^{rt} \, ||\nabla v_1||^2 \mathrm{d}t;$$

$$\mathcal{P}_3 = \int_0^T \left((\nabla \cdot \boldsymbol{\phi}) \frac{\boldsymbol{q}}{H}, \nabla v_1 \right) \mathrm{d}t \leq K_9 \, ||\nabla \boldsymbol{\phi}||^2_{\mathcal{L}^2((0,T);\mathcal{L}^2(\Omega))} + K \int_0^T e^{rt} \, ||\nabla v_1||^2 \mathrm{d}t.$$

From Lemma 4.3, and Lemma 4.2, there exists $K_{10} = K_{10}(\tau_\circ, K_1, K_2, K^*)$ such that the bounds for $\mathcal{P}_2$ and $\mathcal{P}_4$ are as follows:

$$\begin{aligned} \mathcal{P}_2 &= \int_0^T \left(\left[\frac{\boldsymbol{q}}{H} - \frac{\mathcal{Q}}{\Pi} \right] \cdot \nabla \widetilde{\boldsymbol{q}}, \nabla v_1 \right) \mathrm{d}t \leq K \int_0^T e^{-rt} \, ||\theta||^2 \mathrm{d}t + \epsilon \int_0^T e^{-rt} \, ||\psi||^2 \mathrm{d}t \\ &\quad + K \int_0^T e^{-rt} \, ||\boldsymbol{\phi}||^2 \mathrm{d}t + K \int_0^T e^{-rt} \, ||\boldsymbol{\chi}||^2 \mathrm{d}t + K_{10} \int_0^T e^{rt} \, ||\nabla v_1||^2 \mathrm{d}t, \\ \mathcal{P}_3 &= \int_0^T \left((\nabla \cdot \widetilde{\boldsymbol{q}}) \left[\frac{\boldsymbol{q}}{H} - \frac{\mathcal{Q}}{\Pi} \right], \nabla v_1 \right) \mathrm{d}t \leq K \, ||\theta||^2_{\mathcal{L}^2((0,T);\mathcal{L}^2(\Omega))} + \epsilon \int_0^T e^{-rt} \, ||\psi||^2 \mathrm{d}t \\ &\quad + K \, ||\boldsymbol{\phi}||^2_{\mathcal{L}^2((0,T);\mathcal{L}^2(\Omega))} + K \, ||\boldsymbol{\chi}||^2_{\mathcal{L}^2((0,T);\mathcal{L}^2(\Omega))} + K_{10} \int_0^T e^{rt} \, ||\nabla v_1||^2 \mathrm{d}t. \end{aligned}$$

From Lemma 4.3, Lemma 4.2 and Assumptions **A2, A3, A4, B1, B2**, there exists $K_{11} = K_{11} \left(\tau_\circ, K_1, K_2, ||\nabla h_b||_{\mathcal{L}^\infty((0,T);\mathcal{L}^\infty(\Omega))}, C_*, C^* \right)$ such that the bound for $\mathcal{P}_5$ is as follows:

$$\begin{aligned} \mathcal{P}_5 &= \int_0^T \left(\nabla h_b \cdot \left\{ \left[\frac{\boldsymbol{q}}{H} \right]^2 - \left[\frac{\mathcal{Q}}{\Pi} \right]^2 \right\}, \nabla v_1 \right) \mathrm{d}t \\ &= \int_0^T \left(\nabla h_b \cdot \left\{ \left[\frac{\boldsymbol{q}}{H} - \frac{\mathcal{Q}}{\Pi} \right] \left[\frac{\boldsymbol{q}}{H} + \frac{\mathcal{Q}}{\Pi} \right] + \left[\frac{\mathcal{Q}\boldsymbol{q} - \boldsymbol{q}\mathcal{Q}}{H\Pi} \right] \right\}, \nabla v_1 \right) \mathrm{d}t \\ &\leq K \, ||\theta||^2_{\mathcal{L}^2((0,T);\mathcal{L}^2(\Omega))} + \epsilon \int_0^T e^{-rt} \, ||\psi||^2 \mathrm{d}t + K \, ||\boldsymbol{\phi}||^2_{\mathcal{L}^2((0,T);\mathcal{L}^2(\Omega))} \\ &\quad + K \, ||\boldsymbol{\chi}||^2_{\mathcal{L}^2((0,T);\mathcal{L}^2(\Omega))} + K_{11} \int_0^T e^{rt} \, ||\nabla v_1||^2 \mathrm{d}t. \end{aligned}$$

From Assumption **A2,A3**, there exists $K_{12} = K_{12} \left(\left\| \frac{\boldsymbol{u}^2}{H^2} \right\|_{\mathcal{L}^\infty((0,T);\mathcal{L}^\infty(\Omega))} \right)$ such that we obtain the following upper bound on the projection error term $\mathcal{P}_6$:

$$\mathcal{P}_6 = \int_0^T \left((\nabla \theta \cdot \boldsymbol{q}) \frac{\boldsymbol{q}}{H^2}, \nabla v_1 \right) \mathrm{d}t \leq K \, ||\nabla \theta||^2_{\mathcal{L}^2((0,T);\mathcal{L}^2(\Omega))} + K_{12} \int_0^T e^{rt} \, ||\nabla v_1||^2 \mathrm{d}t.$$

From Lemma 4.3, Lemma 4.2, and Assumptions **A2, A3, B1, B2**, there exists $K_{13} = K_{13}(\tau_\circ, K_1, K_2, C_*, C^*, K^*)$ such that the bound for $\mathcal{P}_7$ is given by

$$\begin{aligned} \mathcal{P}_7 &= \int_0^T \left(\nabla \widetilde{\xi} \cdot \left\{ \left[\frac{\boldsymbol{q}}{H} \right]^2 - \left[\frac{\mathcal{Q}}{\Pi} \right]^2 \right\}, \nabla v_1 \right) \mathrm{d}t \\ &= \int_0^T \left(\nabla \widetilde{\xi} \cdot \left\{ \left[\frac{\boldsymbol{q}}{H} - \frac{\mathcal{Q}}{\Pi} \right] \left[\frac{\boldsymbol{q}}{H} + \frac{\mathcal{Q}}{\Pi} \right] + \left[\frac{\mathcal{Q}\boldsymbol{q} - \boldsymbol{q}\mathcal{Q}}{H\Pi} \right] \right\}, \nabla v_1 \right) \mathrm{d}t \\ &\leq K \int_0^T e^{-rt} \, ||\theta||^2 \mathrm{d}t + \epsilon \int_0^T e^{-rt} \, ||\psi||^2 \mathrm{d}t + K \int_0^T e^{-rt} \, ||\boldsymbol{\phi}||^2 \mathrm{d}t \\ &\quad + K \int_0^T e^{-rt} \, ||\boldsymbol{\chi}||^2 \mathrm{d}t + K_{13} \int_0^T e^{rt} \, ||\nabla v_1||^2 \mathrm{d}t. \end{aligned}$$

Term $\mathcal{P}_8$ is treated similarly to $\mathcal{A}_4$ and term $\mathcal{P}_9$ is treated similarly to $\mathcal{A}_5$ if, for both, we treat $\boldsymbol{\phi}$ like $\boldsymbol{\chi}$.

From Assumption **A2**, there exists $K_{14} = K_{14}\left(||Hg||_{\mathcal{L}^\infty((0,T);\mathcal{L}^\infty(\Omega))}\right)$ such that we obtain the the following upper bound on the projection error term $\mathcal{P}_{10}$:

$$\mathcal{P}_{10} \ = \ \int_0^T (Hg\nabla\theta, \nabla v_1)\ \mathrm{d}t \leq K\,||\nabla\theta||^2_{\mathcal{L}^2((0,T);\mathcal{L}^2(\Omega))} + K_{14}\int_0^T e^{rt}\,||\nabla v_1||^2 \mathrm{d}t.$$

From Lemma 4.2, there exists $K_{15} = K_{15}(K^*)$ and $K_{16} = K_{16}(\tau_\circ, K^*)$ such that we obtain the following bounds on $\mathcal{P}_{11}$ and on $\mathcal{P}_{12}$:

$$\begin{aligned}
\mathcal{P}_{11} \ &= \ \int_0^T \left(\theta g\nabla\widetilde{\xi}, \nabla v_1\right)\ \mathrm{d}t \leq K\int_0^T e^{-rt}\,||\theta||^2 \mathrm{d}t + K_{15}\int_0^T e^{rt}\,||\nabla v_1||^2 \mathrm{d}t \\
\mathcal{P}_{12} \ &= \ -\int_0^T \left(\psi g\nabla\widetilde{\xi}, \nabla v_1\right)\ \mathrm{d}t \leq \epsilon\int_0^T e^{-rt}\,||\psi||^2 \mathrm{d}t + K_{16}\int_0^T e^{rt}\,||\nabla v_1||^2 \mathrm{d}t.
\end{aligned}$$

Obtaining a bound on $\mathcal{P}_{13}$ is straightforward:

$$\mathcal{P}_{13} \ = \ E_h\int_0^T \left(\nabla\frac{\partial\theta}{\partial t}, \nabla v_1\right)\ \mathrm{d}t \leq K\left|\left|\nabla\frac{\partial\theta}{\partial t}\right|\right|^2_{\mathcal{L}^2((0,T);\mathcal{L}^2(\Omega))} + K_{17}\int_0^T e^{rt}\,||\nabla v_1||^2 \mathrm{d}t.$$

Finally, term $\mathcal{P}_{14}$ is treated similarly to $\mathcal{A}_7$.

Using $v_2(\cdot,t) = \int_t^T e^{-rs}\frac{\partial}{\partial t}\psi(\cdot,s)\,\mathrm{d}s$ as the test function in (16), integrating in time over $(0,T]$, and using the relations above yields

$$\begin{aligned}
(19)\quad & \int_0^T e^{-rt}\left|\left|\frac{\partial\psi}{\partial t}\right|\right|^2 \mathrm{d}t + e^{-rT}\frac{\tau_\circ}{2}||\psi(\cdot,T)||^2 + r\frac{\tau_\circ}{2}\int_0^T e^{-rt}\,||\psi||^2 \mathrm{d}t \\
& + \frac{E_h}{2}\,||\nabla v_2(\boldsymbol{x},0)||^2 + r\frac{E_h}{2}\int_0^T e^{rt}\,||\nabla v_2||^2 \mathrm{d}t \\
& = -\int_0^T \left(\left(\frac{\mathcal{Q}}{\Pi}\cdot\nabla\boldsymbol{\chi}\right), \nabla v_2\right)\ \mathrm{d}t - \int_0^T \left((\nabla\cdot\boldsymbol{\chi})\frac{\mathcal{Q}}{\Pi}, \nabla v_2\right)\ \mathrm{d}t \\
& + \int_0^T \left((\nabla\psi\cdot\mathcal{Q})\frac{\mathcal{Q}}{\Pi^2}, \nabla v_2\right)\ \mathrm{d}t - \int_0^T ((\tau_{bf} - \tau_\circ)\boldsymbol{\chi}, \nabla v_2)\ \mathrm{d}t \\
& - \int_0^T (\boldsymbol{k}\times f_c\boldsymbol{\chi}, \nabla v_2)\ \mathrm{d}t - \int_0^T (\Pi g\nabla\psi, \nabla v_2)\ \mathrm{d}t + \int_0^T (\psi\boldsymbol{calF}, \nabla v_2)\ \mathrm{d}t \\
& + \int_0^T \left(\left(\frac{\boldsymbol{q}}{H}\cdot\nabla\boldsymbol{\phi}\right), \nabla v_2\right)\ \mathrm{d}t + \int_0^T \left(\left[\frac{\boldsymbol{q}}{H} - \frac{\mathcal{Q}}{\Pi}\right]\cdot\nabla\widetilde{\boldsymbol{q}}, \nabla v_2\right)\ \mathrm{d}t \\
& + \int_0^T \left((\nabla\cdot\boldsymbol{\phi})\frac{\boldsymbol{q}}{H}, \nabla v_2\right)\ \mathrm{d}t + \int_0^T \left((\nabla\cdot\widetilde{\boldsymbol{q}})\left[\frac{\boldsymbol{q}}{H} - \frac{\mathcal{Q}}{\Pi}\right], \nabla v_2\right)\ \mathrm{d}t \\
& + \int_0^T \left(\nabla h_b\cdot\left\{\left[\frac{\boldsymbol{q}}{H}\right]^2 - \left[\frac{\mathcal{Q}}{\Pi}\right]^2\right\}, \nabla v_2\right)\ \mathrm{d}t + \int_0^T \left((\nabla\theta\cdot\boldsymbol{q})\frac{\boldsymbol{q}}{H^2}, \nabla v_2\right)\ \mathrm{d}t \\
& + \int_0^T \left(\nabla\widetilde{\xi}\cdot\left\{\left[\frac{\boldsymbol{q}}{H}\right]^2 - \left[\frac{\mathcal{Q}}{\Pi}\right]^2\right\}, \nabla v_2\right)\ \mathrm{d}t + \int_0^T ((\tau_{bf} - \tau_\circ)\boldsymbol{\phi}, \nabla v_2)\ \mathrm{d}t \\
& + \int_0^T (\boldsymbol{k}\times f_c\boldsymbol{\phi}, \nabla v_2)\ \mathrm{d}t + \int_0^T (Hg\nabla\theta, \nabla v)\ \mathrm{d}t + \int_0^T \left(\theta g\nabla\widetilde{\xi}, \nabla v_2\right)\ \mathrm{d}t
\end{aligned}$$

$$
\begin{aligned}
&- \int_0^T \left(\psi g \nabla \widetilde{\xi}, \nabla v_2\right) \mathrm{d}t + E_h \int_0^T \left(\nabla \frac{\partial \theta}{\partial t}, \nabla v_2\right) \mathrm{d}t + \int_0^T (\theta \boldsymbol{calF}, \nabla v_2) \mathrm{d}t \\
&= \left(\widehat{\mathcal{A}_1} + \cdots + \widehat{\mathcal{A}_7}\right) + \left(\widehat{\mathcal{P}_1} + \cdots + \widehat{\mathcal{P}_{14}}\right).
\end{aligned}
$$

The terms on the right-hand side of the inequality are handled as in (18). Note that term $\widehat{\mathcal{A}_6}$ differs from $\mathcal{A}_6$ by one term. From Assumptions **A2, A9**, there exists $K_{18} = K_{18}(C^*)$ such that the the upper bound on $\widehat{\mathcal{A}_6}$ is as follows

$$
\widehat{\mathcal{A}_6} \;=\; - \int_0^T (\Pi g \nabla \psi, \nabla v_2) \mathrm{d}t \le \epsilon \int_0^T e^{-rt} \|\nabla \psi\|^2 \mathrm{d}t + K_{18} \int_0^T e^{rt} \|\nabla v_2\|^2 \mathrm{d}t.
$$

4.1.3. Bounding the CME Error Equation. Using $\boldsymbol{w} = \boldsymbol{\chi}$ as the test function in (17) followed by integration in time over $(0, T]$, yields

$$
\begin{aligned}
(20) \qquad & \frac{1}{2}\|\boldsymbol{\chi}(\cdot, T)\|^2 + E_h \|\nabla \boldsymbol{\chi}\|^2_{\mathcal{L}^2((0,T);\mathcal{L}^2(\Omega))} + \left\|\sqrt{\tau_{bf}}\, \boldsymbol{\chi}\right\|^2_{\mathcal{L}^2((0,T);\mathcal{L}^2(\Omega))} \\
&\le - \int_0^T \left(\left(\frac{\mathcal{Q}}{\Pi}\cdot\nabla \boldsymbol{\chi}\right), \boldsymbol{\chi}\right) \mathrm{d}t - \int_0^T \left((\nabla\cdot\boldsymbol{\chi})\frac{\mathcal{Q}}{\Pi}, \boldsymbol{\chi}\right) \mathrm{d}t + \int_0^T \left((\nabla \psi\cdot \mathcal{Q})\frac{\mathcal{Q}}{\Pi^2}, \boldsymbol{\chi}\right) \mathrm{d}t \\
&- \int_0^T (\Pi g \nabla \psi, \boldsymbol{\chi}) \mathrm{d}t + \int_0^T (\psi \boldsymbol{calF}, \boldsymbol{\chi}) \mathrm{d}t \\
&+ \int_0^T \left(\left(\frac{\boldsymbol{q}}{H}\cdot\nabla \boldsymbol{\phi}\right), \boldsymbol{\chi}\right) \mathrm{d}t + \int_0^T \left(\left[\frac{\boldsymbol{q}}{H} - \frac{\mathcal{Q}}{\Pi}\right]\cdot\nabla \widetilde{\boldsymbol{q}}, \boldsymbol{\chi}\right) \mathrm{d}t \\
&+ \int_0^T \left((\nabla\cdot\boldsymbol{\phi})\frac{\boldsymbol{q}}{H}, \boldsymbol{\chi}\right) \mathrm{d}t + \int_0^T \left((\nabla\cdot\widetilde{\boldsymbol{q}})\left[\frac{\boldsymbol{q}}{H} - \frac{\mathcal{Q}}{\Pi}\right], \boldsymbol{\chi}\right) \mathrm{d}t \\
&+ \int_0^T \left(\nabla h_b\cdot\left\{\left[\frac{\boldsymbol{q}}{H}\right]^2 - \left[\frac{\mathcal{Q}}{\Pi}\right]^2\right\}, \boldsymbol{\chi}\right) \mathrm{d}t + \int_0^T \left((\nabla \theta\cdot\boldsymbol{q})\frac{\boldsymbol{q}}{H^2}, \boldsymbol{\chi}\right) \mathrm{d}t \\
&+ \int_0^T \left(\nabla \widetilde{\xi}\cdot\left\{\left[\frac{\boldsymbol{q}}{H}\right]^2 - \left[\frac{\mathcal{Q}}{\Pi}\right]^2\right\}, \boldsymbol{\chi}\right) \mathrm{d}t + \tau^* \int_0^T \underbrace{(\boldsymbol{\phi}, \boldsymbol{\chi})}_{0 \text{ by } (14)} | \,\mathrm{d}t \\
&+ \int_0^T (\boldsymbol{k} \times f_c \boldsymbol{\phi}, \boldsymbol{\chi}) \mathrm{d}t + \int_0^T (H g \nabla \theta, \boldsymbol{\chi}) \mathrm{d}t + \int_0^T \left(\theta g \nabla \widetilde{\xi}, \boldsymbol{\chi}\right) \mathrm{d}t \\
&- \int_0^T \left(\psi g \nabla \widetilde{\xi}, \boldsymbol{\chi}\right) \mathrm{d}t + E_h \int_0^T (\nabla \boldsymbol{\phi}, \nabla \boldsymbol{\chi}) \mathrm{d}t + \int_0^T (\theta \boldsymbol{calF}, \boldsymbol{\chi}) \mathrm{d}t \\
&= \left(\widetilde{\mathcal{A}_1} + \widetilde{\mathcal{A}_2} + \widetilde{\mathcal{A}_3}\right) + \left(\widetilde{\mathcal{A}_6} + \widetilde{\mathcal{A}_7}\right) + \left(\widetilde{\mathcal{P}_1} + \cdots + \widetilde{\mathcal{P}_7}\right) + \left(\widetilde{\mathcal{P}_9} + \cdots + \widetilde{\mathcal{P}_{14}}\right).
\end{aligned}
$$

The terms on the right-hand side of the inequality are handled as in (18). Again, observe that term $\widetilde{\mathcal{A}_6}$ differs from $\mathcal{A}_6$ by one component and is thus treated similarly to $\widehat{\mathcal{A}_6}$ in (19). Also, the treatment of term $\widetilde{\mathcal{P}_{13}}$ differs slightly from the treatment of the corresponding terms in the previous equations:

$$
\widetilde{\mathcal{P}_{13}} \;=\; E_h \int_0^T (\nabla \boldsymbol{\phi}, \nabla \boldsymbol{\chi}) \mathrm{d}t \le K_{19} \|\nabla \boldsymbol{\phi}\|^2_{\mathcal{L}^2((0,T);\mathcal{L}^2(\Omega))} + \epsilon \|\nabla \boldsymbol{\chi}\|^2_{\mathcal{L}^2((0,T);\mathcal{L}^2(\Omega))}.
$$

4.1.4. Bounding the Sum of the Error Equations. We observe that the right-hand-side of (18) can be bounded by

$$\begin{aligned}(21)\qquad & \epsilon\int_0^T e^{-rt}\,\|\psi\|^2 \mathrm{d}t+\epsilon\int_0^T e^{-rt}\,\|\nabla\psi\|^2\mathrm{d}t+K_{v1}\int_0^T e^{rt}\,\|\nabla v_1\|^2\mathrm{d}t+K\,\|\boldsymbol{\chi}\|^2_{\mathcal{L}^2((0,T);\mathcal{L}^2(\Omega))}\\ & +\epsilon\,\|\nabla\boldsymbol{\chi}\|^2_{\mathcal{L}^2((0,T);\mathcal{L}^2(\Omega))}+K\,\|\theta\|^2_{\mathcal{L}^2((0,T);\mathcal{L}^2(\Omega))}+K\,\|\nabla\theta\|^2_{\mathcal{L}^2((0,T);\mathcal{L}^2(\Omega))}\\ & +K\left\|\nabla\frac{\partial\theta}{\partial t}\right\|^2_{\mathcal{L}^2((0,T);\mathcal{L}^2(\Omega))}+K\,\|\boldsymbol{\phi}\|^2_{\mathcal{L}^2((0,T);\mathcal{L}^2(\Omega))}+K\,\|\nabla\boldsymbol{\phi}\|^2_{\mathcal{L}^2((0,T);\mathcal{L}^2(\Omega))}\end{aligned}$$

where $K_{v_1}=K_{v_1}\left(K_1,\ldots,K_{17}\right)$.

The right-hand-side of (19) can be bounded by

$$\begin{aligned}(22)\qquad & \epsilon\int_0^T e^{-rt}\,\|\psi\|^2 \mathrm{d}t+\epsilon\int_0^T e^{-rt}\,\|\nabla\psi\|^2\mathrm{d}t+K_{v2}\int_0^T e^{rt}\,\|\nabla v_2\|^2\mathrm{d}t+K\,\|\boldsymbol{\chi}\|^2_{\mathcal{L}^2((0,T);\mathcal{L}^2(\Omega))}\\ & +K\,\|\nabla\boldsymbol{\chi}\|^2_{\mathcal{L}^2((0,T);\mathcal{L}^2(\Omega))}+K\,\|\theta\|^2_{\mathcal{L}^2((0,T);\mathcal{L}^2(\Omega))}+K\,\|\nabla\theta\|^2_{\mathcal{L}^2((0,T);\mathcal{L}^2(\Omega))}\\ & +K\left\|\nabla\frac{\partial\theta}{\partial t}\right\|^2_{\mathcal{L}^2((0,T);\mathcal{L}^2(\Omega))}+K\,\|\boldsymbol{\phi}\|^2_{\mathcal{L}^2((0,T);\mathcal{L}^2(\Omega))}+K\,\|\nabla\boldsymbol{\phi}\|^2_{\mathcal{L}^2((0,T);\mathcal{L}^2(\Omega))}\end{aligned}$$

where $K_{v_2}=K_{v_2}\left(K_1,\ldots,K_6,K_8,\ldots,K_{18}\right)$.

Finally, the right-hand-side of (20) can be bounded by

$$\begin{aligned}(23)\qquad & \epsilon\int_0^T e^{-rt}\,\|\psi\|^2 \mathrm{d}t+\epsilon\int_0^T e^{-rt}\,\|\nabla\psi\|^2\mathrm{d}t+K_{\boldsymbol{w}}\int_0^T e^{rt}\,\|\boldsymbol{\chi}\|^2\mathrm{d}t\\ & +K\,\|\boldsymbol{\chi}\|^2_{\mathcal{L}^2((0,T);\mathcal{L}^2(\Omega))}+\epsilon\,\|\nabla\boldsymbol{\chi}\|^2_{\mathcal{L}^2((0,T);\mathcal{L}^2(\Omega))}+K\,\|\theta\|^2_{\mathcal{L}^2((0,T);\mathcal{L}^2(\Omega))}\\ & +K\,\|\nabla\theta\|^2_{\mathcal{L}^2((0,T);\mathcal{L}^2(\Omega))}+K\,\|\boldsymbol{\phi}\|^2_{\mathcal{L}^2((0,T);\mathcal{L}^2(\Omega))}+K\,\|\nabla\boldsymbol{\phi}\|^2_{\mathcal{L}^2((0,T);\mathcal{L}^2(\Omega))}\end{aligned}$$

where $K_{\boldsymbol{w}}=K_{\boldsymbol{w}}\left(K_1,\ldots,K_4,K_8,\ldots,K_{16},K_{18},K_{19}\right)$.

Now choose $r=\max\left\{2K_{v_1}/\gamma_*,2K_{v_2}/E_h\right\}$, such that

$$r_1=\left(r\frac{\gamma_*}{2}-K_{v_1}\right)\geq 0$$

and

$$r_2=\left(r\frac{E_h}{2}-K_{v_2}\right)\geq 0.$$

Then, summing (18), (19), and (20), using the above choice for r, and using bounds (21), (22), and (23) yields

$$\begin{aligned}(24)\qquad & \frac{r(\tau_\circ+1)}{2}\int_0^T e^{-rt}\,\|\psi\|^2\mathrm{d}t+\left(\frac{\tau_\circ+1}{2}\right)e^{-rT}\,\|\psi(\cdot,T)\|^2\\ & +\int_0^T e^{-rt}\left\|\frac{\partial\psi}{\partial t}\right\|^2\mathrm{d}t+\frac{E_h}{2}\int_0^T e^{-rt}\,\|\nabla\psi\|^2\mathrm{d}t+\underbrace{\frac{\gamma_*}{2}\,\|\nabla v_1(\boldsymbol{x},0)\|^2}_{A}\\ & +\underbrace{\frac{E_h}{2}\,\|\nabla v_2(\boldsymbol{x},0)\|^2}_{B}+\underbrace{r_1\int_0^T e^{rt}\,\|\nabla v_1\|^2\mathrm{d}t}_{C}+\underbrace{r_2\int_0^T e^{rt}\,\|\nabla v_2\|^2\mathrm{d}t}_{D}\end{aligned}$$

$$+\frac{1}{2}\left\|\boldsymbol{\chi}(\cdot,T)\right\|^2+\left\|\sqrt{\tau_{bf}}\;\boldsymbol{\chi}\right\|^2_{\mathcal{L}^2((0,T);\mathcal{L}^2(\Omega))}+\frac{E_h}{2}\left\|\nabla\boldsymbol{\chi}\right\|^2_{\mathcal{L}^2((0,T);\mathcal{L}^2(\Omega))}$$
$$\leq K\left\|\theta\right\|^2_{\mathcal{L}^2((0,T);\mathcal{L}^2(\Omega))}+K\left\|\nabla\theta\right\|^2_{\mathcal{L}^2((0,T);\mathcal{L}^2(\Omega))}+K\left\|\nabla\frac{\partial\theta}{\partial t}\right\|^2_{\mathcal{L}^2((0,T);\mathcal{L}^2(\Omega))}$$
$$+K\left\|\boldsymbol{\phi}\right\|^2_{\mathcal{L}^2((0,T);\mathcal{L}^2(\Omega))}+K\left\|\nabla\boldsymbol{\phi}\right\|^2_{\mathcal{L}^2((0,T);\mathcal{L}^2(\Omega))}+K_{20}\left\|\boldsymbol{\chi}\right\|^2_{\mathcal{L}^2((0,T);\mathcal{L}^2(\Omega))}$$

where $K_{20}=K_{\boldsymbol{w}}e^{rT}+K$.

From (24), we have

$$\left\|\boldsymbol{\chi}(\cdot,T)\right\|^2\leq 2K\left\|\theta\right\|^2_{\mathcal{L}^2((0,T);\mathcal{L}^2(\Omega))}+2K\left\|\nabla\theta\right\|^2_{\mathcal{L}^2((0,T);\mathcal{L}^2(\Omega))}$$
$$+2K\left\|\nabla\frac{\partial\theta}{\partial t}\right\|^2_{\mathcal{L}^2((0,T);\mathcal{L}^2(\Omega))}+2K\left\|\boldsymbol{\phi}\right\|^2_{\mathcal{L}^2((0,T);\mathcal{L}^2(\Omega))}$$
$$+2K\left\|\nabla\boldsymbol{\phi}\right\|^2_{\mathcal{L}^2((0,T);\mathcal{L}^2(\Omega))}+2K_{20}\left\|\boldsymbol{\chi}\right\|^2_{\mathcal{L}^2((0,T);\mathcal{L}^2(\Omega))}. \tag{25}$$

Applying Gronwall's Lemma to (25) yields

$$\left\|\boldsymbol{\chi}(\cdot,T)\right\|^2\leq 2K_{21}\left\{K\left\|\theta\right\|^2_{\mathcal{L}^2((0,T);\mathcal{H}^1(\Omega))}\right.$$
$$\left.+K\left\|\nabla\frac{\partial\theta}{\partial t}\right\|^2_{\mathcal{L}^2((0,T);\mathcal{L}^2(\Omega))}+K\left\|\boldsymbol{\phi}\right\|^2_{\mathcal{L}^2((0,T);\mathcal{H}^1(\Omega))}\right\}, \tag{26}$$

where $K_{21}=e^{2K_{20}T}$.

We now return to (24) to bound it above and below. Let $\beta_1=\min\{E_h,r(\tau_\circ+1)\}$, observe that terms A,B,C,D are all non-negative, and use (26) to obtain,

$$\frac{1}{2}e^{-rT}\left[(\tau_\circ+1)\left\|\psi(\cdot,T)\right\|^2+2\left\|\frac{\partial\psi}{\partial t}\right\|^2_{\mathcal{L}^2((0,T);\mathcal{L}^2(\Omega))}+\beta_1\left\|\psi\right\|^2_{\mathcal{L}^2((0,T);\mathcal{H}^1(\Omega))}\right.$$
$$\left.+\left\|\boldsymbol{\chi}(\cdot,T)\right\|^2+2\left\|\sqrt{\tau_{bf}}\;\boldsymbol{\chi}\right\|^2_{\mathcal{L}^2((0,T);\mathcal{L}^2(\Omega))}+E_h\left\|\nabla\boldsymbol{\chi}\right\|^2_{\mathcal{L}^2((0,T);\mathcal{L}^2(\Omega))}\right]$$
$$\leq(1+2K_{20}K_{21}K)K\left\{\left\|\theta\right\|^2_{\mathcal{L}^2((0,T);\mathcal{H}^1(\Omega))}\right.$$
$$\left.+\left\|\nabla\frac{\partial\theta}{\partial t}\right\|^2_{\mathcal{L}^2((0,T);\mathcal{L}^2(\Omega))}+\left\|\boldsymbol{\phi}\right\|^2_{\mathcal{L}^2((0,T);\mathcal{H}^1(\Omega))}\right\}. \tag{27}$$

Now, letting $\kappa=\min\{\,(\tau_\circ+1)\,,2,\beta_1,1,E_h\}$ and multiplying through by $2e^{rT}$, we obtain

$$\left\|\psi(\cdot,T)\right\|^2+\left\|\frac{\partial\psi}{\partial t}\right\|^2_{\mathcal{L}^2((0,T);\mathcal{L}^2(\Omega))}+\left\|\psi\right\|^2_{\mathcal{L}^2((0,T);\mathcal{H}^1(\Omega))}$$
$$+\left\|\boldsymbol{\chi}(\cdot,T)\right\|^2+\left\|\sqrt{\tau_{bf}}\;\boldsymbol{\chi}\right\|^2_{\mathcal{L}^2((0,T);\mathcal{L}^2(\Omega))}+\left\|\nabla\boldsymbol{\chi}\right\|^2_{\mathcal{L}^2((0,T);\mathcal{L}^2(\Omega))}$$
$$\leq K_{22}\left\{\left\|\theta\right\|^2_{\mathcal{L}^2((0,T);\mathcal{H}^1(\Omega))}+\left\|\nabla\frac{\partial\theta}{\partial t}\right\|^2_{\mathcal{L}^2((0,T);\mathcal{L}^2(\Omega))}+\left\|\boldsymbol{\phi}\right\|^2_{\mathcal{L}^2((0,T);\mathcal{H}^1(\Omega))}\right\} \tag{28}$$

with $K_{22}=\frac{2e^{rT}(1+2K_{20}K_{21}K)K}{\kappa}$. Observe that K_{22} depends on r,s_1,T, and on K^*,C_*,C^*.

Use the approximation result stated in Lemma 4.1,

$$\left\|\theta\right\|_{\mathcal{L}^2((0,T);\mathcal{H}^1(\Omega))},\;\left\|\nabla\frac{\partial\theta}{\partial t}\right\|_{\mathcal{L}^2((0,T);\mathcal{L}^2(\Omega))},\;\left\|\boldsymbol{\phi}\right\|_{\mathcal{L}^2((0,T);\mathcal{H}^1(\Omega))}\quad\leq\quad K_0h^{\ell-1},$$

to obtain

$$(29)\quad ||\psi(\cdot,T)|| + \left|\left|\frac{\partial\psi}{\partial t}\right|\right|_{\mathcal{L}^2((0,T);\mathcal{L}^2(\Omega))} + ||\psi||_{\mathcal{L}^2((0,T);\mathcal{H}^1(\Omega))}$$
$$+ ||\boldsymbol{\chi}(\cdot,T)|| + \left|\left|\sqrt{\tau_{bf}}\,\boldsymbol{\chi}\right|\right|_{\mathcal{L}^2((0,T);\mathcal{L}^2(\Omega))} + ||\nabla\boldsymbol{\chi}||_{\mathcal{L}^2((0,T);\mathcal{L}^2(\Omega))} \le K_{23}h^{\ell-1}$$

where $K_{23} \approx K_0\sqrt{K_{22}}$.

Finally, applying the triangle inequality to the projection error and to the affine error (29) yields the following error estimate.

THEOREM 4.4 (*A Priori* ERROR ESTIMATE). *Let $0 \le s_0 \le k$, $s_0 \le \ell \le s_1$, $0 \le k \le (s_1 - 1)$, and let $\mathcal{H} = \mathcal{H}_0^1(\Omega) \cap \mathcal{H}^\ell(\Omega)$. Let $(\xi, \boldsymbol{q})$ be the solution to (5)-(7). Let $(\Xi, \mathcal{Q})$ be the Galerkin approximations to $(\xi, \boldsymbol{q})$. If $\xi(t) \in \mathcal{H}(\Omega) \cap \mathcal{W}_\infty^1(\Omega)$,, $\boldsymbol{q}(t) \in \boldsymbol{\mathcal{H}}(\Omega) \cap \boldsymbol{\mathcal{W}}_\infty^1(\Omega)$, for each t; if $\Xi(t) \in \mathcal{S}^h(\Omega)$, $\mathcal{Q}(t) \in \boldsymbol{\mathcal{S}}^h(\Omega)$ for each t; and suppose that assumptions* **A2–A10** *and* **B1, B2** *hold; then, $\exists$ a constant $\bar{K} = \bar{K}(T, s_1, r, K^*, C_*, C^*)$ such that*

$$(30)\quad \left|\left|\frac{\partial}{\partial t}(\xi - \Xi)\right|\right|_{\mathcal{L}^2((0,T);\mathcal{L}^2(\Omega))} + ||\,(\xi - \Xi)(\cdot,T)\,|| + ||\xi - \Xi||_{\mathcal{L}^2((0,T);\mathcal{H}^1(\Omega))}$$
$$+ ||\,(\boldsymbol{q} - \mathcal{Q})(\cdot,T)\,|| + \left|\left|\sqrt{\tau_{bf}}(\boldsymbol{q} - \mathcal{Q})\right|\right|_{\mathcal{L}^2((0,T);\mathcal{L}^2(\Omega))}$$
$$+ ||\nabla\boldsymbol{q} - \nabla\mathcal{Q}||_{\mathcal{L}^2((0,T);\mathcal{L}^2(\Omega))} \le \bar{K}h^{\ell-1}.$$

Moreover, for h sufficiently small and $s_1 \ge 3$ then

$$||\Xi||_{\mathcal{L}^\infty((0,T);\mathcal{L}^\infty(\Omega))} + ||\mathcal{Q}||_{\mathcal{L}^\infty((0,T);\mathcal{L}^\infty(\Omega))} \;<\; 2K^* \le C^*,$$

and

$$\Xi \;>\; \frac{H_*}{2} \ge C_*.$$

Thus, the dependence of $\bar{K}$ on C_*, C^* is removed.

4.1.5. Boundedness of Approximations. The proof of the theorem is now complete in the case of linears since we assumed that **B1, B2** holds for this case.

We now complete the proof of the theorem in the case of at least quadratic polynomials ($s_1 \ge 3$).

From the inverse assumptions, the boundedness of the $\mathcal{L}^2$ projection and affine error estimate (29), we obtain

$$\begin{aligned}
||\mathcal{Q}||_{\mathcal{L}^\infty((0,T);\mathcal{L}^\infty(\Omega))} &\le ||\mathcal{Q} - \widetilde{\boldsymbol{q}}||_{\mathcal{L}^\infty((0,T);\mathcal{L}^\infty(\Omega))} + ||\widetilde{\boldsymbol{q}}||_{\mathcal{L}^\infty((0,T);\mathcal{L}^\infty(\Omega))} \\
&\le K_0h^{-1}||\mathcal{Q} - \widetilde{\boldsymbol{q}}||_{\mathcal{L}^\infty((0,T);\mathcal{L}^2(\Omega))} + K^* \\
&\le K_0h^{-1}K_{23}h^{\ell-1} + K^* \\
&= K_{24}h^{\ell-2} + K^*
\end{aligned}$$

For h sufficiently small, viz, $h^{\ell-2} < \frac{K^*}{K_{24}}$, we get

$$||\mathcal{Q}||^2_{\mathcal{L}^\infty((0,T);\mathcal{L}^\infty(\Omega))} < 2K^* \le C^*.$$

The upper bound for $\Pi(x,t)$ is shown similarly.

To get the lower bound for $\Pi(x,t)$, use Assumption **A2**, inverse assumptions, and estimates on the affine and projection errors.

$$\begin{aligned}
\Pi &= H - (H - \Pi) = H - \theta + \psi \\
&\ge H_* - K_{25}h^{\ell-2}.
\end{aligned}$$

For h sufficiently,small, viz. $h^{\ell-2} < \frac{H_*}{2K_{25}}$, we get

$$\Pi > \frac{H_*}{2} \geq C_*.$$

Thus for the case of quadratics and higher, there exists a $\bar{K}$ bounded above independent of C^*, C_*.

This completes the proof of the theorem.

5. Conclusions. We have analyzed a full nonlinear coupled GWCE-CME system of equations. Making physically-realistic assumptions, we derived an *a priori* error estimate for the Galerkin finite element approximation to the solution of GWCE-CME system of equations, in weak form, by using an $\mathcal{L}^2$ projection. This led to a suboptimal estimate. That is, if we use continuous, piecewise polynomials of degree $s_1 - 1$ to approximate the elevation and velocity unknowns on a mesh with grid-spacing h, then the approximations tend to the solutions of the weak form like h^{s_1-1}. To our knowledge, our error analysis of a system of shallow water equations is the first of its kind.

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A. Review of Tensor Notation. Let $\boldsymbol{\varphi}, \boldsymbol{\psi} \in \mathbb{R}^2$.
The dyadic product is defined as $(\boldsymbol{\varphi}\boldsymbol{\psi})_{ij} = \varphi_i \psi_j$. Thus,

$$\boldsymbol{\varphi}^2 = \begin{pmatrix} \varphi_1^2 & \varphi_1\varphi_2 \\ \varphi_2\varphi_1 & \varphi_2^2 \end{pmatrix}.$$

The dot product of a tensor with a vector is the usual matrix-vector multiplication result $[M \cdot \boldsymbol{\varpi}]_i = \sum_j M_{ij}\varpi_j$.

The scalar product (or double-dot product) of two tensors is defined as $S{:}T = \sum_{i,j} S_{ij}T_{ij}$.

The gradient of a vector is defined as $\{\nabla\boldsymbol{\varphi}\}_{ij} = \frac{\partial \varphi_j}{\partial x_i}$. For example,

$$\nabla\boldsymbol{\varphi} = \begin{pmatrix} \frac{\partial\varphi_1}{\partial x_1} & \frac{\partial\varphi_2}{\partial x_1} \\ \frac{\partial\varphi_1}{\partial x_2} & \frac{\partial\varphi_2}{\partial x_2} \end{pmatrix}.$$

The divergence of a tensor is defined as $[\nabla \cdot S]_j = \sum_i \frac{\partial}{\partial x_i} S_{ij}$. Thus,

$$\nabla\cdot\nabla\boldsymbol{\varphi} = \begin{pmatrix} \frac{\partial^2\varphi_1}{\partial x_1^2} + \frac{\partial^2\varphi_1}{\partial x_2^2} \\ \frac{\partial^2\varphi_2}{\partial x_1^2} + \frac{\partial^2\varphi_2}{\partial x_2^2} \end{pmatrix} = \begin{pmatrix} \triangle\varphi_1 \\ \triangle\varphi_2 \end{pmatrix}.$$

Observe that $(\nabla\boldsymbol{\varphi}{:}\nabla\boldsymbol{\varphi}) = \left(\frac{\partial}{\partial x_1}\varphi_1\right)^2 + \left(\frac{\partial}{\partial x_2}\varphi_1\right)^2 + \left(\frac{\partial}{\partial x_1}\varphi_2\right)^2 + \left(\frac{\partial}{\partial x_2}\varphi_2\right)^2$ and that $\nabla\cdot(\nabla\cdot\boldsymbol{\varphi}^2) = \frac{\partial^2}{\partial x_1^2}\varphi_1^2 + \frac{\partial^2}{\partial x_1 \partial x_2}(\varphi_2\varphi_1) + \frac{\partial^2}{\partial x_2 \partial x_1}(\varphi_1\varphi_2) + \frac{\partial^2}{\partial x_2^2}\varphi_2^2$.

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