The Resolution of the Gibbs Phenomenon for "Spliced" Functions in One and Two Dimensions

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THE RESOLUTION OF THE GIBBS PHENOMENON FOR "SPLICED" FUNCTIONS IN ONE AND TWO DIMENSIONS[1]

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Abstract

In this paper we study approximation methods for analytic functions that have been "spliced" into non-intersecting sub-domains. We assume that we are given the first $2N+1$ Fourier coefficients for the functions in each sub-domain. The objective is to approximate the "spliced" function in each sub-domain and then to "glue" the approximations together in order to recover the original function in the full domain.

The Fourier partial sum approximation in each sub-domain yields poor results, as the convergence is slow and spurious oscillations occur at the boundaries of *each* sub-domain. Thus once we "glue" the sub-domain approximations back together, the approximation for the function in the full domain will exhibit oscillations throughout the entire domain.

Recently methods have been developed that successfully eliminate the Gibbs phenomenon for analytic but non-periodic functions in one dimension. These methods are based on the knowledge of the first $2N+1$ Fourier coefficients and use either the Gegenbauer polynomials (Gottlieb et al.) or the Bernoulli polynomials (Abarbanel, Gottlieb, Cai et al., and Eckhoff).

We propose a way to accurately reconstruct a "spliced" function in a full domain by extending the current methods to eliminate the Gibbs phenomenon in each non-intersecting sub-domain and then "gluing" the approximations back together. We solve this problem in both one and two dimensions. In the one-dimensional case we provide two alternative options, the Bernoulli method and the Gegenbauer method, as well as a new hybrid method, the Gegenbauer-Bernoulli method. In the two-dimensional case we prove, for the very first time, exponential convergence of the Gegenbauer method, and then we apply it to solve the "spliced" function problem.

Key Words: Gibbs phenomenon, Fourier series, Gegenbauer polynomials, Bernoulli polynomials, exponential accuracy

AMS(MOS) subject classification: 42A10, 41A10, 41A25

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1 Introduction

Consider an analytic and periodic function $f(x)$ defined on $[a,b]$ that has been split into two non-intersecting functions $f_1(x)$ and $f_2(x)$ in the sub-domains $[a,c]$ and $[c,b]$, where $a < c < b$. The functions in each sub-domain are no longer periodic. Now suppose that the first $2N+1$ Fourier coefficients for each sub-domain

$$\begin{aligned} \hat{f}_1(k) &= \frac{1}{c-a}\int_a^c f(x)e^{\frac{-2ik\pi x}{c-a}}dx \\ \hat{f}_2(k) &= \frac{1}{b-c}\int_c^b f(x)e^{\frac{-2ik\pi x}{b-c}}dx \end{aligned}$$

are known. The Fourier partial sum approximation to the function in each sub-domain

$$f_j(x) = \sum_{k=-N}^{N} \hat{f}_j(k)e^{ik\pi x}, \qquad j = 1,2, \tag{1.1}$$

yields poor results. The convergence rate of equation (1.1) for non-periodic functions is slow with error $O(\frac{1}{N})$ away from the boundaries, and exhibits spurious oscillations of order $O(1)$ at the boundaries. This is called the Gibbs phenomenon [11]. Furthermore, summing the approximations in each sub-domain to obtain an approximation to $f(x)$ over the entire domain $[a,b]$ results in the spurious oscillations at the center point $x = c$ as well as at the exterior boundary points $x = a$ and $x = b$. Therefore, even though $f(x)$ is smooth and periodic on $[a,b]$, the resulting Fourier approximation after "splicing" $f(x)$ into different non-intersecting sub-domains no longer converges!

Recently, methods based on the knowledge of the Fourier coefficients $\hat{f}(k)$ of a smooth but non-periodic function $f(x)$ in a general interval $[a,b]$, have been developed ([1], [10], [12], and later [7]) that successfully eliminate the Gibbs phenomenon.

Here we consider a smooth (not necessarily periodic) function $f(x)$ in an interval $[a,b]$ that has been split into non-intersecting sub-domains. By extending the current theory, we propose a way to accurately reconstruct $f(x)$ in $[a,b]$ by "gluing" the approximations in each sub-domain together.

We solve this problem in both one and two dimensions. In the one-dimensional case we provide two alternative options, the Bernoulli method, [1] and [7], and the Gegenbauer

method, [12], as well as a new hybrid method, called the Gegenbauer-Bernoulli method. In the two-dimensional case we prove, for the very first time, exponential convergence of the Gegenbauer method for analytic but non-periodic two-dimensional functions. Although the focus of this study is on spliced pictures, for simplicity we prove all of our results for the interval $[-1, 1]$, noting that a simple transformation leads to the same results in any general interval. Furthermore, this study applies to any piecewise analytic function, as long as the picture is spliced at the points of discontinuity.

2 The Bernoulli Method

2.1 The Method of Reconstruction

The objective of the Bernoulli method is to employ the Bernoulli polynomials to construct an approximation to $f(x)$ based on the knowledge of $\hat{f}(k)$. This method exploits the rapid decay rate of the Fourier coefficients for smooth and *periodic* functions. We follow the description in [7] closely, and refer readers there for more detail.

We begin by defining the magnitude of the jump discontinuity of $f(x)$ and its first n derivatives,

$$A_n = [f^{(n)}(1)] - [f^{(n)}(-1)], \tag{2.1}$$

which leads to the following lemma.

Lemma 2.1 *Let $f(x)$ be a continuous non-periodic function. Then for $|k| \geq 1$ and any $Q > 0$*

$$\hat{f}(k) = e^{-ik\pi} \sum_{n=0}^{Q} \frac{A_n}{2(\pi ik)^{n+1}} + \frac{1}{2} \int_{-1}^{1} \frac{A_{Q+1}}{(\pi ik)^{(Q+1)}} e^{-ik\pi x} dx. \tag{2.2}$$

The proof is given in [5] and [7].

□

Using the lemma above, the partial Fourier sum,

$$f_N(x) = \sum_{k=-N}^{N} \hat{f}(k) e^{ik\pi x}, \tag{2.3}$$

can be rewritten as

$$f_N(x) = w_N(x) + \sum_{n=0}^{Q} A_n V_n(x), \tag{2.4}$$

where $w_N(x)$ is the partial sum defined as

$$w_N(x) = \sum_{\substack{k=-N \\ k \neq 0}} (\frac{1}{2}\int_{-1}^{1} \frac{A_{Q+1}}{(\pi i k)^{(Q+1)}} e^{-ik\pi x} dx) e^{ik\pi x}, \tag{2.5}$$

and $V_n(x)$ are the Bernoulli polynomials whose Fourier coefficients satisfy

$$(\hat{V_n})_k = \begin{cases} \frac{e^{-ik\pi}}{2(\pi i k)^{n+1}} & k = \pm 1, \pm 2, \dots \\ 0 & k = 0 \end{cases}.$$

The first Bernoulli function, $V_0(x)$, is defined as

$$V_0(x) = -\frac{1}{2}(x+1), \qquad x \in [-1, 1], \tag{2.6}$$

and the subsequent functions $V_n(x)$ satisfy the relationship

$$V_n(x) = \int V_{n-1}(x) dx, \qquad n = 1, 2, \dots, \tag{2.7}$$

where the constants of integration are determined by

$$\int_{-1}^{1} V_n(x) dx = 0, \qquad n = 1, 2, \dots. \tag{2.8}$$

Note that $w_N(x)$ approximates a $Q+1$ times continuously differentiable function $w(x)$ on the interval $[-1, 1]$, and thus the Fourier coefficients of the partial sum $w_N(x)$ decay like $O(\frac{1}{k^{Q+2}})$ as $k \to \pm\infty$. It is now possible to obtain an *algebraically* accurate approximation to $f(x)$ by evaluating $w_N(x)$ and $\sum_{n=0}^{Q} A_n V_n(x)$ separately.

The algorithm of the Bernoulli method of reconstruction can be described as follows:

- Rewrite $f(x)$ as

$$f(x) = w(x) + \sum_{n=0}^{Q} A_n V_n(x), \tag{2.9}$$

  where $w(x)$ is some Q times continuously differentiable function in the interval $[-1, 1]$, A_n is defined in equation (2.1), and $V_n(x)$ is defined in equations (2.6), (2.7), and (2.8). Q represents the total number of Bernoulli polynomials employed.

- Compute the jump functions A_n for $n = 0, \ldots, Q$.
- Evaluate the Fourier coefficients $\hat{w}(k)$ by equation (2.9).

$$\begin{aligned} \hat{w}(k) &= \hat{f}(k) - \sum_{n=0}^{Q} A_n(\hat{V}_n)_k, \qquad k = \pm 1, \pm 2, \ldots \\ \hat{w}(0) &= \hat{f}(0). \end{aligned} \tag{2.10}$$

- Calculate the Fourier partial sum $w_N(x)$ from $\hat{w}(k)$.
- Approximate $f(x)$ by equation (2.4).

To compute A_n, we define (see [7])

$$C_k \equiv (\hat{f}(k) - \hat{w}(k)). \tag{2.11}$$

Thus (2.10) is equivalent to the system of equations

$$\sum_{n=0}^{Q} A_n(\hat{V}_n)_k = C_k. \tag{2.12}$$

Since coefficients $\hat{w}(k)$ decay like $O(\frac{1}{k^{Q+1}})$ and the coefficients $\hat{f}(k)$ decay like $O(\frac{1}{k})$, for $|k|$ large enough we can approximate

$$C_k \approx \hat{f}(k) \equiv \tilde{C}_k. \tag{2.13}$$

The difference $(C_k - \tilde{C}_k)$ is of order $O(\frac{1}{k^{Q+1}})$, and thus the values A_n in the relation (2.12) can be estimated by the system

$$\sum_{n=0}^{Q} A_n(\hat{V}_n)_k = \tilde{C}_k \tag{2.14}$$

for large $|k|$. In [7] it is shown that the error in approximating A_n is of the order $O(\frac{1}{N^{Q-n+1}})$. The different options in choosing the $Q+1$ values of k will be discussed in section 4.

$\hat{w}(k)$ is now evaluated by equation (2.10) and $w(x)$ is consequently approximated by

$$w_N(x) = \sum_{k=-N}^{N} \hat{w}(k) e^{ik\pi x}. \tag{2.15}$$

Finally $f(x)$ is estimated as

$$f_N(x) = w_N(x) + \sum_{n=0}^{Q} A_n V_n(x).$$

2.2 Numerical Examples

We provide a couple of numerical examples for which we are given the first $2N+1$ Fourier coefficients of the "spliced" function in each subinterval, $[-1,0]$ and $[0,1]$.

Example 2.1

$$f(x) = e^{i1.4\pi x}, \qquad x \in [-1,1]. \tag{2.16}$$

The Gibbs phenomenon in the approximation of Example 2.1, prevalent in figure 1(a) particularly at $x = 0, \pm 1$, is a result of "splicing" $f(x)$ into two pieces. The smooth approximation (figure 1(b)) is obtained by employing the Bernoulli method with $Q = 4$ in each subinterval.

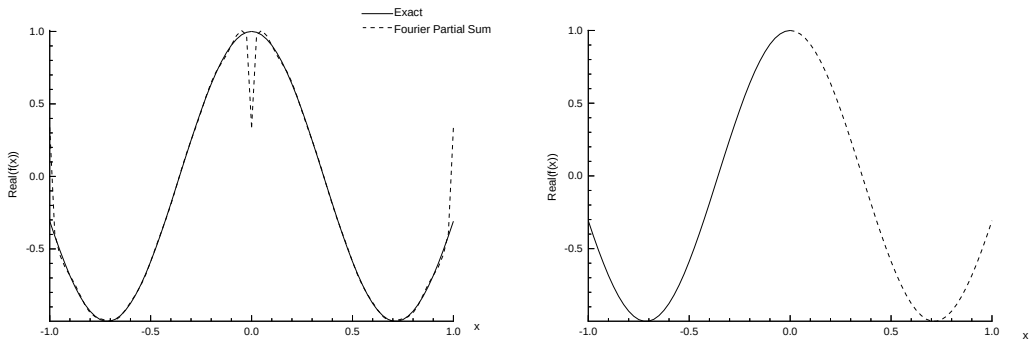

Figure 1: Approximation of $f(x) = e^{i1.4\pi x}$ by the (a) Fourier partial sum for $N = 32$ and (b) Bernoulli method for $N = 8$ and $Q = 4$.

Figure 2 shows the logarithmic pointwise errors using the Bernoulli method for $N = 16$ with respect to Q, the order of Bernoulli polynomial.

Example 2.2

$$f(x) = \frac{e^{i3.8\pi x}}{4} + \frac{e^{i15.8\pi x}}{16}, \qquad x \in [-1,1]. \tag{2.17}$$

This more complicated example provides insight for the resolution properties of the Bernoulli method and the Gegenbauer method (Section 3). Figure 3 shows the logarithmic

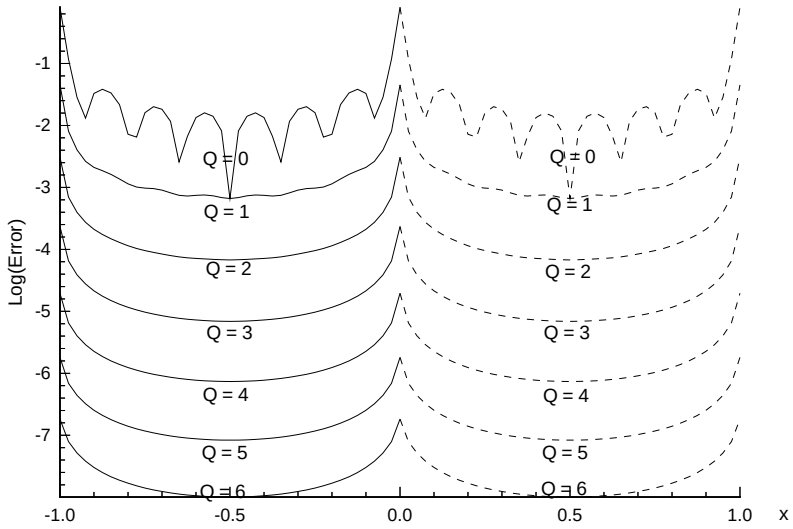

Figure 2: Bernoulli Method pointwise convergence for Example 2.1 with respect to Q for $N = 16$.

pointwise errors for $N = 128$. Notice that the accuracy decreases when $Q = 6$. Unfortunately, this is true even if the jump coefficients are known explicitly and system (2.14), which may be ill-conditioned, need not be solved.

3 The Gegenbauer Method

3.1 Review of the Gegenbauer Method

The Gegenbauer method was developed in [12] where it was shown that knowledge of the Fourier coefficients of a continuous but non-periodic function provide enough information to recover this function with spectral accuracy, even at the boundaries.

The Gegenbauer expansion for such a function $f(x)$, $x \in [-1, 1]$ is defined as

$$f(x) = \sum_{l=0}^{\infty} \hat{f}_l^{\lambda} C_l^{\lambda}(x), \tag{3.1}$$

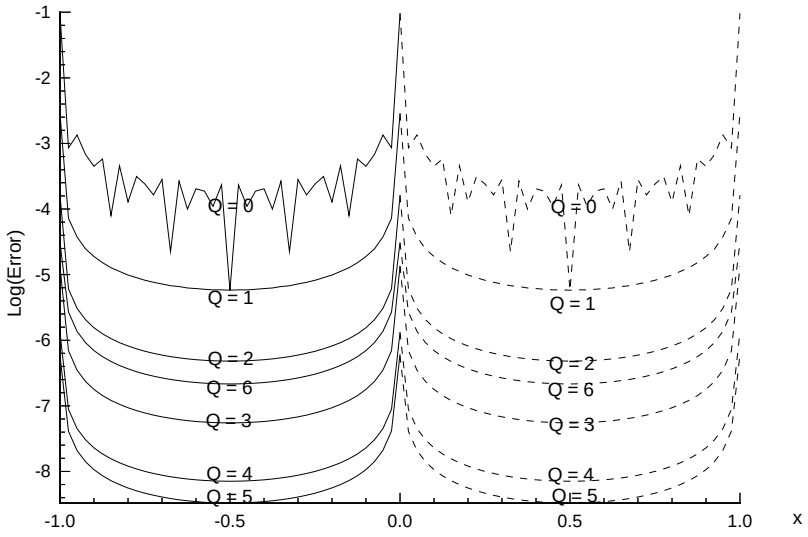

Figure 3: Bernoulli Method pointwise convergence for Example 2.2 with respect to Q for $N = 128$.

where $\hat{f}_l^\lambda$ are the Gegenbauer coefficients based on the function $f(x)$ defined by

$$\hat{f}_l^\lambda = \frac{1}{h_l^\lambda} \int_{-1}^{1} (1-x^2)^{\lambda-\frac{1}{2}} C_l^\lambda(x) f(x) dx.$$

The Gegenbauer method is described in these two steps:

1. An exponentially accurate approximation to the first $m \sim N$ coefficients $\hat{f}_l^\lambda$ in the Gegenbauer expansion is obtained from the first $2N+1$ Fourier coefficients of $f(x)$. The parameter λ must grow with the number of Fourier modes for exponential convergence, but it is possible to yield algebraic convergence for a fixed λ. These approximate coefficients, denoted $\hat{g}_l^\lambda$, are defined as

$$\hat{g}_l^\lambda = \frac{1}{h_l^\lambda} \int_{-1}^{1} (1-x^2)^{\lambda-\frac{1}{2}} C_l^\lambda(x) f_N(x) dx, \tag{3.2}$$

   where $f_N(x) = \sum_{k=-N}^{N} \hat{f}(k) e^{ik\pi x}$.

2. The coefficients $\hat{g}_l^\lambda$ are now used in the partial Gegenbauer sum to approximate the

original function $f(x)$ as

$$f_m^\lambda(x) = \sum_{l=0}^{m} \hat{g}_l^\lambda C_l^\lambda(x). \tag{3.3}$$

There are two errors incurred here. The error between the exact Gegenbauer coefficients and the ones obtained from the Fourier coefficients is called the *truncation error*, and the error between the Gegenbauer expansion of $f(x)$ and its approximated partial sum is known as the *regularization error*. In [12], both approximations are proven to converge exponentially in the maximum norm. Since this paper also proves the exponential convergence of the Gegenbauer method in two dimensions, we have included an appendix that contains some important properties of the Gegenbauer polynomials.

3.2 Numerical Results of the Gegenbauer Method

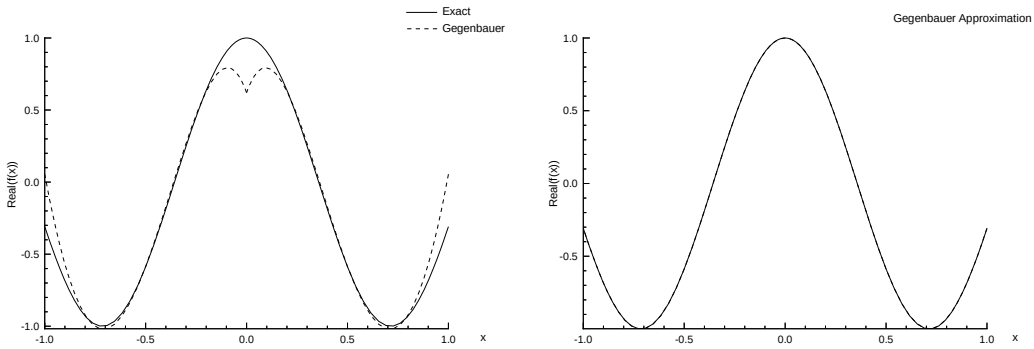

Figure 4: Approximation of $f(x) = e^{i1.4\pi x}$ using the Gegenbauer method for (a) $N = 8$ and (b) $N = 32$.

For simplicity we fix $\lambda = 5$, although this is not optimal, and choose m, the number of Gegenbauer polynomials, to minimize the error at the boundaries. We assume knowledge of the Fourier coefficients in the sub-domains $[-1, 0]$ and $[0, 1]$ for both of the examples in Section 2.2.

Figure 4(a) clearly shows that the Gegenbauer approximation of Example 2.1 is not well enough resolved for $N = 8$. as is particularly evident at $x = \pm 1$ and $x = 0$. However,

smooth results are obtained in figure 4(b) employing the Gegenbauer method with $N = 32$.

The logarithmic values of the pointwise errors for Example 2.1 when $N = 8, 16, 32, 64$ is shown in figure 5, and the errors for Example 2.2 when $N = 32, 64, 128$ and 256 are in figure 6. Clearly we cannot resolve the boundaries in Example 2.2 when $N \leq 128$.

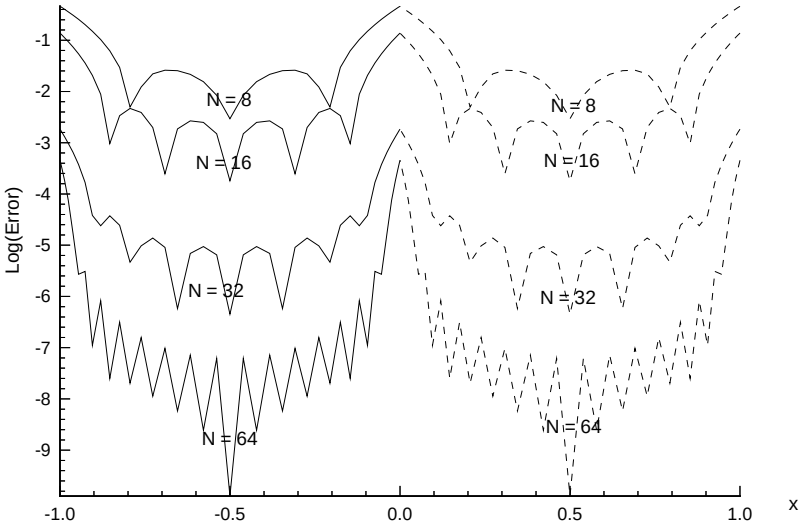

Figure 5: Pointwise error convergence using the Gegenbauer Method for Example 2.1 with respect to N for $N = 8, 16, 32, 64$.

3.3 The Gegenbauer-Bernoulli Method

The Gegenbauer-Bernoulli Method developed here is a hybrid method combining the Gegenbauer and Bernoulli methods. As observed in [15], the Gegenbauer method suffers from round-off error for a large number of Gegenbauer polynomials, m, particularly when using the pseudo-spectral Fourier coefficients. We would like to counteract this error by resolving the function with m as small as possible. The hybrid method suggested here involves first pre-processing the truncated Fourier sum with the Bernoulli method, and then determining the Gegenbauer coefficients, and finally expanding these pre-processed Gegenbauer coefficients in terms of the Gegenbauer polynomials. Again we note that the method is applied

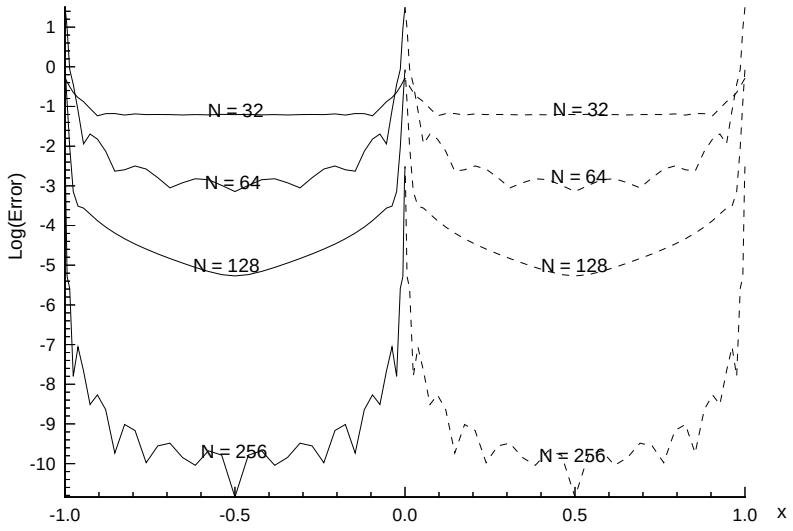

Figure 6: Pointwise error convergence applying the Gegenbauer Method to Example 2.2 for $N = 32, 64, 128$ and 256.

to different sub-domains and then the results are "glued" together to approximate $f(x)$ in the general domain.

We rewrite $f(x)$ as

$$f(x) = w(x) + \sum_{n=0}^{Q} A_n V_n(x),$$

and approximate $f(x)$ by $f_N(x)$ in the following way:

1. Solve the system of equations for the jump coefficients A_n,

$$\sum_{n=0}^{Q} A_n(\hat{V_n})_k = \hat{f}(k).$$

2. Determine the coefficients $\hat{w}(k)$,

$$\hat{w}(k) = \hat{f}(k) - \sum_{n=0}^{Q} A_n(\hat{V_n})_k.$$

3. Evaluate the Fourier partial sum for $w(x)$,

$$w_N(x) = \sum_{k=-N}^{N} \hat{w}(k) e^{ik\pi x}.$$

4. Compute $f_N(x)$ as

$$f_N(x) = w_N(x) + \sum_{n=0}^{Q} A_n V_n(x).$$

5. Using the Bernoulli method approximation as the pre-processed Fourier sum, we can approximate the Gegenbauer coefficients by

$$\hat{g}_l^\lambda = \frac{1}{h_l^\lambda} \int_{-1}^{1} (1-x^2)^{\lambda-\frac{1}{2}} C_l^\lambda(x) f_N(x) dx.$$

6. The Gegenbauer coefficients obtained are then used in the partial Gegenbauer sum to approximate the original function $f(x)$.

$$f_m^\lambda(x) = \sum_{l=0}^{m} \hat{g}_l^\lambda C_l^\lambda(x).$$

3.4 Numerical Results of the Gegenbauer-Bernoulli method

Figure 7(a) shows the convergence rate of the Gegenbauer-Bernoulli method with respect to Q, the highest order of Bernoulli polynomials used in the approximation, for Example 2.1, while figure 7(b) shows the convergence rate for Example 2.2. A clear improvement is made by the Bernoulli pre-processing, although the function is still not resolved for $N \leq 128$. In 7(b), we note that the approximation does not improve after $Q = 5$.

Table 1 compares the numerical errors of Example 2.1 for the Bernoulli and Gegenbauer-Bernoulli method with respect to Q and N. (The Gegenbauer method is equivalent to the Gegenbauer-Bernoulli method with $Q = 0$.) To obtain an accuracy of the order 10^{-2}, we can choose either the Bernoulli method with $N = 8$ and $Q = 2$ or the Gegenbauer-Bernoulli method with $N = 16$ and $Q = 1$. The Bernoulli method yields an accuracy of 10^{-4} with $N = 8$, while the Gegenbauer-Bernoulli method requires $N = 16$. To obtain an accuracy of 10^{-8}, both methods require $N \geq 32$, while only the Gegenbauer-Bernoulli method is able to produce an accuracy better than 10^{-8} and requires $N = 64$. Thus we see that if less accuracy is required, the Bernoulli method works better with fewer points, but for greater accuracy, we need the Gegenbauer-Bernoulli method with more points.

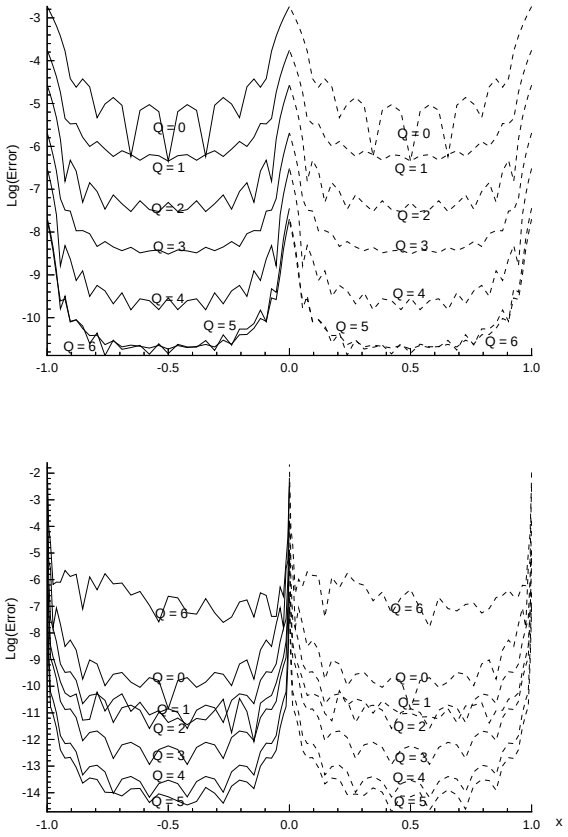

Figure 7: Logarithmic pointwise errors employing the Gegenbauer-Bernoulli method for (a) $f(x) = e^{i1.4\pi x}$ with $N = 32$ and (b) $f(x) = \frac{e^{i3.8\pi x}}{4} + \frac{e^{i15.8\pi x}}{16}$ with $N = 256$. In both cases, $\lambda = 5$ and $Q \leq 6$.

Table 2 corresponds to the convergence rate using the Bernoulli and Gegenbauer-Bernoulli methods for Example 2.2. An accuracy of 10^{-4} can be obtained by the Bernoulli method for $N \geq 32$ and for $N \geq 64$, the Bernoulli method achieves an accuracy of 10^{-7}. Clearly the Gegenbauer-Bernoulli method does not converge for $N \leq 128$.

This table does not reflect the results for larger N, but experiments indicate that the Gegenbauer method will suffer from round-off error for very large N due to the regulartization error (see Section 3).

Figures 8 and 9 correspond to table 2. Figure 8 compares the maximum error convergence rate of the Bernoulli method and the Gegenbauer-Bernoulli method with respect to Q and

Table 1: Absolute maximum error for $f(x) = e^{i1.4\pi x}$, $x \in [-1, 1]$, using the Bernoulli method (B) and the Gegenbauer-Bernoulli method (GB) with $N = 16$, 32, and 64

Q	B—8	GB—8	B—16	GB—16	B—32	GB—32	B—64	GB—64
0	.81	.46	.81	.14	.81	1.8E-3	.81	4.5E-4
1	9.9E-2	4.2E-2	4.5E-2	1.2E-2	2.2E-3	1.8E-4	2.1E-2	5.1E-6
2	1.6E-2	6.3E-2	3.1E-3	4.7E-3	6.9E-4	2.7E-5	1.7E-4	9.8E-8
3	3.2E-3	4.9E-2	2.3E-4	4.5E-4	2.4E-5	2.1E-6	2.7E-6	2.0E-9
4	8.3E-4	5.3E-2	2.0E-5	1.5E-4	8.7E-7	3.0E-7	5.1E-8	5.2E-11
5	3.0E-4	5.2E-2	1.8E-6	5.4E-5	3.3E-8	2.0E-8	2.0E-7	1.9E-10
6	1.8E-4	5.2E-2	1.8E-7	6.1E-5	3.8E-8	3.6E-8	5.9E-6	6.6E-9

Table 2: Maximum error for Example 2.2 in the full domain $[-1, 1]$ using the Bernoulli method (B) and the Gegenbauer-Bernoulli method with $\lambda = 5$(GB) where $N = 64$, 128, and 256.

Q	B—32	GB—32	B—64	GB—64	B—128	GB—128	B—256	GB—256
0	9.7E-2	.52	9.7E-2	31.7	9.7E-2	.85	9.7E-2	3.1E-3
1	1.3E-2	.52	5.9E-3	31.7	2.9E-3	.85	1.4e-3	1.6E-4
2	3.9E-3	.52	7.3E-4	31.7	1.6E-4	.85	3.9e-5	6.4E-6
3	1.8E-3	.52	1.3E-4	31.7	1.3E-5	.85	1.5e-6	3.4E-7
4	9.6E-4	.52	2.9E-5	31.7	1.3E-6	.85	4.1e-7	9.6E-8
5	5.7E-4	.52	6.6E-6	31.7	6.1E-7	.85	4.9e-5	1.3E-5
6	3.6E-4	.52	9.0E-7	31.7	3.1E-5	.85	5.1e-3	1.1E-2

N, while figure 9 depicts the maximum errors convergence with respect to Q for each method when $N = 128$ and $N = 256$.

3.5 Operational Order Comparison

Implementation costs of the Bernoulli and the Gegenbauer methods indicate that while the Gegenbauer method is quite expensive, the Bernoulli method is trivial to compute.

Basically, the cost of the Bernoulli method is only in computing the system (2.14), which is a $Q \times Q$ matrix solving the $2N + 1$ equations for $\hat{w}(k)$ in equation (2.12), and then finding the Fourier partial sum for $w_N(x)$ in equation (2.15). This can be solved using the Fast Fourier Transform (FFT) algorithm.

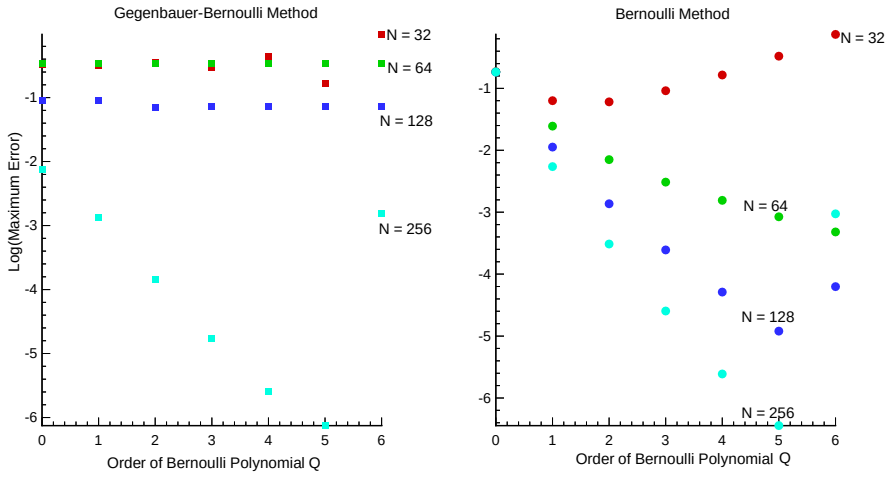

Figure 8: Logarithmic maximum errors for Example 2.2 and $x \in [-1, 1]$ employing (a) Gegenbauer-Bernoulli with $\lambda = 5$ and (b) Bernoulli method for $N = 32, 64, 128, 256$ and $Q \leq 6$.

On the other hand, the Gegenbauer method is rather expensive, where the big expense is incurred when computing the Gegenbauer coefficients $\hat{g}_l^\lambda$,

$$\hat{g}_l^\lambda = \frac{1}{h_l^\lambda} \int_{-1}^{1} (1 - x^2)^{\lambda - \frac{1}{2}} C_l^\lambda(x) f_N(x) dx.$$

This involves solving an integral for *each* $0 \leq l \leq m$. Gauss-Labotto quadrature is necessary in solving this integral with sufficient accuracy [6]. The Gauss-Labotto formula is given by

$$\int_{-1}^{1} \frac{f(x)}{\sqrt{1 - x^2}} dx = \frac{\pi}{N} \sum_{j=0}^{N} \frac{f(x_j)}{c_j}$$

where $c_0 = c_N = 2$ and $c_j = 1$ for $j = 1, \ldots, N - 1$ and $x_j = \cos(\frac{\pi j}{N})$. We therefore solve $f_N(x)$ in equation (3.2) on the Gauss-Labotto points, and hence cannot employ the FFT.

The other minor expenses are in solving the sum (3.3)

$$f_m^\lambda(x) = \sum_{l=0}^{m} \hat{g}_l^\lambda C_l^\lambda(x),$$

and in computing the Gegenbauer polynomials $C_l^\lambda(x)$, which satisfy the recursive relationship,

$$(l + 1) C_{l+1}^\lambda(x) = 2(\lambda + l)(x) C_l^\lambda(x) - (2\lambda + l - 1) C_{l-1}^\lambda(x).$$

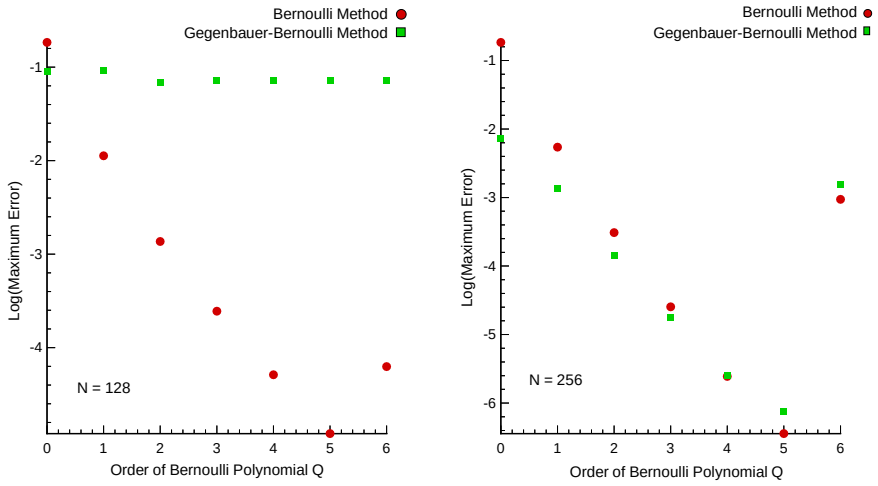

Figure 9: Logarithmic maximum errors for Example 2.2, $x \in [-1, 1]$, using the Gegenbauer-Bernoulli with $\lambda = 5$ (circles) and Bernoulli method (squares) for (a) $N = 128$ and (b) $N = 256$ for $Q \leq 6$.

4 Discussion of the results for the one-dimensional "spliced" functions

Our studies, thus far, point to some interesting observations about approximating one-dimensional smooth functions that have been "spliced" into non-intersecting sub-domains.

It appears that the Bernoulli method yields more satisfactory results than the Gegenbauer-Bernoulli method for these particular examples. The Bernoulli method is easier to employ and significantly less costly than the Gegenbauer method. It also requires far fewer points to resolve the function. It remains to be seen how much further the Bernoulli method may still be improved, particularly for solving the system (2.14). In [8], the $Q \times Q$ system is solved for $k = N, N-1, \ldots, N-Q+1$ and for an over-determined $2Q \times Q$ system for $k = N, N-1, \ldots, N-2Q+1$. Much work was done in order to avoid the inevitable ill-conditioning of the matrix in (2.14). It has been suggested in [15] (in regards to another Gegenbauer-type hybrid approximation method) to use the Gegenbauer method to obtain the jump coefficients. This will eliminate the ill-conditioning of the system (2.14)

at the cost of requiring more points to resolve the function. In our examples we used $k = N, \frac{N}{2}, \frac{2N}{3}, \ldots, \frac{(Q-1)N}{Q}$ and obtained very good results.

Although we have discussed only simple problems, there is already a noticeable trend of deteriorating accuracy in the Bernoulli method when both the number of Fourier coefficients and the order of Bernoulli polynomials is increased. (See tables 1 and 2.) It is also apparent in both Examples 2.1 and 2.2 that this deterioration happens even when the *exact* jump coefficients are explicitly known. Thus it is not only the ill-conditioning of system (2.14) that causes inaccuracy. This may severely impact the Bernoulli method's effectiveness when large N and Q are needed theoretically to resolve the function, and must be investigated further.

Another consideration is that we have assumed knowledge of the jump discontinuity locations. The Bernoulli method has been developed to locate jump discontinuities, but not without affecting the accuracy of the approximation [7]. This is still being investigated for the Gegenbauer method, although it has been shown in [13] that the method works well as we *approach* the discontinuity. This is promising in the sense that even if only an approximate location of the discontinuity is known, accurate (even spectral) results using the Gegenbauer method can still be obtained.

The Gegenbauer method has another promising feature, which is the tolerance of perturbations in input data. Numerical experiments suggest that the Gegenbauer method still retains high accuracy when recovering functions from noisy data. Unfortunately the Bernoulli method does not yield such good results.

The biggest challenge facing the Gegenbauer and hybrid Gegenbauer-Bernoulli methods is the cost and severe resolution restrictions. By "splicing" the picture into different non-intersecting sub-domains, we can data-parallelize and solve for each "spliced" function with smaller N, and thus the Gegenbauer method becomes quite reasonable to compute. This parallelization will be the topic of a future paper.

Still, of course, is the imminent discussion of higher dimensions. It is clear that the

Bernoulli method will be more difficult in higher dimensions since we are no longer computing the jump coefficients of single points, but rather of functions. These are only *algebraic approximations* of the Fourier coefficients, so the one-dimensional theory in [7] does not apply.

More specifically, given $\hat{f}_{k,l}$, the two-dimensional Fourier coefficients, we write the double partial Fourier sum $f_N(x,y)$ as

$$f_N(x,y) = \sum_{k=-N}^{N} \sum_{l=-N}^{N} \hat{f}_{k,l} e^{ik\pi x} e^{il\pi y}.$$

Upon defining

$$\hat{a}_k = \sum_{l=-N}^{N} \hat{f}_{k,l} e^{il\pi y},$$

we can write $f_N(x,y)$ as

$$f_N(x,y) = \sum_{k=-N}^{N} \hat{a}_k e^{ik\pi x}.$$

Unfortunately since $f(x,y)$ is not periodic in y, these $\hat{a}_k$ serve as a poor approximation to the analytic one-dimensional Fourier coefficients, as they already have the Gibbs phenomenon built in. Thus we cannot apply integration by parts to $\hat{a}_k$ which is critical to the Bernoulli method (section 2).

However, the Gegenbauer method extends quite easily into two dimensions, as is shown in the following two sections. The numerical results are shown in section 7.

5 Truncation Error for two-dimensional functions

We now discuss the "spliced" function in two dimensions. We assume that for a smooth function $f(x,y)$ defined on $-1 \le x \le 1$ and $-1 \le y \le 1$, we have the Fourier coefficients for the quadrants: $-1 \le x \le 0, -1 \le y \le 0$; $-1 \le x \le 0, 0 \le y \le 1$; $0 \le x \le 1, -1 \le y \le 0$; and $0 \le x \le 1, 0 \le y \le 1$. We prove our results in the following two sections for the interval $x \in [-1,1]$ and $y \in [-1,1]$ and note that the proofs are valid for any general interval.

Assume that $f(x,y)$ is an analytic but non-periodic function defined for $x \in [-1,1]$ and $y \in [-1,1]$. Also assume that the Fourier coefficients of $f(x,y)$,

$$\hat{a}_{k,l} = \frac{1}{4}\int_{-1}^{1}\int_{-1}^{1} f(x,y)e^{-ik\pi x}e^{-il\pi y}dxdy, \tag{5.1}$$

are given.

The Fourier partial sum is therefore also known,

$$f_N(x,y) = \sum_{k=-N}^{N}\sum_{l=-N}^{N} \hat{a}_{k,l}e^{ik\pi x}e^{il\pi y}. \tag{5.2}$$

Recall that $f(x,y)$ must be periodic in both x or y to ensure rapid convergence of the partial sum $f_N(x,y)$.

The Fourier coefficients $\hat{a}_{k,l}$ satisfy

Assumption 5.1 $|\hat{a}_{k,l}| \leq A \qquad$ *independent of* k, l,

where A is independent of k and l. This is true for any $f(x,y) \in L_1$.

The goal is to recover $f(x,y)$ for $x \in [-1,1]$ and $y \in [-1,1]$ using the two-dimensional Gegenbauer coefficients and Gegenbauer partial sum. We note here that any piecewise analytic function $f(x,y)$ can be recovered in an interval for which the function is analytic, as our examples will show in Section 7. For simplicity of notations, we will prove exponential accuracy only for functions that are analytic for $x \in [-1,1]$ and $y \in [-1,1]$, although the proofs can be easily extended for any general interval.

Definition 5.1 *The two-dimensional Gegenbauer partial sum approximating* $f(x,y)$ *is defined by*

$$f_{m_1,m_2}(x,y) = \sum_{\mu_1=0}^{m_1}\sum_{\mu_2=0}^{m_2} \hat{f}^{\lambda_1,\lambda_2}_{\mu_1,\mu_2} C^{\lambda_1}_{\mu_1}(x)C^{\lambda_2}_{\mu_2}(y), \tag{5.3}$$

where the first $m_1 m_2$ coefficients $\hat{f}^{\lambda_1,\lambda_2}_{\mu_1,\mu_2}$, based upon the Gegenbauer polynomials $C^{\lambda}_{\mu}(\xi)$ with weight function $(1-x^2)^{\lambda_1-\frac{1}{2}}(1-y^2)^{\lambda_2-\frac{1}{2}}$ for any constants $\lambda_1, \lambda_2 \geq 0$, are defined by

$$\hat{f}^{\lambda_1,\lambda_2}_{\mu_1,\mu_2} = \frac{1}{h^{\lambda_1}_{\mu_1}h^{\lambda_2}_{\mu_2}}\int_{-1}^{1}\int_{-1}^{1}(1-x^2)^{\lambda_1-\frac{1}{2}}(1-y^2)^{\lambda_2-\frac{1}{2}}C^{\lambda_1}_{\mu_1}(x)C^{\lambda_2}_{\mu_2}(y)f(x,y)dxdy, \tag{5.4}$$

where $0 \le \mu_1 \le m_1, 0 \le \mu_2 \le m_2$, $x \in [-1,1]$, $y \in [-1,1]$, and the Gegenbauer polynomials $C_\mu^\lambda(\xi)$ are defined by (A.1).

The Gegenbauer expansion of $f(x,y)$ for $-1 \le x \le 1$ and $-1 \le y \le 1$ is given as

$$f(x,y) = \sum_{\mu_1=0}^{\infty} \sum_{\mu_2=0}^{\infty} \hat{f}_{\mu_1,\mu_2}^{\lambda_1,\lambda_2} C_{\mu_1}^{\lambda_1}(x) C_{\mu_2}^{\lambda_2}(y) \tag{5.5}$$

where the coefficients $\hat{f}_{\mu_1,\mu_2}^{\lambda_1,\lambda_2}$ are defined in (5.4).

Of course, $\hat{f}_{\mu_1,\mu_2}^{\lambda_1,\lambda_2}$ is not known, but rather an approximation based on the Fourier partial sum, $f_N(x,y)$, as defined in 5.2. This approximation, $\hat{g}_{\mu_1,\mu_2}^{\lambda_1,\lambda_2}$, is defined by

$$\hat{g}_{\mu_1,\mu_2}^{\lambda_1,\lambda_2} = \frac{1}{h_{\mu_1}^{\lambda_1} h_{\mu_2}^{\lambda_2}} \int_{-1}^{1} \int_{-1}^{1} (1-x^2)^{\lambda_1 - \frac{1}{2}} (1-y^2)^{\lambda_2 - \frac{1}{2}} C_{\mu_1}^{\lambda_1}(x) C_{\mu_2}^{\lambda_2}(y) f_N(x,y) dx dy. \tag{5.6}$$

Definition 5.2 *The truncation error is defined by*

$$TE(\lambda_1, \lambda_2, m_1, m_2, N) = \max_{\substack{-1 \le x \le 1 \\ -1 \le y \le 1}} |\sum_{\mu_1=0}^{m_1} \sum_{\mu_2=0}^{m_2} (\hat{f}_{\mu_1,\mu_2}^{\lambda_1,\lambda_2} - \hat{g}_{\mu_1,\mu_2}^{\lambda_1,\lambda_2}) C_{\mu_1}^{\lambda_1}(x) C_{\mu_2}^{\lambda_2}(y)|, \tag{5.7}$$

where $\hat{f}_{\mu_1,\mu_2}^{\lambda_1,\lambda_2}$ and $\hat{g}_{\mu_1,\mu_2}^{\lambda_1,\lambda_2}$ are defined in equations (5.4) and (5.6). The truncation error describes how well the coefficients $\hat{g}_{\mu_1,\mu_2}^{\lambda_1,\lambda_2}$ approximate the actual Gegenbauer coefficients $\hat{f}_{\mu_1,\mu_2}^{\lambda_1,\lambda_2}$.

In the next two theorems we bound the truncation error in terms of N, m_1, m_2, λ_1 and λ_2. N corresponds to the number of given Fourier coefficients $\hat{a}_{k,l}$, while m_1 and m_2 are the number of Gegenbauer polynomials given in the double sum Gegenbauer expansion. λ_1 and λ_2 are the orders of the Gegenbauer polynomials. Since the theorems are a direct extension of the one-dimensional case, we will simply sketch the proofs and refer readers to [12] for details.

Theorem 5.1 *If $f(x,y)$ is an L_2 function for $-1 \le x \le 1$ and $-1 \le y \le 1$, then there exists a constant A, independent of λ_1, λ_2, m_1, m_2, and N, such that the truncation error defined in (5.7) satisfies the following estimate:*

$$TE(\lambda_1, \lambda_2, m_1, m_2, N) \le A \frac{(m_1+\lambda_1)(m_2+\lambda_2)\Gamma(m_1+2\lambda_1)\Gamma(m_2+2\lambda_2)\Gamma(\lambda_1)\Gamma(\lambda_2)}{(m_1-1)!(m_2-1)!\Gamma(2\lambda_1)\Gamma(2\lambda_2)} \left(\frac{2}{\pi N}\right)^{\lambda_1+\lambda_2-2} \tag{5.8}$$

Proof

Consider a special function $f(x,y) = e^{in_1\pi x}e^{in_2\pi y}$ with $|n_1|, |n_2| > N$. In this particular case, $f_N(x,y) = 0$ causing

$$\begin{aligned}(\hat{f}^{\lambda_1,\lambda_2}_{\mu_1,\mu_2} - \hat{g}^{\lambda_1,\lambda_2}_{\mu_1,\mu_2})C^{\lambda_1}_{\mu_1}(1)C^{\lambda_2}_{\mu_2}(1) &= \frac{C^{\lambda_1}_{m_1}(1)}{h^{\lambda_1}_{\mu_1}}\int_{-1}^{1}(1-x^2)^{\lambda_1-\frac{1}{2}}e^{in_1\pi x}C^{\lambda_1}_{\mu_1}(x)dx \\ &\times \frac{C^{\lambda_2}_{m_2}(1)}{h^{\lambda_2}_{\mu_2}}\int_{-1}^{1}(1-y^2)^{\lambda_2-\frac{1}{2}}e^{in_2\pi y}C^{\lambda_2}_{\mu_2}(y)dy. \qquad (5.9)\end{aligned}$$

The explicit expression given in [3] for the one-dimensional integral in equation (5.9) is

$$\frac{1}{h^{\lambda}_{\mu}}\int_{-1}^{1}(1-\xi^2)^{\lambda-\frac{1}{2}}e^{in\pi\xi}C^{\lambda}_{\mu}(\xi)dx = \Gamma(\lambda)\left(\frac{2}{\pi n}\right)^{\lambda} i^{\mu}(\mu+\lambda)J_{\mu+\lambda}(\pi n), \qquad (5.10)$$

where $J_\nu(\xi)$ is the Bessel function. Since $|J_\nu(\xi)| \leq 1$ for all ξ and $\nu \geq 0$, each integral in equation (5.9) can be estimated by

$$\frac{C^{\lambda}_{\mu}(1)}{h^{\lambda}_{\mu}}\int_{-1}^{1}(1-\xi^2)^{\lambda-\frac{1}{2}}e^{in\pi\xi}C^{\lambda}_{\mu}(\xi)d\xi \leq \frac{(m+\lambda)\Gamma(m+2\lambda)\Gamma(\lambda)}{m!\Gamma(2\lambda)}\left(\frac{2}{\pi|n|}\right)^{\lambda}, \qquad (5.11)$$

where $0 \leq \mu \leq m$. The estimate (5.11) is obtained first by using the inequality (A.4) on $C^{\lambda}_{\mu}(\xi)$ and then noting that $\frac{(\mu+\lambda)\Gamma(\mu+2\lambda)}{\mu!}$ is an increasing function of μ.

Applying the results of the estimate (5.11) to the two-dimensional special case function $f(x,y) = e^{in_1\pi x}e^{in_2\pi y}$ with $|n_1|, |n_2| > N$, we see that

$$\begin{aligned}|(\hat{f}^{\lambda_1,\lambda_2}_{\mu_1,\mu_2} - \hat{g}^{\lambda_1,\lambda_2}_{\mu_1,\mu_2})C^{\lambda_1}_{\mu_1}(1)C^{\lambda_2}_{\mu_2}(1)| &\leq \frac{(m_1+\lambda_1)\Gamma(m_1+2\lambda_1)\Gamma(\lambda_1)}{m_1!\Gamma(2\lambda_1)}\left(\frac{2}{\pi|n_1|}\right)^{\lambda_1} \\ &\times \frac{(m_2+\lambda_2)\Gamma(m_2+2\lambda_2)\Gamma(\lambda_2)}{m_2!\Gamma(2\lambda_2)}\left(\frac{2}{\pi|n_2|}\right)^{\lambda_2}. \qquad (5.12)\end{aligned}$$

Returning to the general function $f(x,y)$, we have

$$f(x,y) - f_N(x,y) = \sum_{|k|>N}\sum_{|l|>N}\hat{a}_{k,l}e^{ik\pi x}e^{il\pi y}, \qquad (5.13)$$

and thus

$$TE(\lambda_1,\lambda_2,m_1,m_2,N) \leq \max_{\substack{0\leq\mu_1\leq m_1\\ 0\leq\mu_2\leq m_2}}\ \max_{\substack{-1\leq x\leq 1\\ -1\leq y\leq 1}} |(\hat{f}^{\lambda_1,\lambda_2}_{\mu_1,\mu_2} - \hat{g}^{\lambda_1,\lambda_2}_{\mu_1,\mu_2})C^{\lambda_1}_{\mu_1}(x)C^{\lambda_2}_{\mu_2}(y)|$$

$$
\begin{aligned}
&\leq m_1 m_2 \max_{\substack{0 \leq \mu_1 \leq m_1 \\ 0 \leq \mu_2 \leq m_2}} |(\hat{f}^{\lambda_1,\lambda_2}_{\mu_1,\mu_2} - \hat{g}^{\lambda_1,\lambda_2}_{\mu_1,\mu_2}) C^{\lambda_1}_{m_1}(1) C^{\lambda_2}_{m_2}(1)| \\
&\leq A \frac{(m_1+\lambda_1)\Gamma(m_1+2\lambda_1)\Gamma(\lambda_1)}{(m_1-1)!\Gamma(2\lambda_1)} \\
&\times \frac{(m_2+\lambda_2)\Gamma(m_2+2\lambda_2)\Gamma(\lambda_2)}{(m_2-1)!\Gamma(2\lambda_2)} \left(\frac{2}{\pi N}\right)^{\lambda_1+\lambda_2-2},
\end{aligned}
$$

where in the second step we used the fact that $C^{\lambda}_{\mu}(\xi) \leq C^{\lambda}_{\mu}(1)$ for all $-1 \leq \xi \leq 1$ [2], and in the third step we applied the estimates (5.12) and (5.13).

□

The results of Theorem 5.1 leads directly to the exponential convergence of the truncation error.

Theorem 5.2 *Assume that $f(x,y)$ is an L_2 function with known Fourier coefficients $\hat{a}_{k,l}$ where $-N \leq k,l \leq N$. Let $\lambda_1 = \alpha_1 N$, $\lambda_2 = \alpha_2 N$, $m_1 = \beta_1 N$, and $m_2 = \beta_2 N$ where $\alpha_1, \alpha_2, \beta_1, \beta_2$ are positive constants. Then the truncation error defined in (5.7) satisfies*

$$
TE(\lambda_1, \lambda_2, m_1, m_2, N) \leq A N^4 q_1^N q_2^N, \tag{5.14}
$$

where

$$
\begin{aligned}
q_1 &= \frac{(\beta_1 + 2\alpha_1)^{\beta_1+2\alpha_1}}{(2\pi e)^{\alpha_1} \alpha_1^{\alpha_1} \beta_1^{\beta_1}} \\
q_2 &= \frac{(\beta_2 + 2\alpha_2)^{\beta_2+2\alpha_2}}{(2\pi e)^{\alpha_2} \alpha_2^{\alpha_2} \beta_2^{\beta_2}}
\end{aligned} \tag{5.15}
$$

In particular, if $\alpha_1 = \beta_1 = \alpha_2 = \beta_2 = \frac{2\pi}{27} \approx \frac{1}{4}$, then

$$
q_1 = q_2 = e^{-\frac{2\pi}{27}} \approx 0.8 < 1 \tag{5.16}
$$

Proof

The proof is attained simply by applying Stirling formula (A.7) and some straight-forward algebra to the estimate (5.8).

□

We should note that the choices here for the parameters $\alpha_1, \alpha_2, \beta_1, \beta_2$ are made to simplify the proof of exponential convergence of the truncation error, but they are not optimal.

6 Regularization Error

As shown in [12] for the one dimensional case, the second part of the Gegenbauer approximation error, called the regularization error, is caused by the Gegenbauer partial sum approximation to the analytic function $f(x,y)$. The regularization error is estimated in the maximum norm.

Definition 6.1 *The regularization error is defined by*

$$RE(\lambda_1,\lambda_2,m_1,m_2) = \max_{\substack{-1\le x\le 1\\ -1\le y\le 1}} |f(x,y) - \sum_{\mu_1=0}^{m_1}\sum_{\mu_2=0}^{m_2} \hat{f}^{\lambda_1,\lambda_2}_{\mu_1,\mu_2} C^{\lambda_1}_{\mu_1}(x) C^{\lambda_2}_{\mu_2}(y)|. \tag{6.1}$$

The exponential convergence of the regularization error has already been proved for the one dimensional case in [12], and here we just extend the results into two-dimensions. We start by stating the following assumption and lemma.

Assumption 6.1 *Let $f(x,y)$ be an analytic function for $-1\le x\le 1$ and $-1\le y\le 1$. Then there exists constants $\rho_1\ge 1$, $\rho_2\ge 1$, $C(\rho_1,\rho_2)$ such that, for every $\mu_1,\mu_2\ge 0$,*

$$\max_{\substack{-1\le x\le 1\\ -1\le y\le 1}} \left|\frac{d^{\mu_1+\mu_2} f}{dx^{\mu_1} dy^{\mu_2}}(x,y)\right| \le C(\rho_1,\rho_2)\frac{\mu_1!}{\rho_1^{\mu_1}}\frac{\mu_2!}{\rho_2^{\mu_2}}. \tag{6.2}$$

This is a standard assumption for analytic functions [14].

Lemma 6.1 *The Gegenbauer coefficient $\hat{f}^{\lambda_1,\lambda_2}_{\mu_1,\mu_2}$ as defined in equation (5.4) of an analytic function satisfying Assumption 6.1 is bounded by*

$$|\hat{f}^{\lambda_1,\lambda_2}_{\mu_1,\mu_2}| \le AC(\rho_1,\rho_2)\frac{\Gamma(\lambda_1+\frac{1}{2})\Gamma(\mu_1+2\lambda_1)}{h^{\lambda_1}_{\mu_1}(2\rho_1)^{\mu_1}\Gamma(2\lambda_1)\Gamma(\mu_1+\lambda_1+1)}\frac{\Gamma(\lambda_2+\frac{1}{2})\Gamma(\mu_2+2\lambda_2)}{h^{\lambda_1}_{\mu_1}(2\rho_2)^{\mu_2}\Gamma(2\lambda_2)\Gamma(\mu_2+\lambda_2+1)}. \tag{6.3}$$

Proof

Applying Rodrigues' formula (A.1) and equation (A.2) to the definition of $\hat{f}^{\lambda_1,\lambda_2}_{\mu_1,\mu_2}$ in (5.4) and performing integration by parts $\mu_1+\mu_2$ times yields

$$\hat{f}^{\lambda_1,\lambda_2}_{\mu_1,\mu_2} = \frac{G(\lambda_1,\mu_1)G(\lambda_2,\mu_2)}{2^{\mu_1+\mu_2}\mu_1!\mu_2! h^{\lambda_1}_{\mu_1} h^{\lambda_2}_{\mu_2}}\int_{-1}^{1}\int_{-1}^{1}\frac{d^{\mu_1+\mu_2}}{dx^{\mu_1}dy^{\mu_2}} f(x,y)(1-x^2)^{\mu_1+\lambda_1-\frac{1}{2}}(1-y^2)^{\mu_2+\lambda_2-\frac{1}{2}}dxdy.$$

Assumption 6.1, equations (A.5) and (A.6), and the fact that $C_0^{\mu}(\xi) = 1$ lead to the result

$$|\hat{f}^{\lambda_1,\lambda_2}_{\mu_1,\mu_2}| \leq \frac{G(\lambda_1,\mu_1)G(\lambda_2,\mu_2)C(\rho_1,\rho_2)\pi\Gamma(\mu_1+\lambda_1+\frac{1}{2})\Gamma(\mu_2+\lambda_2+\frac{1}{2})}{h_{\mu_1}^{\lambda_1}h_{\mu_2}^{\lambda_2}2^{\mu_1+\mu_2}\rho_1{}^{\mu_1}\rho_2{}^{\mu_2}(\mu_1+\lambda_1)(\mu_2+\lambda_2)\Gamma(\mu_1+\lambda_1)\Gamma(\mu_2+\lambda_2)}.$$

Substituting in the value for $\Gamma(\mu+\lambda)$ we obtain equation (6.3).

□

We use Lemma 6.1 and Assumption 6.1 to establish an estimate for the regularization error in the maximum norm.

Theorem 6.1 *If $f(x,y)$ is an analytic function defined above satisfying Assumption 6.1, then the regularization error defined in equation (6.1) can be bounded by*

$$RE(\lambda_1,\lambda_2,m_1,m_2) \leq \frac{C(\rho_1,\rho_2)\Gamma(\lambda_1+\frac{1}{2})\Gamma(\lambda_2+\frac{1}{2}))\Gamma(m_1+2\lambda_1+1)\Gamma(m_2+2\lambda_2+1)}{m_1m_2 2^{m_1+m_2}\rho_1^{m_1}\rho_2^{m_2}\Gamma(2\lambda_1)\Gamma(2\lambda_2)\Gamma(m_1+\lambda_1)\Gamma(m_2+\lambda_2)}. \quad (6.4)$$

Proof

Using the estimate (A.8) and equation (6.3) we obtain

$$|\hat{f}^{\lambda_1,\lambda_2}_{\mu_1,\mu_2}|C^{\lambda_1}_{\mu_1}(1)C^{\lambda_2}_{\mu_2}(1) \leq A\frac{C(\rho_1,\rho_2)\Gamma(\lambda_1+\frac{1}{2})\Gamma(\lambda_2+\frac{1}{2})\Gamma(\mu_1+2\lambda_1)\Gamma(\mu_2+2\lambda_2)}{\sqrt{\lambda_1}\sqrt{\lambda_2}(2\rho_1)^{\mu_1}(2\rho_2)^{\mu_2}\Gamma(2\lambda_1)\Gamma(2\lambda_2)\Gamma(\mu_1+\lambda_1)\Gamma(\mu_2+\lambda_2)}. \quad (6.5)$$

Let us define

$$B(\mu) = A\frac{C(\rho)\Gamma(\lambda+\frac{1}{2})\Gamma(\mu+2\lambda)}{\sqrt{\lambda}(2\rho)^{\mu}\Gamma(2\lambda)\Gamma(\mu+\lambda)}.$$

Then by applying the bound (6.5) and the using fact that $|C_{\mu}^{\lambda}(\xi)| \leq C_{\mu}^{\lambda}(1)$ for all $-1 \leq \xi \leq 1$, we obtain

$$\begin{aligned}
RE(\lambda_1,\lambda_2,m_1,m_2) &= \left| \max_{\substack{-1\leq x\leq 1\\ -1\leq y\leq 1}} \sum_{\mu_1=m_1+1}^{\infty}\sum_{\mu_2=m_2+1}^{\infty} \hat{f}^{\lambda_1,\lambda_2}_{\mu_1,\mu_2}C^{\lambda_1}_{\mu_1}(x)C^{\lambda_2}_{\mu_2}(y)\right| \\
&\leq \sum_{\mu_1=m_1+1}^{\infty}\sum_{\mu_2=m_2+1}^{\infty} B(\mu_1)B(\mu_2) \\
&\leq B(m_1+1)B(m_2+1)\frac{4(m_1+\lambda_1)(m_2+\lambda_2)}{m_1m_2} \\
&\leq A\frac{C(\rho_1,\rho_2)\Gamma(\lambda_1+\frac{1}{2})\Gamma(\lambda_2+\frac{1}{2})\Gamma(\mu_1+2\lambda_1)\Gamma(\mu_2+2\lambda_2)}{\sqrt{\lambda_1}\sqrt{\lambda_2}(2\rho_1)^{\mu_1}(2\rho_2)^{\mu_2}\Gamma(2\lambda_1)\Gamma(2\lambda_2)\Gamma(\mu_1+\lambda_1)\Gamma(\mu_2+\lambda_2)}.
\end{aligned}$$

□

We now prove that the regularization error defined in (6.1) is *exponentially small* when λ_1 and λ_2 grow linearly with m_1 and m_2 respectively.

Theorem 6.2 *If $\lambda_1 = \gamma_1 m_1$ and $\lambda_2 = \gamma_2 m_2$ where γ_1 and γ_2 are positive constant, then the regularization error defined in (6.1) satisfies*

$$RE(\lambda_1, \lambda_2, m_1, m_2) \leq A q_1^{m_1} q_2^{m_2} \tag{6.6}$$

where q_i for $i = 1, 2$ is given by

$$q_i = \frac{(1 + 2\gamma_i)^{1+2\gamma_i}}{\rho_i 2^{1+2\gamma_i} \gamma_i^{\gamma_i} (1 + \gamma_i)^{1+\gamma_i}}, \tag{6.7}$$

which is always less than 1. In particular, if $\gamma_i = 1$ and $m_i = \beta_i N$ where β_i is a positive constant, then

$$RE(\lambda_1, \lambda_2, m_1, m_2) \leq A q_1^N q_2^N \tag{6.8}$$

with

$$q_i = \left(\frac{27}{32\rho_i} \right)^{\beta_i}. \tag{6.9}$$

Proof

The proof follows from the application of Stirling's formula (A.7) to the bound proved in Theorem 6.1. Some algebra leads to equations (6.6) and (6.7), where A involves contributions from ρ_i, where $i = 1, 2$. The value q_i defined in equation (6.7) is a strictly increasing function of γ_i. As $\gamma_i \to \infty$, we have $q_i \to \frac{1}{\rho_i} \leq 1$. Hence $q_i < \frac{1}{\rho_i} \leq 1$ for all $\gamma_i > 0$. Now, by substituting in the value $\gamma_i = 1$ and $m_i = \beta_i N$ we obtain the estimates in (6.8) and (6.9).

□

Summarizing the theorems proved thus far in this section, the following theorem states the exponential decay of the regularization error.

Theorem 6.3 *Assume that $f(x,y)$ is an analytic function for $-1 \le x \le 1$ and $-1 \le y \le 1$ that satisfies Assumption 6.1. Let $\hat{f}^{\lambda_1,\lambda_2}_{\mu_1,\mu_2}$ be the Gegenbauer coefficients defined in equation (5.4) for $0 \le \mu_1 \le m_1$ and $0 \le \mu_2 \le m_2$. For simplicity let $\lambda_1 = m_1$ and $\lambda_2 = m_2$. Then*

$$\max_{\substack{-1 \le x \le 1 \\ -1 \le y \le 1}} |f(x,y) - \sum_{\mu_1=0}^{m_1} \sum_{\mu_2=0}^{m_2} \hat{f}^{\lambda_1,\lambda_2}_{\mu_1,\mu_2} C^{\lambda_1}_{\mu_1}(x) C^{\lambda_2}_{\mu_2}(y)| \le A q_1^N q_2^N \tag{6.10}$$

where q_i, $i = 1,2$ is defined in equation (6.9).

The proof is simply a combination of all the previous results.

□

We can now combine the results we obtain from Section 5 and Section 6 to establish an exponentially convergent approximation to a piecewise analytic and non-periodic two-dimensional function in the maximum norm with information of the first $(2N+1)^2$ Fourier coefficients. (We did not actually examine piecewise analytic functions here, but the theory is just a generalization of the results established here.) In Section 5 we established that the Gegenbauer coefficients can be effectively (with exponential convergence) approximated using the information provided by the Fourier coefficients, and in Section 6 we showed that the Gegenbauer partial sum converges exponentially to $f(x,y)$. Combining these two pertinent results we state the following theorem.

Theorem 6.4 *Consider and analytic and non-periodic function $f(x,y)$ where $-1 \le x \le 1$ and $-1 \le y \le 1$ that satisfies Assumption 6.1. Assume we are given the Fourier coefficients*

$$\hat{a}_{k,l} = \frac{1}{4} \int_{-1}^{1} \int_{-1}^{1} f(x,y) e^{-ik\pi x} e^{-il\pi y} dx dy$$

for $-N \le k,l \le N$. Then for $\lambda_1 = m_1 = \beta_1 N$ and $\lambda_2 = m_2 = \beta_2 N$, we have

$$\max_{\substack{-1 \le x \le 1 \\ -1 \le y \le 1}} |f(x,y) - \sum_{\mu_1=0}^{m_1} \sum_{\mu_2=0}^{m_2} \hat{g}^{\lambda_1,\lambda_2}_{\mu_1,\mu_2} C^{\lambda_1}_{\mu_1}(x) C^{\lambda_2}_{\mu_2}(y)| \le A^T N^4 q_1^T q_2^T + A^R (q_1^R)^N (q_2^R)^N, \tag{6.11}$$

where q_1^T and q_2^T are defined in equation (5.15) and q_1^R and q_2^R are defined in equation (6.9), and $\beta_1 = \beta_2 < \frac{2\pi e}{27}$.

Proof

The total error of the partial Gegenbauer expansion approximation, $g^{\lambda_1,\lambda_2}_{m_1,m_2}(x,y)$, to the function $f(x,y)$, is defined as

$$\begin{aligned} E(\lambda_1,\lambda_2,m_1,m_2) &= \max_{\substack{-1\le x\le 1\\ -1\le y\le 1}} |f(x,y) - g^{\lambda_1,\lambda_2}_{m_1,m_2}(x,y)| \\ &= \max_{\substack{-1\le x\le 1\\ -1\le y\le 1}} |f(x,y) - \sum_{\mu_1=0}^{m_1}\sum_{\mu_2=0}^{m_2} \hat{g}^{\lambda_1,\lambda_2}_{\mu_1,\mu_2} C^{\lambda_1}_{\mu_1}(x) C^{\lambda_2}_{\mu_2}(y)|. \end{aligned}$$

Thus the total error is bounded by the sum of the *regularization error* and the *truncation error*, i.e.,

$$\begin{aligned} E(\lambda_1,\lambda_2,m_1,m_2) \le & \max_{\substack{-1\le x\le 1\\ -1\le y\le 1}} |f(x,y) - \sum_{\mu_1=0}^{m_1}\sum_{\mu_2=0}^{m_2} \hat{f}^{\lambda_1,\lambda_2}_{\mu_1,\mu_2} C^{\lambda_1}_{\mu_1}(x) C^{\lambda_2}_{\mu_2}(y)| \\ + & \max_{\substack{-1\le x\le 1\\ -1\le y\le 1}} |\sum_{\mu_1=0}^{m_1}\sum_{\mu_2=0}^{m_2} \hat{f}^{\lambda_1,\lambda_2}_{\mu_1,\mu_2} C^{\lambda_1}_{\mu_1}(x) C^{\lambda_2}_{\mu_2}(y) - \sum_{\mu_1=0}^{m_1}\sum_{\mu_2=0}^{m_2} \hat{g}^{\lambda_1,\lambda_2}_{\mu_1,\mu_2} C^{\lambda_1}_{\mu_1}(x) C^{\lambda_2}_{\mu_2}(y)|, \end{aligned}$$

where the first term is estimated by Theorem 6.3 and the second by Theorem 5.2. This concludes the proof.

□

7 Numerical results for the two-dimensional problem

We consider a simple example that clearly shows how effective the Gegenbauer method is in two dimensions.

Example 7.1

$$f(x) = e^{i2.3\pi x + i1.2\pi y},$$

where we are given the first $2N+1$ Fourier coefficients of the "spliced" function in each quadrant: $-1 \le x \le 0, -1 \le y \le 0$; $-1 \le x \le 0, 0 \le y \le 1$; $0 \le x \le 1, -1 \le y \le 0$ and $0 \le x \le 1, 0 \le y \le 1$.

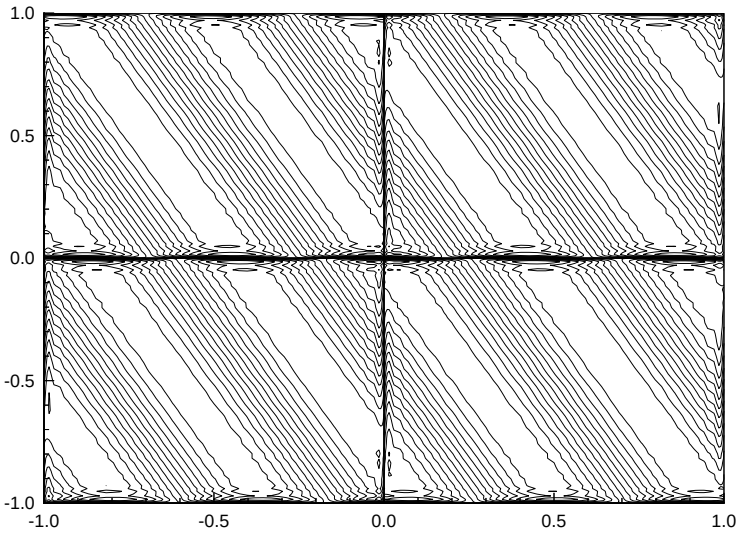

Figure 10: Approximation of $f(x) = e^{i2.3\pi x + i1.2\pi y}$ for $N = 32$ using the Fourier partial sum in each sub-domain.

Figure 10 shows the contour plot of Example 7.1 using the Fourier partial sum for $N = 32$. Notice how the Gibbs' phenomenon affects the boundaries of each quadrant. The Gegenbauer method completely eliminates this phenomenon for $N = 32$, as shown in figure 11. We see a more colorful representation of this improvement in figures 12 and 13.

In fact, there is a dramatic improvement of the Gegenbauer method over the Fourier partial sum for $N = 8$ and $N = 16$, as shown in figure 14.

Table 3 shows the convergence of the Gegenbauer method as applied to each sub-domain in the maximum norm. The maximum error using the Fourier partial sum is 1.077.

8 Conclusion

Our results in this paper show that the Gibbs phenomenon can be eliminated for "spliced" functions in one and two dimensions, as long as the first $2N + 1$ Fourier coefficients of each sub-domain function are known. The impact of these results can be considered in two ways.

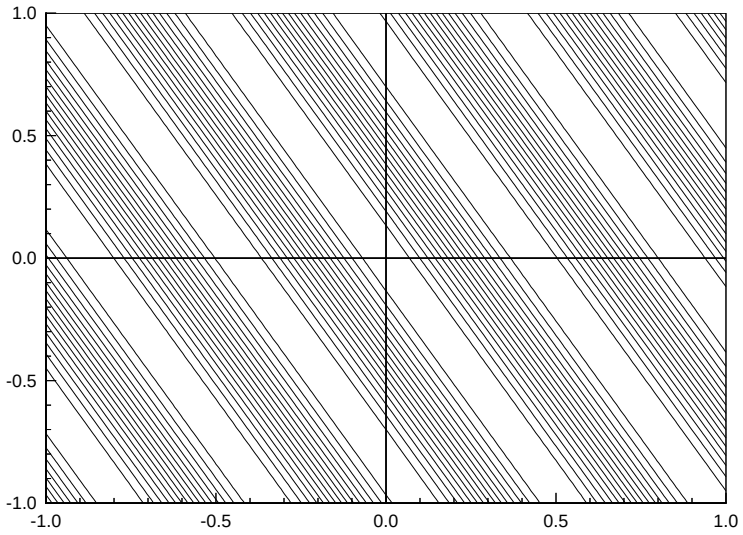

Figure 11: Approximation of $f(x) = e^{i2.3\pi x + i1.2\pi y}$ for $N = 32$ using the Gegenbauer method in each sub-domain.

Firstly, we can consider a smooth but non-periodic function on a general domain $[a, b]$ that is too "rough" to be resolved using any approximation method over the entire domain. Now we are able to split the function into different pieces and solve for the "smoother" parts of the function in smaller sub-domains. This question was also addressed in [15], but the approach was different than ours. Secondly, we can assume that we only have access to the Fourier coefficients of a function in various sub-domains, and hence our method is directly applicable. In any case, parallelization can obviously be applied to this type of approach, cutting costs

Table 3: Absolute maximum error of $f(x) = e^{i2.3\pi x + i1.2\pi y}$ using the the Gegenbauer method for $N = 8, 16, 32$, and 64 in each sub-domain.

N	max error
8	.12
16	4.1E-2
32	3.0E-3
64	2.0E-5

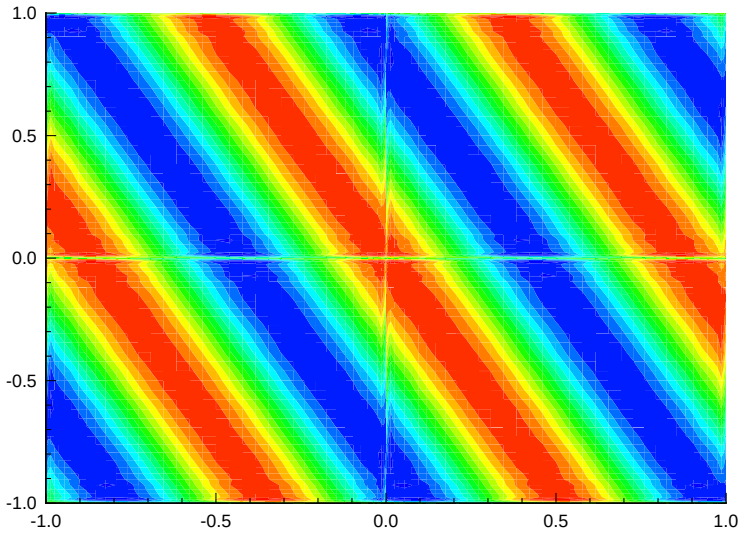

Figure 12: Approximation of $f(x) = e^{i2.3\pi x + i1.2\pi y}$ for $N = 32$ using the Fourier partial sum in each sub-domain.

significantly. Although we did not prove results for the pseudospectral Fourier coefficients, our results indicate that they will also suffice in the Gegenbauer, Bernoulli, and Gegenbauer-Bernoulli methods. Furthermore, the Gegenbauer method will tolerate perturbations in the pseudospectral Fourier coefficients.

This paper did not address a few things that will be discussed in future papers.

1. The parameters λ and m have not been optimized in the two-dimensional Gegenbauer method.

2. In [15] the Gegenbauer method is used to find the jump coefficients. It may be possible to use this approach in creating a hybrid method combining the Gegenbauer and Bernoulli methods in two dimensions.

3. We are assuming explicit knowledge of the points of discontinuity. When this information is not available, methods for locating discontinuities must be employed, such as [7] or [9], which will affect the overall convergence.

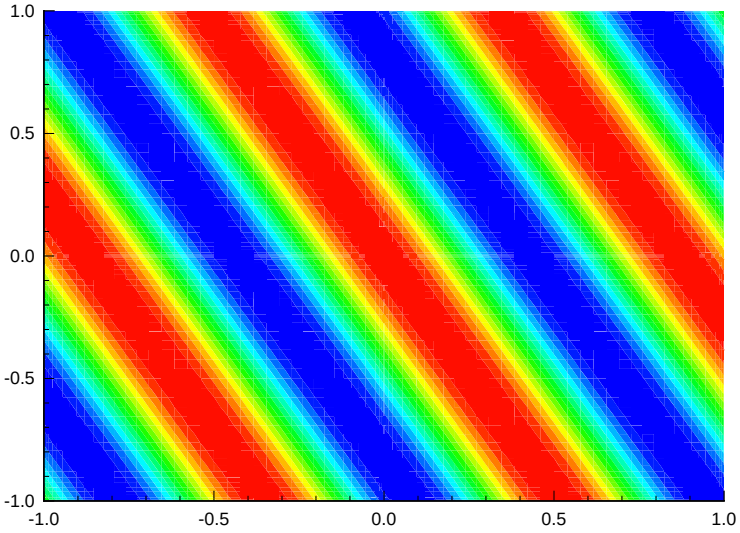

Figure 13: Approximation of $f(x) = e^{i2.3\pi x + i1.2\pi y}$ for $N = 32$ using the Gegenbauer method in each sub-domain.

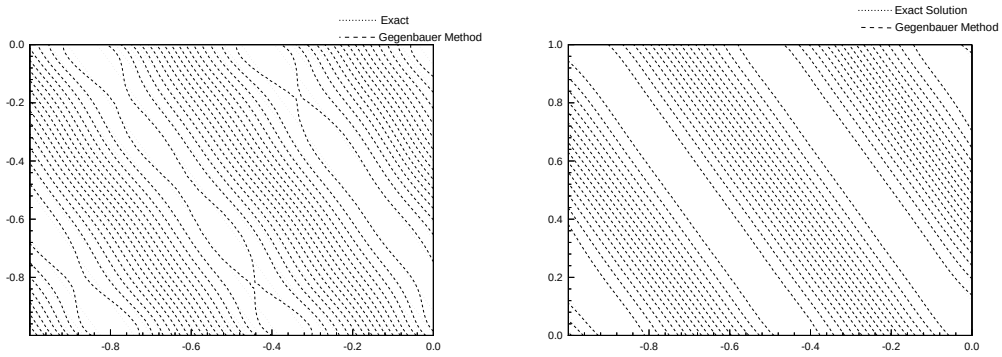

Figure 14: Approximation of $f(x) = e^{i2.3\pi x + i1.2\pi y}$ for (a) $N = 8$ and $-1 \leq x \leq 0, -1 \leq y \leq 0$ (b) $N = 16$ and $-1 \leq x \leq 0, 0 \leq y \leq 1$ using the Gegenbauer method in different sub-domains.

A Gegenbauer polynomials

In this section some useful results are collected about the Gegenbauer polynomials, to be used in later sections. The standardization in Bateman [3] is heavily used.

Definition A.1 *The Gegenbauer polynomial $C_n^\lambda(x)$, for $\lambda \geq 0$, is defined by*

$$(1-x^2)^{\lambda-\frac{1}{2}} C_n^\lambda(x) = G(\lambda,n) \frac{d^n}{dx^n}\left[(1-x^2)^{n+\lambda-\frac{1}{2}}\right], \tag{A.1}$$

where $G(\lambda,n)$ is given by

$$G(\lambda,n) = \frac{(-1)^n \Gamma(\lambda+\frac{1}{2})\Gamma(n+2\lambda)}{2^n n!\Gamma(2\lambda)\Gamma(n+\lambda+\frac{1}{2})}. \tag{A.2}$$

Formula (A.1) is also called the Rodrigues' formula [2, page 175].

Under this definition, for $\lambda > 0$,

$$C_n^\lambda(1) = \frac{\Gamma(n+2\lambda)}{n!\Gamma(2\lambda)} \tag{A.3}$$

and

$$|C_n^\lambda(x)| \leq C_n^\lambda(1), \qquad -1 \leq x \leq 1. \tag{A.4}$$

The Gegenbauer polynomials are orthogonal under the weight function $(1-x^2)^{\lambda-\frac{1}{2}}$, thus

$$\int_{-1}^{1} (1-x^2)^{\lambda-\frac{1}{2}} C_k^\lambda(x) C_n^\lambda(x) dx = \delta_{k,n} h_n^\lambda, \tag{A.5}$$

where, for $\lambda > 0$,

$$h_n^\lambda = \pi^{\frac{1}{2}} C_n^\lambda(1) \frac{\Gamma(\lambda+\frac{1}{2})}{\Gamma(\lambda)(n+\lambda)}. \tag{A.6}$$

The approximation of the Gegenbauer polynomials for large n and λ is dependent upon the well-known Stirling's formula for $\Gamma(x)$ given by

$$(2\pi)^{\frac{1}{2}} x^{x+\frac{1}{2}} e^{-x} \leq \Gamma(x+1) \leq (2\pi)^{\frac{1}{2}} x^{x+\frac{1}{2}} e^{-x} e^{\frac{1}{12x}}, \qquad x \geq 1. \tag{A.7}$$

Lemma A.1 *There exists a constant A independent of λ and n such that*

$$A^{-1} \frac{\lambda^{\frac{1}{2}}}{(n+\lambda)} C_n^\lambda(1) \leq h_n^\lambda \leq A \frac{\lambda^{\frac{1}{2}}}{(n+\lambda)} C_n^\lambda(1). \tag{A.8}$$

The proof follows from (A.6) and the Stirling's formula (A.7).

□

The following lemma to be used later is easily obtained from the Rodrigues' formula (A.1).

Lemma A.2 *For any* $\lambda \geq 1$

$$\frac{d}{dx}\left[(1-x^2)^{\lambda-\frac{1}{2}}C_n^\lambda(x)\right] = \frac{G(\lambda,n)}{G(\lambda-1,n+1)}(1-x^2)^{\lambda-\frac{3}{2}}C_{n+1}^{\lambda-1}(x). \tag{A.9}$$

The proof follows from taking the derivative on both sides of the Rodrigues' formula (A.1), and then using it again on the right hand side.

□

The following formula [1, page 176] will also be needed:

$$C_n^\lambda(x) = \frac{1}{2(n+\lambda)}(\frac{d}{dx}\left[C_{n+1}^\lambda(x) - C_{n-1}^\lambda(x)\right]), \tag{A.10}$$

which is true for all $\lambda \geq 0$.

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