# Parallel Computing for Finite Element Models on Surface Water Flows

*S. Chippada*

*C. Dawson*

*M. Martinez*

*M. Wheeler*

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# PARALLEL COMPUTING FOR FINITE ELEMENT MODELS OF SURFACE WATER FLOW

Srinivas Chippada, Clint N. Dawson, Mary F. Wheeler
*Texas Institute for Computational and Applied Mathematics*
*The University of Texas at Austin*
*Austin, Texas 78712*
*U.S.A.*
*chippada@ticam.utexas.edu*
*clint@ticam.utexas.edu*
*mfw@ticam.utexas.edu*


Mónica L. Martínez
*Department of Computational and Applied Mathematics*
*Rice University*
*6100 Main Street*
*Houston, Texas 77005-1892*
*U.S.A.*
*monicam@caam.rice.edu*

Various sophisticated finite element models for surface water flow exist in the literature. Gray, Kolar, Luettich, Lynch and Westerink have developed a hydrodynamic model based on the generalized wave continuity equation (GWCE) formulation, and have formulated a Galerkin finite element procedure based on combining the GWCE with the nonconservative momentum equations (NCME). Numerical experiments suggest that this method is robust, accurate and suppresses spurious oscillations which plague other models.

A simulator developed by Westerink et al, based on the GWCE-NCME formulation, has been parallelized to run on distributed memory parallel platforms. Speed-up efficiencies on clusters of workstations as well as on the Intel Paragon are presented.

We have analyzed a slightly modified Galerkin model which uses the conservative momentum equations (CME). For this GWCE-CME system of equations, we present an *a priori* error estimate.

# 1 Introduction

In recent years, there has been much interest in the numerical solution to shallow water equations. Simulation of shallow water systems can serve numerous purposes, including modeling tidal fluctuations and computing tidal ranges and surges such as tsunamis and hurricanes. The shallow water hydrodynamic model can also be coupled to a transport model in considering flow and transport phenomenon, thus making it possible, for example, to study remediation options for polluted bays and estuaries.

The 2-dimensional shallow water equations are obtained by depth averaging of the continuum mass and momentum balances given by the 3-dimensional incompressible Navier-Stokes equations. We denote by $\xi(\boldsymbol{x}, t)$ the free surface elevation over a reference plane and by $h_b(\boldsymbol{x})$ the bathymetric depth under that reference plane so that $H(\boldsymbol{x}, t) = \xi + h_b$ is the total water column. Also, we denote by $U(\boldsymbol{x}, t), V(\boldsymbol{x}, t)$ the depth-averaged horizontal velocities. Letting $\boldsymbol{v} = (UH, VH)^T$, the 2-dimensional governing equations, in operator form [2], are the primitive continuity equation (CE)

$$\mathrm{CE}(\xi, U, V; h_b) \quad \equiv \quad \frac{\partial \xi}{\partial t} + \nabla \cdot \boldsymbol{v} = 0,$$

and the primitive non-conservative momentum equations (NCME) [4],

$$\mathrm{NCME}(\xi, U, V; \Phi) \quad \equiv \quad \frac{\partial}{\partial t}\left(\frac{\boldsymbol{v}}{H}\right) + \left(\frac{\boldsymbol{v}}{H} \cdot \nabla\right)\frac{\boldsymbol{v}}{H} + \tau_{bf}\left(\frac{\boldsymbol{v}}{H}\right) + \boldsymbol{k} \times f_c\left(\frac{\boldsymbol{v}}{H}\right)$$

$$+ g\nabla\xi - \frac{1}{H}E_h\nabla \cdot \nabla \boldsymbol{v} - \frac{1}{H}\boldsymbol{\tau}_{ws} + \nabla p_a - g\nabla\eta = 0,$$

where $\Phi = (h_b, \tau_{bf}, f_c, g, E_h, \boldsymbol{\tau}_{ws}, p_a, \eta)$. In particular, $\tau_{bf}(\xi, U, V)$ is a bottom friction function, $\boldsymbol{k}$ is a unit vector in the vertical direction, $f_c$ is the Coriolis parameter, $g$ is acceleration due to gravity, $E_h$ is the horizontal eddy diffusion/dispersion (constant) coefficient, $\boldsymbol{\tau}_{ws}$ is the applied free surface wind stress relative to the reference density of water, $p_a(\boldsymbol{x}, t)$ is the atmospheric pressure at the free surface relative to the reference density of water, and $\eta(\boldsymbol{x}, t)$ is the Newtonian equilibrium tide potential relative to the effective Earth elasticity factor. The primitive conservative momentum equations (CME) are derived from the NCME as

$$\mathrm{CME} \equiv H(\mathrm{NCME}) + \boldsymbol{v}(\mathrm{CE})/H = 0.$$

The numerical procedure used to solve the shallow water equations must resolve the physics of the problem without introducing spurious oscillations or excessive numerical diffusion. Westerink et al note a need for greater grid refinement near land boundaries to resolve important processes and to prevent energy from aliasing. Permitting a high degree of grid flexibility, the finite element method is a good candidate.

There has been substantial effort over the past two decades in applying finite element methods to the CE coupled with either the NCME or the

CME, see [1] for a discussion of much of this prior work. Early finite element simulations of shallow water systems were plagued by spurious oscillations. Various methods were introduced to eliminate these oscillations through artificial diffusion. In this paper, we will examine a finite element approximation to a modified shallow water model described below. Computational and experimental evidence in the literature suggest that this formulation leads to approximate solutions with reduced oscillations. Moreover, these approximate solutions have accurately matched actual tidal data. This modified shallow water model is based on a reformulation of the CE.

The generalized wave continuity equation (GWCE) [2] is an extension of the wave continuity equation derived by Lynch and Gray [3]. It is derived by

$$\text{GWCE}(\xi, U, V; \Phi) \quad \equiv \quad \frac{\partial(\text{CE})}{\partial t} - \nabla\cdot(\text{CME}) + \tau_\mathrm{o}(\text{CE}) = 0,$$

where $\tau_\mathrm{o}$ is a time-independent positive constant. The GWCE can be coupled to the CME or to the NCME. This formulation has led to the development of robust finite element algorithms for depth-integrated coastal circulation models. A finite element simulator (*ADCIRC*) based on the GWCE-NCME has been developed by Luettich, et al [4].

Recently, we have derived an *a priori* error estimate for a finite element approximation to the GWCE-CME formulation, assuming equal order polynomial approximating spaces for elevation and velocities. We have also parallelized the *ADCIRC* simulator for distributed-memory parallel platforms. The results of this work are summarized in this paper.

The rest of the paper is outlined as follows. In section 2, we introduce the weak formulation associated with the GWCE-CME systems of equations and summarize, in a theorem, the results of our error analysis. In section 3, we discuss the the numerical approximation strategy taken in *ADCIRC* to solve the GWCE-NCME system of equations. A parallelized versions of *ADCIRC* has been developed, and is referred to as *PADCIRC*. In section 4, numerical results for *PADCIRC* are discussed. Finally, we conclude with closing remarks and future directions in Section 5.

# 2 A Priori Error Estimate for the GWCE-CME Formulation

For the *a priori* error estimate analysis, we consider the coupled system given by the GWCE-CME with, for simplicity, homogeneous Dirichlet boundary conditions

$$\xi(\boldsymbol{x}, t) = 0, \ \ U(\boldsymbol{x}, t) = 0, V(\boldsymbol{x}, t) = 0, \quad \boldsymbol{x} \in \partial\Omega, \ t > 0, \tag{1}$$

and with the compatible initial conditions

$$\left.\begin{array}{ll} \xi(\boldsymbol{x}, 0) = \xi_0(\boldsymbol{x}), & \frac{\partial\xi}{\partial t}(\boldsymbol{x}, 0) = \xi_1(\boldsymbol{x}), \\ U(\boldsymbol{x}, 0) = U_0(\boldsymbol{x}), & V(\boldsymbol{x}, 0) = V_0(\boldsymbol{x}), \end{array}\right\} \boldsymbol{x} \in \bar{\Omega}, \tag{2}$$

where $\partial\Omega$ is the boundary of $\Omega \subset \mathbb{R}^2$ and $\bar{\Omega} = \Omega \cup \partial\Omega$. Extensions to more general land and sea boundary conditions will be treated in a later paper. As noted in Kinnmark [2], the condition necessary for the solution of the GWCE-CME system of equations to be the same as the solution of the primitive form is

$$\xi_1(\boldsymbol{x}) \;=\; -\nabla\cdot\boldsymbol{v}(\boldsymbol{x},0).$$

We denote by $\mathcal{H}^\ell(\Omega)$ and $\mathcal{H}_0^1(\Omega)$ the standard Sobolev spaces. Denote by $(\varphi,\varsigma)$ the usual Lebesgue integral over $\Omega$ of the dot product of functions $\varphi$ and $\varsigma$.

The weak form of this system is the following: For $t \in (0,T]$, find $\xi(\boldsymbol{x},t) \in \mathcal{H}_0^1(\Omega)$ and $\boldsymbol{v}(\boldsymbol{x},t) \in \boldsymbol{\mathcal{H}}_0^1(\Omega)$ satisfying

$$\left(\frac{\partial^2\xi}{\partial t^2},v\right) + \tau_\circ\left(\frac{\partial\xi}{\partial t},v\right) + \left(\nabla\cdot\left\{\frac{1}{H}\boldsymbol{v}^2\right\},\nabla v\right) + ((\tau_{bf}-\tau_\circ)\boldsymbol{v},\nabla v)$$

$$+\,(\boldsymbol{k}\times f_c\boldsymbol{v},\nabla v) + (Hg\nabla\xi,\nabla v) + E_h\left(\nabla\frac{\partial\xi}{\partial t},\nabla v\right)$$

$$-\,(\boldsymbol{\tau}_{ws},\nabla v) + (H\nabla p_a,\nabla v) - (Hg\nabla\eta,\nabla v) = 0, \quad \forall v \in \mathcal{H}_0^1(\Omega), t > 0, \quad (3)$$

$$\left(\frac{\partial\boldsymbol{v}}{\partial t},\boldsymbol{w}\right) + \left(\nabla\cdot\left\{\frac{1}{H}\boldsymbol{v}^2\right\},\boldsymbol{w}\right) + (\tau_{bf}\boldsymbol{v},\boldsymbol{w}) + (\boldsymbol{k}\times f_c\boldsymbol{v},\boldsymbol{w})$$

$$+\,(Hg\nabla\xi,\boldsymbol{w}) + E_h\left(\nabla\boldsymbol{v},\nabla\boldsymbol{w}\right) - (\boldsymbol{\tau}_{ws},\boldsymbol{w})$$

$$+\,(H\nabla p_a,\boldsymbol{w}) - (Hg\nabla\eta,\boldsymbol{w}) = 0, \quad \forall\boldsymbol{w} \in \boldsymbol{\mathcal{H}}_0^1(\Omega), t > 0, \quad (4)$$

with initial conditions

$$\begin{aligned}
(\xi(\boldsymbol{x},0),v) &= (\xi_0(\boldsymbol{x}),v), \quad \forall v \in \mathcal{H}_0^1(\Omega),\\
(\tfrac{\partial\xi}{\partial t}(\boldsymbol{x},0),v) &= (\xi_1(\boldsymbol{x}),v), \quad \forall v \in \mathcal{H}_0^1(\Omega),\\
(\boldsymbol{v}(\boldsymbol{x},0),\boldsymbol{w}) &= (\boldsymbol{v}_0(\boldsymbol{x}),\boldsymbol{w}), \quad \forall\boldsymbol{w} \in \boldsymbol{\mathcal{H}}_0^1(\Omega),
\end{aligned} \quad (5)$$

where $\boldsymbol{v}_0 = (H_0U_0, H_0V_0)^T$. Here we have used tensor notation.

The result stated below is proved in [1]. Details on the assumptions needed in the proof can be found there, in particular, we assume for $(\boldsymbol{x},t) \in \bar{\Omega}\times(0,T]$ that the solutions $(\xi,\,\boldsymbol{v})$ to (3)-(5) exist and are unique. Furthermore, we make physically reasonable assumptions about the elevation and velocity solutions and on the force terms. We also make smoothness assumptions on the initial data.

Let $\mathcal{S}^h$ denote a finite dimensional subspace of $\mathcal{H}_0^1(\Omega)$ (defined on a triangulation of $\Omega$ into elements whose maximum diameter is $h$) consisting of piecewise polynomials of degree $k-1$. Define $\mathcal{H}(\Omega) = \mathcal{H}_0^1(\Omega) \cap \mathcal{H}^\ell(\Omega)$, and assume $\mathcal{S}^h$ satisfies the standard approximation property and an inverse assumption.

**Theorem 1 (A Priori Error Estimate in Continuous Time)** *Let* $0 \leq s \leq \ell \leq k$, $\ell, k \geq 2$. *Let* $(\xi, \boldsymbol{v})$ *be the solution to (3)-(5). Let* $(\Xi, \boldsymbol{\Upsilon})$ *be the continuous-time Galerkin finite element approximations to* $(\xi, \boldsymbol{v})$.

*If* $\xi(t) \in \mathcal{H}(\Omega)$, $\boldsymbol{v}(t) \in \mathcal{H}(\Omega)$ *for each* $t$, *and if* $\Xi(t) \in \mathcal{S}^h(\Omega)$, $\boldsymbol{\Upsilon}(t) \in \mathcal{S}^h(\Omega)$ *for each* $t$, *then under physically reasonable assumptions,* $\exists$ *a constant* $C = C(T, k)$ *such that*

$$|| (\xi - \Xi)(\cdot, T) || + \left\| \frac{\partial}{\partial t}(\xi - \Xi) \right\|_{\mathcal{L}^2((0,T);\mathcal{L}^2(\Omega))} + ||\xi - \Xi||_{\mathcal{L}^2((0,T);\mathcal{H}^1(\Omega))}$$
$$+ || (\boldsymbol{v} - \boldsymbol{\Upsilon})(\cdot, T) || + ||\boldsymbol{v} - \boldsymbol{\Upsilon}||_{\mathcal{L}^2((0,T);\mathcal{H}^1(\Omega))} \leq C \, h^{\ell-1}.$$

# 3    A GWCE-NCME Simulator

The Advanced Circulation Model *ADCIRC* uses continuous, piecewise linear triangular elements for $\xi, U, V$ in the finite element spatial discretization.

Westerink et al chose to discretize the GWCE in time using a three-level implicit scheme for linear terms with explicit treatment of non-linear terms. A global (time-dependent) linear system results at each time step.

The NCME are discretized in time using Crank-Nicolson, except that diffusive terms are handled with a two-level implicit scheme and advective terms are treated explicitly. A diagonal linear system results at each time step.

Consequently, the GWCE-CME is decoupled. That is, the GWCE is solved first for elevation $\xi$. Then the NCME is solved for velocities $U, V$ using updated values of $\xi$. Initial and boundary conditions are applied as follows. At land boundaries, normal flux is specified. At open (ocean or river) boundaries, free surface elevation is specified, a discharge is specified, or a radiation boundary condition is used to allow waves to enter or exit domain.

A grid generation package is used to provide an efficient triangular mesh which fits the domain and refines the mesh near land boundaries to capture steep gradient information.

The parallelized version of *ADCIRC* is referred to as *PADCIRC*. In preserving the spirit of the simulator, modifications to *ADCIRC* were kept to a minimum. To that end, a single program-multiple data (SPMD) approach was taken. Specifically, a domain (data) decomposition with message-passing was implemented. The message passing calls were needed in the Jacobi Conjugate Gradient routine (used in the GWCE system solve) as well as at the end of each time step to pass elevation and velocity information.

In the parallelization of the simulator, no global arrays are used. Each processor reads and writes its own data. A preprocessor was developed to take a global input file and divide the triangle vertices as evenly as possible among the processors (see Figure 1). The preprocessor also constructs local connectivity tables and establishes inter-processor communication. Because elements share vertices, there is roughly one layer of overlap of elements in

Figure 1: Data Decomposition over 3 processors on a grid consisting of 50 elements, 36 nodes
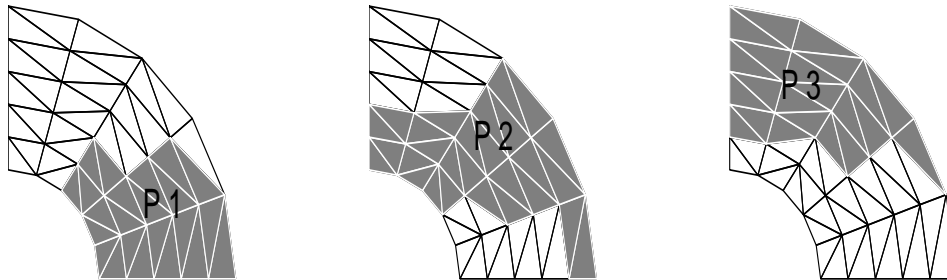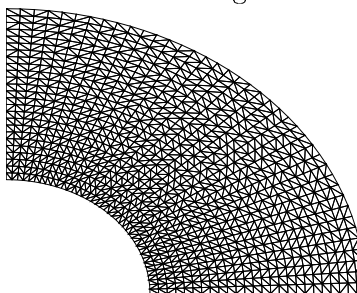


Figure 2: Quarter donut harbor - fine grid: 2000 elements, 1066 nodes



our decomposition, as indicated in Figure 1. This leads to some unnecessary computation, but becomes negligible for larger problem sizes.

A postprocessor was developed to take output generated by each processor and write global output files for plotting and visualization. The preprocessor and postprocessor run on workstations.

## 4  Numerical Results

A series of standard test cases were run to determine the efficiency of the parallel code. Figures 1 – 3 show the spatial domains used in the parallelization experiments.

For the coarse grid test case, we employed a time step size of $\Delta t = 86.4$sec. with final time of $T = 10$ days. In the fine grid test case, we used $\Delta t = 86.4$ sec., and $T = 10$ days. And finally, for the east coast grid test case, we used $\Delta t = 37.5$ sec. and $T = 1$ day.

Table 1, shows CPU times for each computational domain on a 40 node Intel Paragon. The upper bound for the number of processors used in each test case was determined by the parallel efficiency of the case being considered. In particular, when efficiency began to decline, which inevitably happens with a fixed problem size, further decompositions were not attempted. This phenomenon suggests that employing larger domain data sets per processor will yield improved speed-up efficiencies. For example, the coarse grid test case was more efficient on 1 processor than on 3, because of the surface/volume

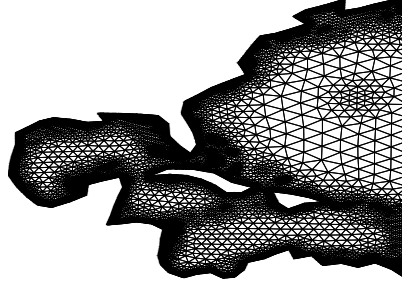Figure 3: United States east coast grid: 18578 elements, 10147 nodes



Table 1: CPU times on the Intel Paragon (in minutes)

| | Number of Processors | | | | | |
|---|---|---|---|---|---|---|
| | 1 | 3 | 4 | 10 | 20 | 30 |
| Coarse grid | 5.5 | 6.3 | - | - | - | - |
| Fine grid | 84.0 | - | 27.6 | 16.5 | - | - |
| East Coast | 755.0 | - | 55.5 | 27.5 | 16.35 | 13.5 |

Here, '-' signifies test not actualized.

ratio of the domains and the amount of message passing relative to computation on each processor.

In the test case of Figure 2, we see a speed-up of 3.04 on 4 processors and a speed-up of 5.09 on 10 processors. Optimal speed-up factors would be 4 and 10, respectively. These results indicate again that as the number of processors increases for a fixed problem size, the speed-up factor eventually decreases.

The CPU time for 1 processor in the east coast test case actually is the CPU time on a CRAY Y-MP. Due to memory limitations it was difficult to run the simulator on 1 processor on the Intel Paragon. The minimum number of processors from which speed-up could be measured was 4 processors. We see speed-ups (relative to 4 processors) of 2.16, 3.39, and 4.11 on 10, 20 and 30 processors, respectively. Optimal speed-up factors would be 2.5, 5 and 7.5.

Table 2 shows, for the fine grid test case, the definite advantage of the parallelized simulator over the sequential simulator on a network of workstations. We observed a speed-up of 2.68 in using 4 processors over 1 processor. The use of 10 processors did not yield additional speed-up, but this number is highly dependent on network traffic at the time the simulation is being performed.

Table 2: CPU times on a Cluster of Sun Workstations (in minutes)

| | Number of Processors | | |
| --- | --- | --- | --- |
| | 1 | 4 | 10 |
| Fine grid | 138.0 | 51.4 | 51.1 |

# 5 Conclusions

In conclusion, analysis and parallelization of an existing shallow water simulator has been completed. The parallelized simulator shows great promise for solving larger data sets than ever attempted before in a reasonable amount of time. The authors are currently working on various projects extending this research. Included in this effort is the development of an *a priori error estimate* for the GWCE-CME formulation in discrete time. Additionally, the authors are developing numerical algorithms for solving the shallow water equations based on the primitive formulation.

# 6 Acknowledgments

**Keywords:** shallow water equations, surface flow, mass conservation, momentum conservation, finite element model, error estimate, projection

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