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Two Dimensional Domains on the
Sphere**

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**CRPC-TR97681
February 1997**

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RECOVERING GRID-POINT VALUES WITHOUT GIBBS OSCILLATIONS IN TWO DIMENSIONAL DOMAINS ON THE SPHERE ¹

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Abstract

Spectral methods using spherical harmonic basis functions have proven to be very effective in geophysical and astrophysical simulations. It is an unfortunate fact, however, that spurious oscillations, known as the Gibbs phenomenon, contaminate these spectral solutions, particularly in regions where discontinuities or steep gradients occur. They are also apparent in the polar regions even when considering analytical periodic functions.

These undesirable artificial oscillations have been the topic of several recent articles [8],[17],[19]. Navarra et.al. [19] alleviates the problem by employing various filters in one and two dimensions. Lindberg and Broccoli [17] implement a nonuniform spherical smoothing spline and zonal filtering, while Gelb [8] applies the Gegenbauer method [13] in the latitudinal direction for fixed longitudinal coordinates.

Since the physical problems solved on spheres often involve discontinuities or steep gradients in the longitudinal direction, and since spherical harmonic spectral methods always introduce oscillations in the polar regions, it is clear that an ideal numerical method should incorporate the removal of the Gibbs phenomenon in both directions, as suggested in both [17] and [19]. This paper offers a two-dimensional approach to the problem by simultaneously applying the Gegenbauer method in both directions. Assuming only the knowledge of the first $(N + 1)^2$ spherical harmonic coefficients, we prove an *exponentially convergent* approximation to a piecewise smooth function in regions composed of arbitrary rectangles for which the function is continuous, thereby entirely removing the Gibbs phenomenon.

Key Words: Gibbs phenomenon, Gegenbauer polynomials, spherical harmonics.

AMS(MOS) subject classification: 42A10, 42A20, 33C55, 65M70, 85-08.

¹The research was supported in part by the National Science Foundation under Cooperative Agreement No. CCR-9120008. The research was also partially supported by the Environment Program grant ENV4-CT95-0109.

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1 Introduction

Spectral methods using the two-dimensional spherical harmonic basis set provide the most natural way of solving problems that occur on spheres and offer many advantages over finite difference methods [4],[15] and [20]. Weather forecasts are now routinely performed by using spectral models that solve the atmospheric equation of motion which includes all the intricacies of the complex radiative and thermo-dynamical forcing processes. The core of the algorithm is the pseudo-spectral method [20]. Spectral models have also proven to be quite capable of simulating the long range behavior of the climate system [18]. In this latter application, spectral models describe the circulation of the atmosphere component and they are coupled with ocean models of various complexity, such as sea-ice models and biosphere subsystems of various kinds. Spectral models have proven to be very robust, accurate, and flexible for this class of problems [5].

However, the spherical transform involved in the pseudo-spectral method generally used in these applications is not exempt from the Gibbs phenomenon. The particular formulation of several models require a transform of the surface mountains that represent a bottom boundary condition for the equations of motion. The resolution used is usually not sufficient to properly resolve steep gradients. The Gibbs waves, visible in the model mountains field as a series of altitude oscillations over the oceans, are especially prominent in the vicinity of sharp mountain ridges like the Andes or, to a lesser extent, the Himalayas.

Early investigators were well aware of the presence of these waves, but they were considered a minor problem, or something that could be easily alleviated by compensating the surface temperature in some way. However in the case of an atmospheric model coupled to an ocean model the error causes large consequences. In this case, the wind's spurious oscillations in the mountain field caused by the Gibbs error interacts with the ocean in a nonlinear feedback, thereby intensifying the error. Thus it is particularly important to have an algorithm that recovers the grid-point values of the field of interest with better accuracy, i.e. without Gibbs oscillations.

To formulate the problem mathematically, we define the spherical harmonic spectral expansion of a function $f(\theta, \phi)$ with colatitude coordinate $\theta \in [0, \pi]$ and longitude coordinate $\phi \in [0, 2\pi]$ as

Definition 1.1

$$f(\theta, \phi) = \sum_{q=0}^{\infty} \sum_{|\nu| \leq q} a_q^\nu Y_q^\nu(\theta, \phi) \quad (1.1)$$

where the spherical harmonic $Y_q^\nu(\theta, \phi)$ of degree q and order ν is

$$Y_q^\nu(\theta, \phi) = \sqrt{\frac{(2q+1)(q-\nu)!}{4\pi(q+\nu)!}} P_q^\nu(\cos \theta) e^{i\nu\phi}$$

in terms of the associate Legendre functions $P_q^\nu(\cos \theta)$ [6].

The orthonormality of the spherical harmonics $Y_q^\nu(\theta, \phi)$ over the sphere imply that the coefficients a_q^ν are given by

$$a_q^\nu = \int_0^{2\pi} \int_0^\pi f(\theta, \phi) [Y_q^\nu(\theta, \phi)]^* \sin \theta d\theta d\phi, \quad (1.2)$$

where $[Y_q^\nu(\theta, \phi)]^*$ are the complex conjugates of $Y_q^\nu(\theta, \phi)$.

The truncated spectral representation of $f(\theta, \phi)$ used in the spectral models is

$$g_N(\theta, \phi) = \sum_{q=0}^N \sum_{|\nu| \leq q} a_q^\nu Y_q^\nu(\theta, \phi). \quad (1.3)$$

Lemma 1.1 *If $f(\theta, \phi)$ is infinitely differentiable, then $g_N(\theta, \phi)$ converges spectrally to $f(\theta, \phi)$.*

The proof of this lemma is presented in [20].

It is an unfortunate fact that the truncated sum (1.3) will yield poor results if $f(\theta, \phi)$ is discontinuous (or in this case where the steep gradients are not properly resolved), by introducing spurious oscillations in the regions of the discontinuities. These artificial ripples, known as the Gibbs phenomenon, will eventually contaminate the solution over the entire sphere. It has also been noticed [17] that a Gibbs-like phenomenon is evident in the polar

regions, regardless of the analyticity of $f(\theta, \phi)$. This is due to nature of the associated Legendre functions, and the error is amplified with increased latitudinal resolution.

Attempts made to reduce the effect of the Gibbs phenomenon have met with some success. Navarra et.al. [19] employs one and two-dimensional filtering to remove the phenomenon, while Lindberg and Broccoli [17] implement a nonuniform spherical smoothing spline in conjunction with zonal filtering. One major drawback in utilizing filters is the possibility of losing the finer features of a function along with the Gibbs phenomenon, thereby reducing the overall accuracy.

The Gegenbauer method, [8], [9], [11], [12], [13], enables exponential convergence for piecewise smooth functions, assuming knowledge of their spectral coefficients. This paper presents an algorithm to eliminate the Gibbs oscillations in a region composed of arbitrary rectangles on the sphere by combining the method in [11] to the longitudinal direction and the method in [8] to the latitudinal direction. We confirm that the first $(N + 1)^2$ spherical harmonic coefficients contain enough information to reconstruct a *spectrally* accurate approximation to a piecewise analytic function. The results shows that grid-point values free of Gibbs oscillations can be obtained in arbitrary regions on the sphere with a robust and flexible algorithm. The procedure consists of the same two steps that were originally created in [11]:

1. The two-dimensional Gegenbauer expansion coefficients are approximated using the first $(N + 1)^2$ spherical harmonic coefficients, a_q'' . Exponential accuracy for these approximated Gegenbauer coefficients is attainable for any L_1 function as long as the Gegenbauer parameters meet certain requirements. The error incurred at this stage is called the *truncation error*, and is investigated in Section 3.
2. The exponential convergence of the Gegenbauer expansion partial sum to a piecewise analytic function in a continuous subinterval has been previously established in [11] and [12]. The error at this stage is labeled the *regularization error* and the results are quoted in Section 4.

The combination of these two errors is the total error between the piecewise analytic function and the Gegenbauer approximation. Section 5 includes a numerical example to illustrate our results. Section 2 contains the properties of Gegenbauer polynomials necessary in proving the exponential accuracy of the Gegenbauer method.

Throughout this paper, A denotes a generic constant or at most a polynomial in the growing parameters, as is indicated in the text.

2 Preliminaries

2.1 Gegenbauer polynomials

This section contains some useful results about the Gegenbauer polynomials previously compiled in [13]. The standardization comes from Bateman [3].

Definition 2.1 *The Gegenbauer polynomial $C_n^\lambda(x)$, for $\lambda \geq 0$, is defined by*

$$(1-x^2)^{\lambda-\frac{1}{2}} C_n^\lambda(x) = G(\lambda, n) \frac{d^n}{dx^n} \left[(1-x^2)^{n+\lambda-\frac{1}{2}} \right], \quad (2.1)$$

where $G(\lambda, n)$ is given by

$$G(\lambda, n) = \frac{(-1)^n \Gamma(\lambda + \frac{1}{2}) \Gamma(n+2\lambda)}{2^n n! \Gamma(2\lambda) \Gamma(n+\lambda+\frac{1}{2})}. \quad (2.2)$$

Formula (2.1) is also called the Rodrigues' formula [1, page 175].

Under this definition, for $\lambda > 0$,

$$C_n^\lambda(1) = \frac{\Gamma(n+2\lambda)}{n! \Gamma(2\lambda)} \quad (2.3)$$

and

$$|C_n^\lambda(x)| \leq C_n^\lambda(1), \quad -1 \leq x \leq 1. \quad (2.4)$$

The Gegenbauer polynomials are orthogonal under the weight function $(1-x^2)^{\lambda-\frac{1}{2}}$, thus

$$\int_{-1}^1 (1-x^2)^{\lambda-\frac{1}{2}} C_k^\lambda(x) C_n^\lambda(x) dx = \delta_{k,n} h_n^\lambda, \quad (2.5)$$

where, for $\lambda > 0$,

$$h_n^\lambda = \pi^{\frac{1}{2}} C_n^\lambda(1) \frac{\Gamma(\lambda + \frac{1}{2})}{\Gamma(\lambda)(n + \lambda)}. \quad (2.6)$$

The approximation of the Gegenbauer polynomials for large n and λ is dependent upon the well-known Stirling's formula for $\Gamma(x)$ given by

$$(2\pi)^{\frac{1}{2}} x^{x+\frac{1}{2}} e^{-x} \leq \Gamma(x+1) \leq (2\pi)^{\frac{1}{2}} x^{x+\frac{1}{2}} e^{-x} e^{\frac{1}{12x}}, \quad x \geq 1. \quad (2.7)$$

Lemma 2.1 *There exists a constant A independent of λ and n such that*

$$A^{-1} \frac{\lambda^{\frac{1}{2}}}{(n + \lambda)} C_n^\lambda(1) \leq h_n^\lambda \leq A \frac{\lambda^{\frac{1}{2}}}{(n + \lambda)} C_n^\lambda(1). \quad (2.8)$$

The proof follows from (2.6) and the Stirling's formula (2.7). □

The following lemma to be used later is easily obtained from the Rodrigues' formula (2.1).

Lemma 2.2 *For any $\lambda \geq 1$*

$$\frac{d}{dx} \left[(1 - x^2)^{\lambda - \frac{1}{2}} C_n^\lambda(x) \right] = \frac{G(\lambda, n)}{G(\lambda - 1, n + 1)} (1 - x^2)^{\lambda - \frac{3}{2}} C_{n+1}^{\lambda-1}(x). \quad (2.9)$$

The proof follows from taking the derivative on both sides of the Rodrigues' formula (2.1), and then using it again on the right hand side. □

The following formula [1, page 176] will also be needed:

$$C_n^\lambda(x) = \frac{1}{2(n + \lambda)} \left(\frac{d}{dx} \left[C_{n+1}^\lambda(x) - C_{n-1}^\lambda(x) \right] \right), \quad (2.10)$$

which is true for all $\lambda \geq 0$.

The associated Legendre functions are defined as

Definition 2.2

$$P_l^m(x) = (1 - x^2)^{\frac{m}{2}} \frac{d^m P_l(x)}{dx^m}, \quad 0 \leq m \leq l, \quad -1 \leq x \leq 1. \quad (2.11)$$

Employing Rodrigues formula for $P_l(x)$, the corresponding Rodrigues formula for $P_l^m(x)$ is

$$P_l^m(x) = \frac{(1-x^2)^{\frac{m}{2}}}{2^l l!} \frac{d^{l+m}}{dx^{l+m}} (x^2-1)^l, \quad 0 \leq m \leq l.$$

The equation (2.11) allows rapid development of many properties of the P_l^m , particularly the recurrence relations [14, page 1005],

for l

$$(l-m+1)P_{l+1}^m - (2l+1)xP_l^m + (l+m)P_{l-1}^m = 0,$$

and for m

$$\sqrt{1-x^2}P_l^{m+1} + 2mxP_l^m + (l+m)(l-m+1)\sqrt{1-x^2}P_l^{m-1} = 0.$$

These equations lead to other useful relations as well, notably those used in the truncation error proofs in Section (3).

3 Truncation Error in a subinterval

Let $f(\theta, \phi)$ be an L_1 function defined for $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi]$, such that $f(\theta, \phi)$ is continuous in the subinterval $0 \leq \phi_1 \leq \phi \leq \phi_2 \leq 2\pi$, $0 \leq \theta_1 \leq \theta \leq \theta_2 \leq \pi$.

The spherical harmonic partial sum of $f(\theta, \phi)$,

$$g_N(\theta, \phi) = \sum_{q=0}^N \sum_{|\nu| \leq q} a_q^\nu Y_q^\nu(\theta, \phi) \quad (3.1)$$

where the spherical harmonics $Y_q^\nu(\theta, \phi)$ are defined by

$$Y_q^\nu(\theta, \phi) = \sqrt{\frac{(2q+1)(q-\nu)!}{4\pi(q+\nu)!}} P_q^\nu(\cos \theta) e^{i\nu\phi} \quad (3.2)$$

and $P_q^\nu(\cos \theta)$ are the associated Legendre functions in (2.11), are computed from the given coefficients

$$a_q^\nu = \int_0^{2\pi} \int_0^\pi f(\theta, \phi) [Y_q^\nu(\theta, \phi)]^* \sin \theta d\theta d\phi \quad (3.3)$$

where $[Y_q^\nu(\theta, \phi)]^*$ denote the complex conjugates of $Y_q^\nu(\theta, \phi)$.

Note that the coefficients of $f(\theta, \phi) \in L_1$ must satisfy the following assumption:

Assumption 3.1 $|a_q^\nu| \leq A$ independent of q .

Unfortunately, while $g_N(\theta, \phi)$ converges rapidly for continuous functions, $g_N(\theta, \phi)$ will not converge fast if $f(\theta, \phi)$ has any discontinuities, and furthermore spurious oscillations will form near the discontinuities.

The goal is to accurately recover $f(\theta, \phi)$ for θ in $[\theta_1, \theta_2] \subset [0, \pi]$, and $\phi \in [\phi_1, \phi_2] \subset [0, 2\pi]$ using the spherical harmonic partial sum given in (3.1) and the Gegenbauer polynomials defined in (2.1). Hence $\theta \in [0, \pi]$, $\phi \in [0, 2\pi]$ are transformed to $\xi_1, \xi_2 \in [-1, 1]$ and the following definitions are formulated.

Definition 3.1 The local variables ξ_1 and ξ_2 are defined by

$$\begin{aligned} \theta(\xi_1) &= \epsilon_1 \xi_1 + \delta_1 \\ \epsilon_1 &= \frac{\theta_2 - \theta_1}{2}, & \delta_1 &= \frac{\theta_2 + \theta_1}{2} \\ \phi(\xi_2) &= \epsilon_2 \xi_2 + \delta_2 \\ \epsilon_2 &= \frac{\phi_2 - \phi_1}{2}, & \delta_2 &= \frac{\phi_2 + \phi_1}{2}. \end{aligned} \quad (3.4)$$

The Gegenbauer expansion of $f(\theta, \phi)$ in the subinterval $[\theta_1, \theta_2], [\phi_1, \phi_2]$ is

$$f(\theta(\xi_1), \phi(\xi_2)) = \sum_{\mu_1=0}^{\infty} \sum_{\mu_2=0}^{\infty} \hat{f}_{\mu_1, \mu_2}^{\lambda_1, \lambda_2} C_{\mu_1}^{\lambda_1}(\xi_1) C_{\mu_2}^{\lambda_2}(\xi_2) \quad (3.5)$$

where

$$\hat{f}_{\mu_1, \mu_2}^{\lambda_1, \lambda_2} = \frac{1}{h_{\mu_1}^{\lambda_1} h_{\mu_2}^{\lambda_2}} \int_{-1}^1 \int_{-1}^1 f(\theta(\xi_1), \phi(\xi_2)) (1 - \xi_1^2)^{\lambda_1 - \frac{1}{2}} (1 - \xi_2^2)^{\lambda_2 - \frac{1}{2}} C_{\mu_1}^{\lambda_1}(\xi_1) C_{\mu_2}^{\lambda_2}(\xi_2) d\xi_1 d\xi_2 \quad (3.6)$$

and $h_{\mu_1}^{\lambda_1}, h_{\mu_2}^{\lambda_2}$ are given in equation (2.6).

Of course $\hat{f}_{\mu_1, \mu_2}^{\lambda_1, \lambda_2}$ is unknown, so it is necessary to construct the approximation

$$\hat{g}_{\mu_1, \mu_2}^{\lambda_1, \lambda_2} = \frac{1}{h_{\mu_1}^{\lambda_1} h_{\mu_2}^{\lambda_2}} \int_{-1}^1 \int_{-1}^1 g_N(\theta(\xi_1), \phi(\xi_2)) (1 - \xi_1^2)^{\lambda_1 - \frac{1}{2}} (1 - \xi_2^2)^{\lambda_2 - \frac{1}{2}} C_{\mu_1}^{\lambda_1}(\xi_1) C_{\mu_2}^{\lambda_2}(\xi_2) d\xi_1 d\xi_2 \quad (3.7)$$

where $g_N(\theta(\xi_1), \phi(\xi_2))$ is the partial sum in (3.1) based on the transformation in definition (3.1).

The truncation error describes how well the coefficients $\hat{g}_{\mu_1, \mu_2}^{\lambda_1, \lambda_2}$ approximate $\hat{f}_{\mu_1, \mu_2}^{\lambda_1, \lambda_2}$.

Definition 3.2 The truncation error is defined by

$$TE(\lambda_1, \lambda_2, m_1, m_2, N) = \max_{\substack{-1 \leq \xi_1 \leq 1 \\ -1 \leq \xi_2 \leq 1}} \left| \sum_{\mu_1=0}^{m_1} \sum_{\mu_2=0}^{m_2} (\hat{f}_{\mu_1, \mu_2}^{\lambda_1, \lambda_2} - \hat{g}_{\mu_1, \mu_2}^{\lambda_1, \lambda_2}) C_{\mu_1}^{\lambda_1}(\xi_1) C_{\mu_2}^{\lambda_2}(\xi_2) \right| \quad (3.8)$$

where $\hat{f}_{\mu_1, \mu_2}^{\lambda_1, \lambda_2}$ and $\hat{g}_{\mu_1, \mu_2}^{\lambda_1, \lambda_2}$ are defined in equations (3.6) and (3.7).

The following lemma is required to minimize $TE(\lambda_1, \lambda_2, m_1, m_2, N)$.

Lemma 3.1

$$\begin{aligned} |(\hat{f}_{\mu_1, \mu_2}^{\lambda_1, \lambda_2} - \hat{g}_{\mu_1, \mu_2}^{\lambda_1, \lambda_2}) C_{\mu_1}^{\lambda_1}(1) C_{\mu_2}^{\lambda_2}(1)| &\leq \frac{A(m_2 + \lambda_2) (m_1 + 2\lambda_1)^{m_1+2\lambda_1} (m_2 + 2\lambda_2)^{m_2+2\lambda_2}}{N^{\kappa-\frac{1}{2}} (2\epsilon_1 \lambda_1)^{\lambda_1} m_1^{m_1}} \\ &\quad \frac{\Gamma(m_2 + 2\lambda_2) \Gamma(\lambda_2)}{m_2! \Gamma(2\lambda_2)} \left(\frac{2}{\epsilon_2 N} \right)^{\lambda_2-1} \end{aligned} \quad (3.9)$$

for $\mu_1 \leq m_1$, $\mu_2 \leq m_2$, A is a constant, and $\lambda > \kappa \geq 1$.

Proof

$$\begin{aligned} |(\hat{f}_{\mu_1, \mu_2}^{\lambda_1, \lambda_2} - \hat{g}_{\mu_1, \mu_2}^{\lambda_1, \lambda_2}) C_{\mu_1}^{\lambda_1}(1) C_{\mu_2}^{\lambda_2}(1)| &= \sum_{q=N+1}^{\infty} \left| \frac{C_{\mu_1}^{\lambda_1}(1) C_{\mu_2}^{\lambda_2}(1)}{h_{\mu_1}^{\lambda_1} h_{\mu_2}^{\lambda_2}} \sum_{|\nu| \leq N} \sqrt{\frac{(2q+1)(q-\nu)!}{4\pi(q+\nu)!}} a_q^{\nu} e^{i\nu\delta_2} \right. \\ &\quad \int_{-1}^1 (1 - \xi_1^2)^{\lambda_1-\frac{1}{2}} C_{\mu_1}^{\lambda_1}(\xi_1) P_q^{\nu}(\epsilon_1 \xi_1 + \delta_1) d\xi_1 \\ &\quad \left. \int_{-1}^1 (1 - \xi_2^2)^{\lambda_2-\frac{1}{2}} C_{\mu_2}^{\lambda_2}(\xi_2) e^{i\nu\epsilon_2 \xi_2} d\xi_2 \right|. \end{aligned} \quad (3.10)$$

There is an explicit expression given in [3] for

$$\frac{1}{h_{\mu}^{\lambda}} \int_{-1}^1 (1 - \xi^2)^{\lambda-\frac{1}{2}} e^{i\nu\pi\xi} C_{\mu}^{\lambda}(\xi) dx = \Gamma(\lambda) \left(\frac{2}{\pi\nu} \right)^{\lambda} i^{\mu} (\mu + \lambda) J_{\mu+\lambda}(\pi\nu) \quad (3.11)$$

where $J_{\mu+\lambda}(\pi\nu)$ is the Bessel function. Subsequently it was shown in [11] that

$$\frac{C_{\mu_2}^{\lambda_2}(1)}{h_{\mu_2}^{\lambda_2}} \int_{-1}^1 (1 - \xi_2^2)^{\lambda_2-\frac{1}{2}} C_{\mu_2}^{\lambda_2}(\xi_2) e^{i\nu\epsilon_2 \xi_2} d\xi_2 \leq \frac{(m_2 + \lambda_2) \Gamma(m_2 + 2\lambda_2) \Gamma(\lambda_2)}{m_2! \Gamma(2\lambda_2)} \left(\frac{2}{\epsilon_2 |\nu|} \right)^{\lambda_2} \quad (3.12)$$

for $\mu_2 \leq m_2$.

In [8] it was estimated that

$$\begin{aligned} S_{q, \nu}^{\lambda, \mu} &= \int_{-1}^1 (1 - \xi^2)^{\lambda-\frac{1}{2}} C_{\mu}^{\lambda}(\xi) P_q^{\nu}(\epsilon\xi + \delta) d\xi \\ &\leq A \sqrt{\frac{(q+\nu)! (q-\mu)^{(\lambda-\kappa)} \Gamma(q-\lambda)}{(q-\nu)! \epsilon^{\lambda} \Gamma(q)}} \frac{|G(\lambda, \mu)|}{|G(0, \mu + \lambda)|} \end{aligned} \quad (3.13)$$

where A is a constant, $\lambda \geq \kappa > 1$, and $G(\lambda, \mu)$ is defined in (2.2). Equation (3.13) and Stirling's formula (2.7) imply that

$$\sqrt{\frac{(2q+1)(q-\nu)!}{4\pi(q+\nu)!} \frac{C_{\mu_1}^{\lambda_1}(1)}{h_{\mu_1}^{\lambda_1}}} S_{q,\nu}^{\lambda_1, \mu_1} \leq A \frac{(m_1 + 2\lambda_1)^{m_1+2\lambda_1}}{(2\epsilon\lambda_1)^{\lambda_1} m_1^{m_1}} \frac{1}{q^{\kappa-\frac{1}{2}}} \quad (3.14)$$

for $\mu_1 \leq m_1$.

Applying the assumption 3.1 and substituting equations (3.12), (3.13) and (3.14) into equation (3.10) yields the desired result (3.9). □

Theorem 3.1 *Suppose $\lambda_1 = \alpha_1 \epsilon_1 N$, $\lambda_2 = \alpha_2 \epsilon_2 N$, $m_1 = \beta_1 \epsilon_1 N$, and $m_2 = \beta_2 \epsilon_2 N$, where $\alpha_i, \beta_i < 1$, $i = 1, 2$. Then the truncation error defined in (3.8) can be estimated by*

$$TE(\lambda_1, \lambda_2, m_1, m_2, N) \leq A \frac{N^3}{N^{\kappa-\frac{1}{2}}} \left(\frac{(\beta_1 + 2\alpha_1)^{\beta_1+2\alpha_1}}{(2\alpha_1)^{\alpha_1} \beta_1^{\beta_1}} \right)^{\epsilon_1 N} \left(\frac{(\beta_2 + 2\alpha_2)^{\beta_2+2\alpha_2}}{(2\alpha_2 \epsilon_2 e)^{\alpha_2} \beta_2^{\beta_2}} \right)^{\epsilon_2 N} \quad (3.15)$$

where $\lambda_1 > \kappa \geq 1$ and A is a constant.

Proof

From the estimate (2.4) and the equations (3.9), (3.12) and (3.14) it is easy to show that

$$TE(\lambda_1, \lambda_2, m_1, m_2, N) \leq \max_{\substack{-1 \leq \xi_1 \leq 1 \\ -1 \leq \xi_2 \leq 1}} \left| \sum_{\mu_1=0}^{m_1} \sum_{\mu_2=0}^{m_2} \frac{A(m_2 + \lambda_2)}{N^{\kappa-\frac{1}{2}}} \frac{(m_1 + 2\lambda_1)^{m_1+2\lambda_1} (m_2 + 2\lambda_2)^{m_2+2\lambda_2}}{(2\epsilon_1 \lambda_1)^{\lambda_1} m_1^{m_1}} \frac{\Gamma(m_2 + 2\lambda_2) \Gamma(\lambda_2)}{m_2! \Gamma(2\lambda_2)} \left(\frac{2}{\epsilon_2 N} \right)^{\lambda_2-1} \right|. \quad (3.16)$$

Simply applying Stirling's formula (2.7) and a bit of algebra yields the estimate (3.15). □

Thus, for Gegenbauer parameters $\mu_1, \mu_2, \lambda_1, \lambda_2 \sim N$, the truncation error converges exponentially.

4 Regularization Error

The regularization error is the difference between the Gegenbauer partial sum approximation and the piecewise analytic function $f(\theta, \phi)$ in the maximum norm.

Definition 4.1 *The regularization error is defined by*

$$RE(\lambda_1, \lambda_2, m_1, m_2) = \max_{\substack{-1 \leq \xi_1 \leq 1 \\ -1 \leq \xi_2 \leq 1}} |f(\theta(\xi_1), \phi(\xi_2)) - \sum_{\mu_1=0}^{m_1} \sum_{\mu_2=0}^{m_2} \hat{f}_{\mu_1, \mu_2}^{\lambda_1, \lambda_2} C_{\mu_1}^{\lambda_1}(\xi_1) C_{\mu_2}^{\lambda_2}(\xi_2)|. \quad (4.1)$$

Exponential convergence was proved for the one dimension case in [11], and then extended to include two dimensions in [9]. For brevity, we simply quote the results in the transformed coordinates ξ_1 and ξ_2 .

The following assumption and lemma are needed.

Assumption 4.1 *Let $f(\theta(\xi_1), \phi(\xi_2))$ be a piecewise analytic function for $-1 \leq \xi_1 \leq 1$ and $-1 \leq \xi_2 \leq 1$. Then there exists constants $\rho_1 \geq 1$, $\rho_2 \geq 1$, $C(\rho_1, \rho_2)$ such that, for every $\mu_1, \mu_2 \geq 0$,*

$$\max_{\substack{-1 \leq \xi_1 \leq 1 \\ -1 \leq \xi_2 \leq 1}} \left| \frac{d^{\mu_1 + \mu_2} f}{d\xi_1^{\mu_1} d\xi_2^{\mu_2}}(\theta(\xi_1), \phi(\xi_2)) \right| \leq C(\rho_1, \rho_2) \frac{\mu_1!}{\rho_1^{\mu_1}} \frac{\mu_2!}{\rho_2^{\mu_2}}. \quad (4.2)$$

This is a standard assumption for analytic functions [16].

Lemma 4.1 *The Gegenbauer coefficient $\hat{f}_{\mu_1, \mu_2}^{\lambda_1, \lambda_2}$ as defined in equation (3.6) of an analytic function satisfying Assumption 4.1 is bounded by*

$$|\hat{f}_{\mu_1, \mu_2}^{\lambda_1, \lambda_2}| \leq AC(\rho_1, \rho_2) \frac{\Gamma(\lambda_1 + \frac{1}{2})\Gamma(\mu_1 + 2\lambda_1)}{h_{\mu_1}^{\lambda_1}(2\rho_1)^{\mu_1}\Gamma(2\lambda_1)\Gamma(\mu_1 + \lambda_1 + 1)} \frac{\Gamma(\lambda_2 + \frac{1}{2})\Gamma(\mu_2 + 2\lambda_2)}{h_{\mu_2}^{\lambda_2}(2\rho_2)^{\mu_2}\Gamma(2\lambda_2)\Gamma(\mu_2 + \lambda_2 + 1)}. \quad (4.3)$$

Proof

The proof follows from applying Rodrigues' formula (2.1) and equation (2.2) to the definition of $\hat{f}_{\mu_1, \mu_2}^{\lambda_1, \lambda_2}$ in (3.6) and performing integration by parts $\mu_1 + \mu_2$ times. Then employing assumption 4.1, equation (2.6), and the fact that $C_0^\mu(\xi) = 1$, yields the above estimate.

□

The combination of Lemma 4.1 and Assumption 4.1 establishes an estimate for the regularization error in the maximum norm.

Theorem 4.1 *If $f(\theta(\xi_1), \phi(\xi_2))$ is a piecewise analytic function satisfying Assumption 4.1, then the regularization error defined in equation (4.1) can be bounded by*

$$RE(\lambda_1, \lambda_2, m_1, m_2) \leq \frac{C(\rho_1, \rho_2)\Gamma(\lambda_1 + \frac{1}{2})\Gamma(\lambda_2 + \frac{1}{2})\Gamma(m_1 + 2\lambda_1 + 1)\Gamma(m_2 + 2\lambda_2 + 1)}{m_1 m_2 2^{m_1+m_2} \rho_1^{m_1} \rho_2^{m_2} \Gamma(2\lambda_1)\Gamma(2\lambda_2)\Gamma(m_1 + \lambda_1)\Gamma(m_2 + \lambda_2)}. \quad (4.4)$$

Proof

Applying estimate (2.8) to equation (4.3) in conjunction with the fact that $|C_t^\lambda(\xi)| \leq C_t^\lambda(1)$ for all $-1 \leq \xi \leq 1$ leads to the desired outcome.

□

It follows that the regularization error defined in (4.1) is *exponentially small* when λ_1 and λ_2 grow linearly with m_1 and m_2 .

Theorem 4.2 *If $\lambda_1 = \gamma_1 m_1$ and $\lambda_2 = \gamma_2 m_2$ where γ_1 and γ_2 are positive constant, then the regularization error defined in (4.1) satisfies*

$$RE(\lambda_1, \lambda_2, m_1, m_2) \leq A q_1^{m_1} q_2^{m_2} \quad (4.5)$$

where q_i , $i = 1, 2$ is given by

$$q_i = \frac{(1 + 2\gamma_i)^{1+2\gamma_i}}{\rho_i 2^{1+2\gamma_i} \gamma_i^{\gamma_i} (1 + \gamma_i)^{1+\gamma_i}}, \quad (4.6)$$

which is always less than 1. In particular, if $\gamma_i = 1$ and $m_i = \beta_i N$ where β_i is a positive constant, then

$$RE(\lambda_1, \lambda_2, m_1, m_2) \leq A q_1^N q_2^N \quad (4.7)$$

with

$$q_i = \left(\frac{27}{32\rho_i} \right)^{\beta_i}. \quad (4.8)$$

Proof

The proof follows from the application of Stirling's formula (2.7) to the bound proved in Theorem 4.1. Some algebra leads to equations (4.5) and (4.6), where A involves contributions from ρ_i , $i = 1, 2$. The value q_i defined in equation (4.6) is a strictly increasing function of γ_i . As $\gamma_i \rightarrow \infty$, we have $q_i \rightarrow \frac{1}{\rho_i} \leq 1$. Hence $q_i < \frac{1}{\rho_i} \leq 1$ for all $\gamma_i > 0$. Now, by substituting in the value $\gamma_i = 1$ and $m_i = \beta_i N$ the estimates in (4.7) and (4.8) are obtained.

□

Summarizing the theorems cited thus far, the following theorem states the exponential decay of the regularization error.

Theorem 4.3 *Assume that $f(\theta(\xi_1), \phi(\xi_2))$ is a piecewise analytic function for $-1 \leq \xi_1 \leq 1$ and $-1 \leq \xi_2 \leq 1$ that satisfies Assumption 4.1. Let $\hat{f}_{\mu_1, \mu_2}^{\lambda_1, \lambda_2}$ be the Gegenbauer coefficients defined in equation (3.6) for $0 \leq \mu_1 \leq m_1$ and $0 \leq \mu_2 \leq m_2$. For simplicity let $\lambda_1 = m_1$ and $\lambda_2 = m_2$. Then*

$$\max_{\substack{-1 \leq \xi_1 \leq 1 \\ -1 \leq \xi_2 \leq 1}} |f(\theta(\xi_1), \phi(\xi_2)) - \sum_{\mu_1=0}^{m_1} \sum_{\mu_2=0}^{m_2} \hat{f}_{\mu_1, \mu_2}^{\lambda_1, \lambda_2} C_{\mu_1}^{\lambda_1}(\xi_1) C_{\mu_2}^{\lambda_2}(\xi_2)| \leq A q_1^N q_2^N \quad (4.9)$$

where q_i , $i = 1, 2$ is defined in equation (4.8).

□

Section 3 established that the Gegenbauer coefficients can be approximated with exponential convergence provided that $\lambda_1, \lambda_2, m_1, m_2 \sim N$, and Section 4 confirms that the Gegenbauer partial sum converges exponentially to $f(\theta(\xi_1), \phi(\xi_2))$. These two pertinent factors determine the following result.

Theorem 4.4 *Consider a piecewise analytic function $f(\theta, \phi)$ that satisfies Assumption 4.1. Assume we are given the spherical harmonic coefficients (3.6). Then for $\lambda_1 = m_1 = \beta_1 N$ and $\lambda_2 = m_2 = \beta_2 N$, we have, within a constant,*

$$\max_{\substack{\theta_1 \leq \theta \leq \theta_2 \\ \phi_1 \leq \phi \leq \phi_2}} |f(\theta, \phi) - \sum_{\mu_1=0}^{m_1} \sum_{\mu_2=0}^{m_2} \hat{g}_{\mu_1, \mu_2}^{\lambda_1, \lambda_2} C_{\mu_1}^{\lambda_1}(\xi_1) C_{\mu_2}^{\lambda_2}(\xi_2)| \leq N^3 (q_1^T)^N (q_2^T)^N + (q_1^R)^N (q_2^R)^N \quad (4.10)$$

where

$$\begin{aligned} q_1^T &= \left(\frac{(\beta_1 + 2\alpha_1)^{\beta_1 + 2\alpha_1}}{(2\alpha_1)^{\alpha_1} \beta_1^{\beta_1}} \right)^{\epsilon_1} \\ q_2^T &= \left(\frac{(\beta_2 + 2\alpha_2)^{\beta_2 + 2\alpha_2}}{(2\alpha_2 \epsilon_2 e)^{\alpha_2} \beta_2^{\beta_2}} \right)^{\epsilon_2} \end{aligned} \quad (4.11)$$

and q_1^R and q_2^R are defined in equation (4.8) for $\beta_1 = \beta_2 < \frac{2\pi e}{27}$.

Proof

The total error of the partial Gegenbauer expansion approximation, $g_N(\theta, \phi)$, to the function $f(x, y)$, is defined as

$$E(\lambda_1, \lambda_2, m_1, m_2, N) = \max_{\substack{\theta_1 \leq \theta \leq \theta_2 \\ \phi_1 \leq \phi \leq \phi_2}} |f(\theta, \phi) - \sum_{\mu_1=0}^{m_1} \sum_{\mu_2=0}^{m_2} \hat{g}_{\mu_1, \mu_2}^{\lambda_1, \lambda_2} C_{\mu_1}^{\lambda_1}(\xi_1) C_{\mu_2}^{\lambda_2}(\xi_2)|.$$

Thus the total error is bounded by the sum of the *regularization error* and the *truncation error*, i.e.,

$$\begin{aligned} E(\lambda_1, \lambda_2, m_1, m_2, N) &\leq \max_{\substack{\theta_1 \leq \theta \leq \theta_2 \\ \phi_1 \leq \phi \leq \phi_2}} |f(\theta, \phi) - \sum_{\mu_1=0}^{m_1} \sum_{\mu_2=0}^{m_2} \hat{f}_{\mu_1, \mu_2}^{\lambda_1, \lambda_2} C_{\mu_1}^{\lambda_1}(\xi_1) C_{\mu_2}^{\lambda_2}(\xi_2)| \\ &\quad + \max_{\substack{\theta_1 \leq \theta \leq \theta_2 \\ \phi_1 \leq \phi \leq \phi_2}} \left| \sum_{\mu_1=0}^{m_1} \sum_{\mu_2=0}^{m_2} \hat{f}_{\mu_1, \mu_2}^{\lambda_1, \lambda_2} C_{\mu_1}^{\lambda_1}(\xi_1) C_{\mu_2}^{\lambda_2}(\xi_2) - \sum_{\mu_1=0}^{m_1} \sum_{\mu_2=0}^{m_2} \hat{g}_{\mu_1, \mu_2}^{\lambda_1, \lambda_2} C_{\mu_1}^{\lambda_1}(\xi_1) C_{\mu_2}^{\lambda_2}(\xi_2) \right| \end{aligned}$$

where the first term is estimated by Theorem 4.3 and the second by Theorem 3.1. This concludes the proof. □

5 Numerical Results

The following simple numerical example is provided to establish the efficacy the Gegenbauer approximation method.

Example 5.1 Consider the function

$$f(\theta, \phi) = \begin{cases} \cos 3.2\theta + \sin 2.7\phi & \frac{\pi}{4} \leq \theta \leq \frac{3\pi}{4}, \quad \frac{\pi}{2} \leq \phi \leq \frac{3\pi}{2} \\ \cos 1.5\theta \sin 2\phi + \cos 3.2\phi & \text{otherwise} \end{cases} \quad (5.1)$$

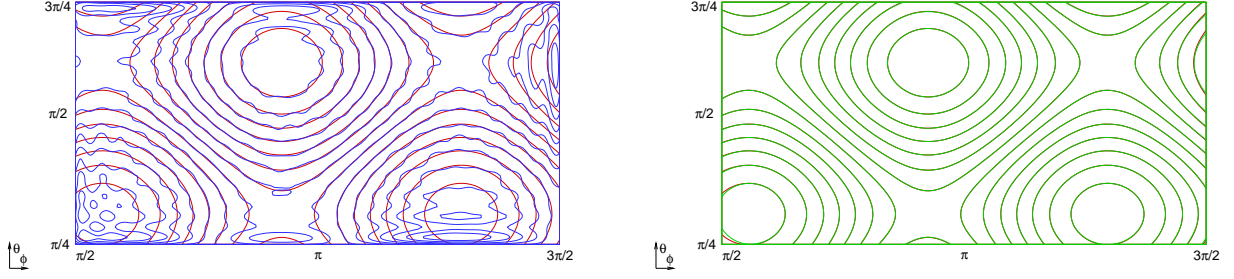


Figure 1: (1a)The spherical harmonic partial sum approximation and the (1b)Gegenbauer approximation in the prescribed subinterval.

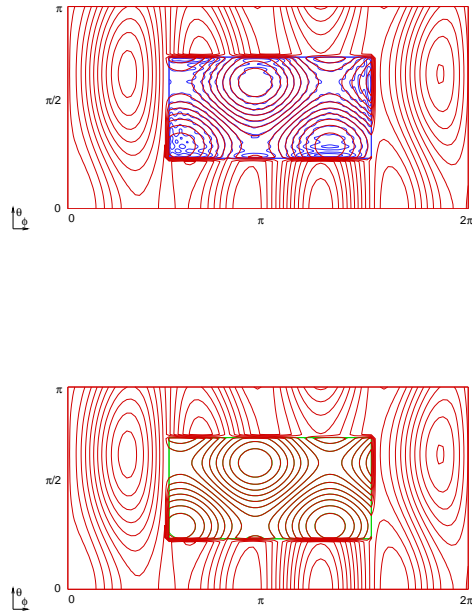


Figure 2: (2a)The spherical harmonic partial sum approximation and the (2b)Gegenbauer approximation for the entire globe.

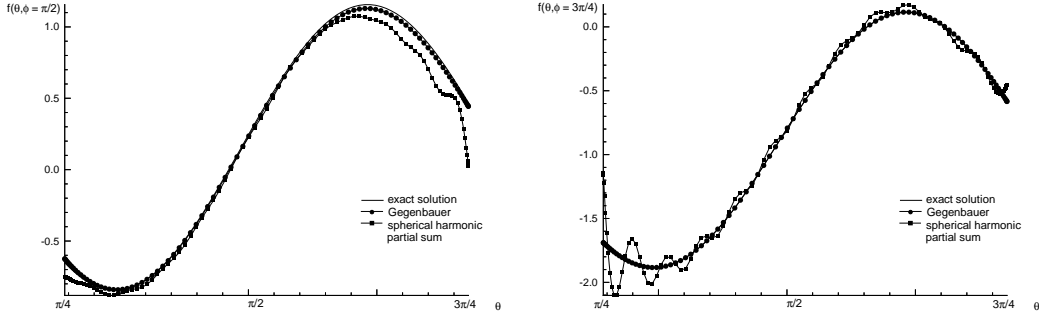


Figure 3: Approximation for $\theta \in [\frac{\pi}{4}, \frac{3\pi}{4}]$ at (3a) $\phi = \frac{\pi}{2}$ and at (3b) $\phi = 0$

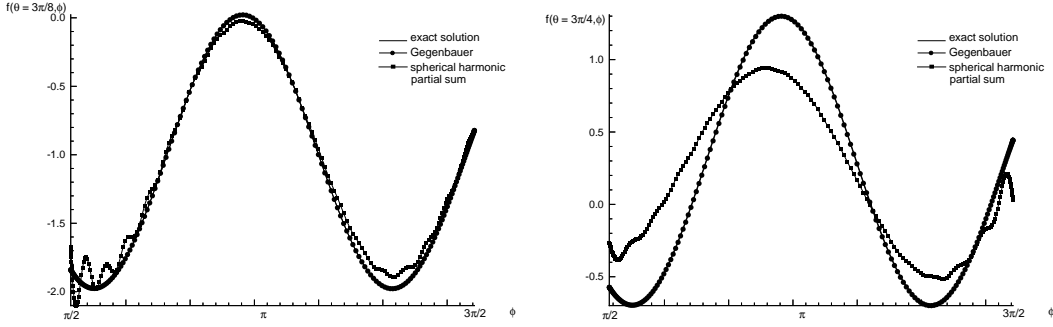


Figure 4: Approximation for $\phi \in [\frac{\pi}{2}, \frac{3\pi}{2}]$ at (4a) $\theta = \frac{3\pi}{8}$ and at (4b) $\theta = \frac{3\pi}{4}$

The contour plot of this example is shown in figures 1 and 2.

We used 48 latitudinal points for $\theta \in [0, \pi]$, and 96 longitudinal points $\phi \in [0, 2\pi]$ with $\lambda_1 = 8$, $\lambda_2 = 4$, $\mu_1 = 6$, and $\mu_2 = 10$. No attempt was made to optimize these parameters. The Gibbs phenomenon clearly dominates the region for the spherical harmonic partial sum approximation, but is completely eliminated by the Gegenbauer method! Figures 3 and 4 show the one dimensional cross sections.

One major drawback of spectral methods is that they are typically limited to boxes and are not suited for general domains. This is true for the Gegenbauer reconstruction as well. However since exponential accuracy is obtained for any continuous region, it can be applied

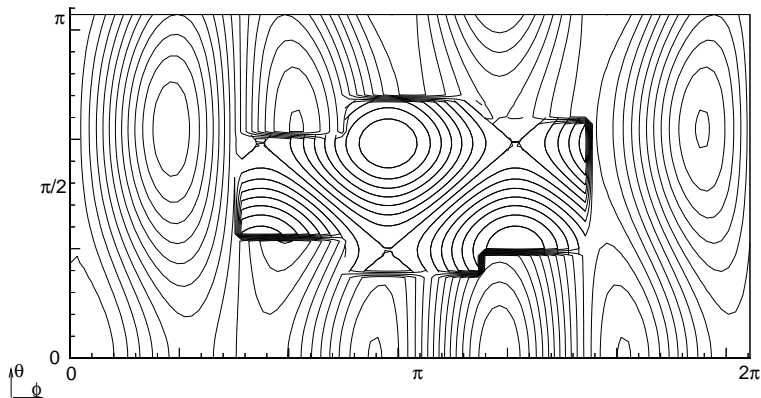


Figure 5: Contour plot of example 5.1 where the region that $f(\theta, \phi)$ is continuous is not a simple “box”.

to boxes of any size and thus essentially cover any domain. This is not as trivial as it may seem, as other filtering techniques [2],[7], must be applied to an entire analytic sub-domain, since the discontinuity jump value and location are both essential in the reconstruction. Yet they are limited to boxes as well, making them more impractical in covering general domains.

The Gegenbauer method is therefore ideal for parallel computers, and practical for “real” problems. To demonstrate, we use the same example 5.1, only in a more generic subregion shown in figure 5.

We simultaneously applied the Gegenbauer method in smaller boxes, as depicted in figure 6. The results are displayed in figure 7.

Assessing the numerical convergence is difficult for the following reasons.

1. There is quite a bit of error introduced in computing the spherical harmonic coefficients, resulting from the singularities at the poles. This is true even for *continuous* and periodic functions. The Gegenbauer method requires an accurate evaluation of the

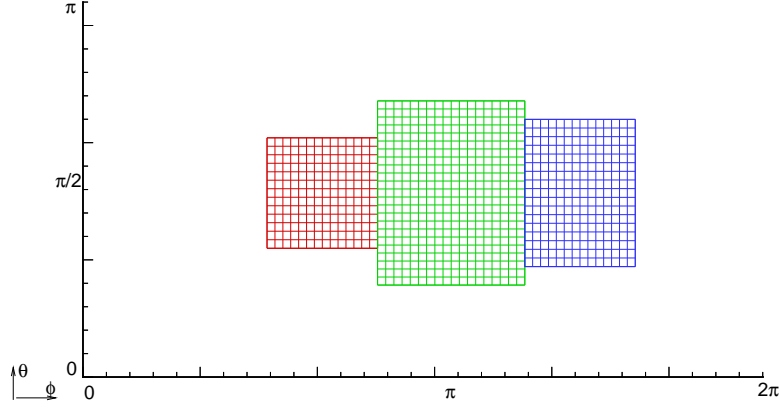


Figure 6: Mesh of example 5.1 where the region that $f(\theta, \phi)$ is continuous is not a simple “box”, we divide the problem into three different boxes and solve simultaneously.

spherical harmonic coefficients and consequently this inaccuracy affects every region on the sphere.

2. Introducing *more* latitudinal points amplifies the oscillations in the spherical harmonic coefficient partial sum approximation at the poles, which in turn affects the Gegenbauer approximation. Thus too much resolution is an additional source of error.
3. Theoretically, μ and λ should grow with N , where $(N + 1)^2$ is the number of spherical harmonic coefficients in the triangular spherical harmonic truncation. Unfortunately, the value of the Gegenbauer polynomials increase rapidly as the order and degree of the Gegenbauer polynomials grow and causes computational roundoff errors for larger m and λ .

One other important consideration is that this paper assumes explicit knowledge of the discontinuity locations. If this information is ambiguous, the results may be skewed. The Gegenbauer method works well in any analytic subinterval, but will produce Gibbs oscil-

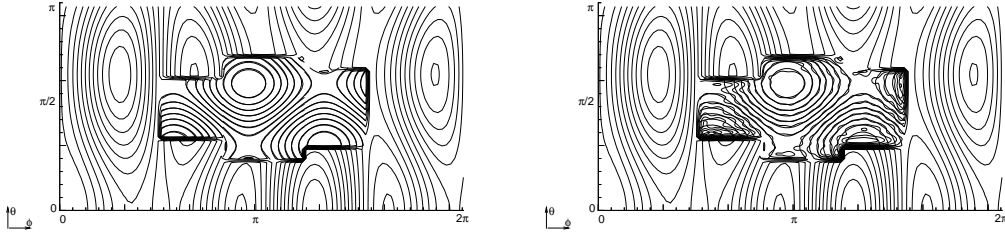


Figure 7: Approximation for example 5.1 in three different subregions (a) Gegenbauer reconstruction (b) Fourier partial sum

lations in a region that contains discontinuities. Therefore it is useful to have on hand a discontinuity locator, which will be the subject of future papers.

6 Conclusion

This paper has shown that the Gegenbauer approach can be successfully applied simultaneously to eliminate Gibbs oscillations in both the meridional and longitudinal direction on the sphere. The results is of great practical consequence for numerical simulations of the atmospheric motion often carried out using spectral solution techniques. The method is robust and flexible, the only inconvenience being that the position of the discontinuity must be known a priori. In the case of the climate system this is not a major setback because the main source of Gibbs errors are the surface mountains whose positions can of course be established well in advance. Further work is under way however to develop an algorithm that would include a discontinuity locator in the Gegenbauer approach.

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