

**Mixed Finite Element Methods on
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MIXED FINITE ELEMENT METHODS ON NON-MATCHING MULTIBLOCK GRIDS*

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Abstract. We consider mixed finite element methods for second order elliptic equations on non-matching multiblock grids. A mortar finite element space is introduced on the non-matching interfaces. We approximate in this mortar space the trace of the solution, and we impose weakly a continuity of flux condition. A standard mixed finite element method is used within the blocks. Optimal order convergence is shown for both the solution and its flux. Moreover, at certain discrete points, superconvergence is obtained for the solution, and also for the flux in special cases. Computational results using an efficient parallel domain decomposition algorithm are presented in confirmation of the theory.

Key words. Mixed finite element, mortar finite element, error estimates, superconvergence, multiblock, non-conforming grids

AMS subject classifications. 65N06, 65N12, 65N15, 65N22, 65N30

1. Introduction. Mixed finite element methods have become popular due to their local (mass) conservation property and good approximation of the flux variable. In many applications the complexity of the geometry or the behavior of the solution may warrant using a multiblock domain structure, wherein the domain is decomposed into non-overlapping blocks or subdomains with grids defined independently on each block. Typical examples include modeling faults and wells in subsurface applications. Faults are natural discontinuities in material properties. Locally refined grids are needed for accurate approximation of high gradients around wells.

In this work we consider second order linear elliptic equations that in porous medium applications model single phase Darcy flow. We solve for the pressure p and the velocity \mathbf{u} satisfying

$$\mathbf{u} = -K\nabla p \quad \text{in } \Omega, \quad (1.1)$$

$$\nabla \cdot \mathbf{u} = f \quad \text{in } \Omega, \quad (1.2)$$

$$p = g \quad \text{on } \partial\Omega, \quad (1.3)$$

where $\Omega \subset \mathbf{R}^d$, $d = 2$ or 3 , is a multiblock domain and K is a symmetric, uniformly positive definite tensor with $L^\infty(\Omega)$ components representing the permeability divided by the viscosity. The Dirichlet boundary conditions are considered merely for simplicity.

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A number of papers deal with the analysis and the implementation of the mixed methods applied to the above problem on conforming grids (see, e.g., [25, 23, 22, 7, 5, 6, 9, 12, 21, 26, 13, 15, 2, 1] and [24, 8]). Mixed methods on nested locally refined grids are considered in [14, 16]. These works apply the notion of “slave” or “worker” nodes to force continuity of fluxes across the interfaces. The results rely heavily on the fact that the grids are nested and cannot be extended to non-matching grids.

In the present work we employ a partially hybridized form [3, 8] of the mixed method to obtain accurate approximations on non-matching grids. We assume that Ω is a union of non-overlapping polygonal blocks, each covered by a conforming, affine finite element partition. Lagrange multiplier pressures are introduced on the interblock boundaries [3, 8, 17]. Since the grids are different on the two sides of the interface, the Lagrange multiplier space can no longer be the normal trace of the velocity space. A different boundary space is needed, which we call a mortar finite element space, using terminology from previous works on Galerkin and spectral methods (see [4] and references therein). As we show later in the analysis, the method is optimally convergent if the boundary space has one order higher approximability than the normal trace of the velocity space. Moreover, superconvergence for the pressure and, in the case of rectangular grids, for the velocity is obtained at certain discrete points.

We allow the mortar space to consist of either continuous or discontinuous piecewise polynomials and obtain the same order of convergence in both cases. The method using discontinuous mortars provides better local mass conservation across the interfaces, but numerical observations suggest that this may lead to slightly bigger numerical error.

The method presented here has also been considered in [27] in the case of the lowest order Raviart-Thomas spaces [23, 22]. Here we take a somewhat different approach in the analysis, which allows us to relax a condition on the mortar grids needed to obtain optimal convergence and superconvergence. The relaxed condition is easily satisfied in practice.

An attractive feature of the scheme is that it can be implemented efficiently in parallel using non-overlapping domain decomposition algorithms. In particular, we modify the Glowinski-Wheeler algorithm [17, 11] to handle non-matching grids. Since this algorithm uses Lagrange multipliers on the interface, the only additional cost is computing projections of the mortar space onto the normal trace of the local velocity spaces and vice-versa.

The rest of the paper is organized as follows. The mixed finite element method with mortar elements is presented in the next section. In Section 3 we construct a projection operator onto the space of weakly continuous (with respect to the mortars) velocities and analyze its approximation properties. Sections 4 and 5 are devoted to the error analysis of the velocity and the pressure, respectively. In Section 6 the method is reformulated as an interface problem. A substructuring domain decomposition algorithm for the solution of the interface problem is discussed in Section 7. Numerical results confirming the theory are presented in Section 8.

2. Formulation of the method. A weak solution of (1.1)–(1.3) is a pair $\mathbf{u} \in H(\text{div}; \Omega)$, $p \in L^2(\Omega)$ such that

$$(K^{-1}\mathbf{u}, \mathbf{v}) = (p, \nabla \cdot \mathbf{v}) - \langle g, \mathbf{v} \cdot \nu \rangle_{\partial\Omega}, \quad \mathbf{v} \in H(\text{div}; \Omega), \quad (2.1)$$

$$(\nabla \cdot \mathbf{u}, w) = (f, w), \quad w \in L^2(\Omega), \quad (2.2)$$

It is well known (see, e.g., [8, 24]) that (2.1)–(2.2) has a unique solution.

Let $\Omega = \cup_{i=1}^n \Omega_i$ be decomposed into non overlapping subdomains Ω_i , and let $\Gamma_{i,j} = \partial\Omega_i \cap \partial\Omega_j$, $\Gamma = \cup_{i,j=1}^n \Gamma_{i,j}$, and $\Gamma_i = \partial\Omega_i \cap \Gamma = \partial\Omega_i \setminus \partial\Omega$. Let

$$\mathbf{V}_i = H(\text{div}; \Omega_i), \quad \mathbf{V} = \bigoplus_{i=1}^n \mathbf{V}_i,$$

and

$$W_i = L^2(\Omega_i), \quad W = \bigoplus_{i=1}^n W_i = L^2(\Omega).$$

If the solution (\mathbf{u}, p) of (2.1)–(2.2) belongs to $H(\text{div}; \Omega) \times H^1(\Omega)$, it is easy to see that it satisfies, for $1 \leq i \leq n$,

$$(K^{-1} \mathbf{u}, \mathbf{v})_{\Omega_i} = (p, \nabla \cdot \mathbf{v})_{\Omega_i} - \langle p, \mathbf{v} \cdot \nu_i \rangle_{\Gamma_i} - \langle g, \mathbf{v} \cdot \nu_i \rangle_{\partial\Omega_i \setminus \Gamma}, \quad \mathbf{v} \in \mathbf{V}_i, \quad (2.3)$$

$$(\nabla \cdot \mathbf{u}, w)_{\Omega_i} = (f, w)_{\Omega_i}, \quad w \in W_i, \quad (2.4)$$

where ν_i is the outer unit normal to $\partial\Omega_i$.

Let $\mathcal{T}_{h,i}$ be a conforming, quasi-uniform finite element partition of Ω_i , $1 \leq i \leq n$, allowing for the possibility that $\mathcal{T}_{h,i}$ and $\mathcal{T}_{h,j}$ need not align on $\Gamma_{i,j}$. Let $\mathcal{T}_h = \cup_{i=1}^n \mathcal{T}_{h,i}$. Let

$$\mathbf{V}_{h,i} \times W_{h,i} \subset \mathbf{V}_i \times W_i$$

be any of the usual mixed finite element spaces, (i.e., the RTN spaces [25, 23, 22]; BDM spaces [7]; BDFM spaces [6]; BDDF spaces [5], or CD spaces [9]). We assume that the order of the spaces is the same on every subdomain. Let

$$\mathbf{V}_h = \bigoplus_{i=1}^n \mathbf{V}_{h,i}, \quad W_h = \bigoplus_{i=1}^n W_{h,i}.$$

Recall that

$$\nabla \cdot \mathbf{V}_{h,i} = W_{h,i},$$

and that there exists a projection Π_i onto $\mathbf{V}_{h,i}$, satisfying amongst other properties that for any $\mathbf{q} \in (H^{1/2+\varepsilon}(\Omega_i))^d \cap \mathbf{V}_i$,

$$(\nabla \cdot (\Pi_i \mathbf{q} - \mathbf{q}), w)_{\Omega_i} = 0, \quad w \in W_{h,i} \quad (2.5)$$

$$\langle (\mathbf{q} - \Pi_i \mathbf{q}) \cdot \nu_i, \mathbf{v} \cdot \nu_i \rangle_{\partial\Omega_i} = 0, \quad \mathbf{v} \in \mathbf{V}_{h,i}. \quad (2.6)$$

Note that, since $\mathbf{q} \in (H^{1/2+\varepsilon}(\Omega_i))^d$, $\mathbf{q} \cdot \nu|_e \in H^\varepsilon(e)$ for any element face (edge) e ; therefore $\Pi_i \mathbf{q}$ is well defined.

Let $\mathcal{T}_{h,i,j}$ be a quasi-uniform finite element partition of $\Gamma_{i,j}$. Denote by $\Lambda_{h,i,j} \subset L^2(\Gamma_{i,j})$ the space of either continuous or discontinuous piecewise polynomials of degree $k+1$ on $\mathcal{T}_{h,i,j}$, where k is associated with the degree of the polynomials in $\mathbf{V}_h \cdot \nu$. More precisely, if $d=3$, on any boundary element K , $\Lambda_{h,i,j}|_K = P_{k+1}(K)$, if K is a triangle, and $\Lambda_{h,i,j}|_K = Q_{k+1}(K)$, if K is a rectangle. Let

$$\Lambda_h = \bigoplus_{1 \leq i < j \leq n} \Lambda_{h,i,j}.$$

In the following we treat any function $\mu \in \Lambda_h$ as extended by zero on $\partial\Omega$. An additional assumption on the space Λ_h and hence $\mathcal{T}_{h,i,j}$ will be made below in (2.11) and (3.18). We remark that $\mathcal{T}_{h,i,j}$ need not be conforming if a discontinuous space is used.

In the mixed finite element approximation of (2.1)–(2.2), we seek $\mathbf{u}_h \in \mathbf{V}_h$, $p_h \in W_h$, $\lambda_h \in \Lambda_h$ such that, for $1 \leq i \leq n$,

$$(K^{-1} \mathbf{u}_h, \mathbf{v})_{\Omega_i} = (p_h, \nabla \cdot \mathbf{v})_{\Omega_i} - \langle \lambda_h, \mathbf{v} \cdot \nu_i \rangle_{\Gamma_i} - \langle g, \mathbf{v} \cdot \nu_i \rangle_{\partial\Omega_i \setminus \Gamma}, \quad \mathbf{v} \in \mathbf{V}_{h,i}, \quad (2.7)$$

$$(\nabla \cdot \mathbf{u}_h, w)_{\Omega_i} = (f, w)_{\Omega_i}, \quad w \in W_{h,i}, \quad (2.8)$$

$$\sum_{i=1}^n \langle \mathbf{u}_h \cdot \nu_i, \mu \rangle_{\Gamma_i} = 0, \quad \mu \in \Lambda_h. \quad (2.9)$$

For each subdomain Ω_i , define a projection $\mathcal{Q}_{h,i} : L^2(\Gamma_i) \rightarrow \mathbf{V}_{h,i} \cdot \nu_i|_{\Gamma_i}$ such that, for any $\phi \in L^2(\Gamma_i)$,

$$\langle \phi - \mathcal{Q}_{h,i}\phi, \mathbf{v} \cdot \nu_i \rangle_{\Gamma_i} = 0, \quad \mathbf{v} \in \mathbf{V}_{h,i}. \quad (2.10)$$

Let, for $\phi \in L^2(\Gamma)$, $\mathcal{Q}_h\phi = \bigoplus_{i=1}^n \mathcal{Q}_{h,i}\phi$.

LEMMA 2.1. *Assume that for any $\phi \in \Lambda_h$,*

$$\mathcal{Q}_{h,i}\phi = 0, \quad 1 \leq i \leq n, \quad \text{implies that } \phi = 0. \quad (2.11)$$

Then there exists a unique solution of (2.7)–(2.9).

Proof. Since (2.7)–(2.9) is a square system, it is enough to show uniqueness. Let $f = 0$, $g = 0$. Setting $\mathbf{v} = \mathbf{u}_h$, $w = p_h$, and $\mu = -\lambda_h$, adding (2.7)–(2.9) together, and summing over $1 \leq i \leq n$, implies that $\mathbf{u}_h = 0$. Denote, for $1 \leq i \leq n$,

$$\overline{p_{h,i}} = \frac{1}{|\Omega_i|} \int_{\Omega_i} p_h \, dx, \quad \overline{\mathcal{Q}_{h,i}\lambda_h} = \frac{1}{|\partial\Omega_i|} \int_{\partial\Omega_i} \mathcal{Q}_{h,i}\lambda_h \, ds,$$

and consider the auxiliary problem

$$\begin{aligned} -\nabla \cdot K \nabla \varphi_i &= p_h - \overline{p_{h,i}} && \text{in } \Omega_i, \\ -K \nabla \varphi_i \cdot \nu &= -(\mathcal{Q}_{h,i}\lambda_h - \overline{\mathcal{Q}_{h,i}\lambda_h}) && \text{on } \partial\Omega_i, \end{aligned}$$

where $\lambda_h = 0$ on $\partial\Omega \cap \partial\Omega_i$. Note that the problem is well posed and regular with φ_i determined up to a constant. Setting $\mathbf{v} = -\Pi_i K \nabla \varphi_i$ in (2.7), we have

$$(p_h, p_h - \overline{p_{h,i}})_{\Omega_i} + \langle \mathcal{Q}_{h,i}\lambda_h, \mathcal{Q}_{h,i}\lambda_h - \overline{\mathcal{Q}_{h,i}\lambda_h} \rangle_{\partial\Omega_i} = 0,$$

implying

$$p_h|_{\Omega_i} = \overline{p_{h,i}}, \quad \mathcal{Q}_{h,i}\lambda_h = \overline{\mathcal{Q}_{h,i}\lambda_h}.$$

Since now (2.7) is

$$p_h|_{\Omega_i} (1, \nabla \cdot \mathbf{v})_{\Omega_i} - \mathcal{Q}_{h,i}\lambda_h \langle 1, \mathbf{v} \cdot \nu \rangle_{\partial\Omega_i} = 0,$$

the divergence theorem implies $p_h|_{\Omega_i} = \mathcal{Q}_{h,i}\lambda_h$.

Since $\lambda_h = 0$ on $\partial\Omega$, $p_h|_{\Omega_i} = \mathcal{Q}_{h,i}\lambda_h = 0$ for those domains i with $\partial\Omega_i \cap \partial\Omega \neq \emptyset$. For any j such that $\partial\Omega_i \cap \partial\Omega_j = \Gamma_{i,j} \neq \emptyset$, (2.10) implies that

$$0 = \mathcal{Q}_{h,i}\lambda_h|_{\Gamma_{i,j}} = \mathcal{Q}_{h,j}\lambda_h|_{\Gamma_{i,j}} = \frac{1}{|\Gamma_{i,j}|} \int_{\Gamma_{i,j}} \lambda_h \, ds.$$

We conclude that $\mathcal{Q}_{h,i}\lambda_h = 0$ for all $1 \leq i \leq n$; hence, $p_h = 0$ and $\lambda_h = 0$ by the hypothesis of the lemma. \square

3. The space of weakly continuous velocities. We first introduce some projection operators needed in the analysis. Let \mathcal{P}_h be the $L^2(\Gamma)$ projection onto Λ_h satisfying for any $\psi \in L^2(\Gamma)$,

$$\langle \psi - \mathcal{P}_h \psi, \mu \rangle_\Gamma = 0, \quad \mu \in \Lambda_h.$$

For any $\varphi \in L^2(\Omega)$, let $\hat{\varphi} \in W_h$ be its $L^2(\Omega)$ projection satisfying

$$(\varphi - \hat{\varphi}, w) = 0, \quad w \in W_h.$$

These operators and the projections Π and $\mathcal{Q}_{h,i}$ defined earlier have the following approximation properties, wherein l is associated with the degree of the polynomials in W_h :

$$\|\psi - \mathcal{P}_h \psi\|_{-s, \Gamma} \leq C \|\psi\|_{r, \Gamma} h^{r+s}, \quad 0 \leq r \leq k+2, \quad 0 \leq s \leq k+2, \quad (3.1)$$

$$\|\varphi - \hat{\varphi}\|_0 \leq C \|\varphi\|_r h^r, \quad 0 \leq r \leq l+1, \quad (3.2)$$

$$\|\mathbf{q} - \Pi_i \mathbf{q}\|_{0, \Omega_i} \leq C \|\mathbf{q}\|_{r, \Omega_i} h^r, \quad 1 \leq r \leq k+1, \quad (3.3)$$

$$\|\nabla \cdot (\mathbf{q} - \Pi_i \mathbf{q})\|_{0, \Omega_i} \leq C \|\nabla \cdot \mathbf{q}\|_{r, \Omega_i} h^r, \quad 1 \leq r \leq l+1, \quad (3.4)$$

$$\|\psi - \mathcal{Q}_{h,i} \psi\|_{-s, \Gamma_i} \leq C \|\psi\|_{r, \Gamma_i} h^{r+s}, \quad 0 \leq r \leq k+1, \quad 0 \leq s \leq k+1, \quad (3.5)$$

$$\|(\mathbf{q} - \Pi_i \mathbf{q}) \cdot \nu_i\|_{-s, \Gamma_i} \leq C \|\mathbf{q}\|_{r, \Gamma_i} h^{r+s}, \quad 0 \leq r \leq k+1, \quad 0 \leq s \leq k+1, \quad (3.6)$$

where $\|\cdot\|_r$ is the H^r -norm and $\|\cdot\|_{-s}$ is the norm of H^{-s} , the dual of H^s . Moreover,

$$\nabla \cdot \Pi_i \mathbf{q} = \widehat{\nabla \cdot \mathbf{q}}, \quad (3.7)$$

$$(\Pi_i \mathbf{q}) \cdot \nu_i = \mathcal{Q}_{h,i}(\mathbf{q} \cdot \nu_i). \quad (3.8)$$

Let

$$\mathbf{V}_{h,0} = \left\{ \mathbf{v} \in \mathbf{V}_h : \sum_{i=1}^n \langle \mathbf{v}|_{\Omega_i} \cdot \nu_i, \mu \rangle_{\Gamma_i} = 0 \quad \forall \mu \in \Lambda_h \right\}$$

be the space of weakly continuous velocities, with respect to the mortar space. Then the mixed method (2.7)–(2.9) can be rewritten in the following way. Find $\mathbf{u}_h \in \mathbf{V}_{h,0}$, $p_h \in W_h$ such that

$$(K^{-1} \mathbf{u}_h, \mathbf{v}) = \sum_{i=1}^n (p_h, \nabla \cdot \mathbf{v})_{\Omega_i} - \langle g, \mathbf{v} \cdot \nu \rangle_{\partial \Omega}, \quad \mathbf{v} \in \mathbf{V}_{h,0}, \quad (3.9)$$

$$\sum_{i=1}^n (\nabla \cdot \mathbf{u}_h, w)_{\Omega_i} = (f, w), \quad w \in W_h. \quad (3.10)$$

Our goal for the rest of this section is to construct a projection operator Π_0 onto $\mathbf{V}_{h,0}$ with optimal approximation properties such that, for any $\mathbf{q} \in (H^1(\Omega))^d$,

$$(\nabla \cdot (\Pi_0 \mathbf{q} - \mathbf{q}), w)_\Omega = 0, \quad w \in W_h. \quad (3.11)$$

By an abuse of notation, define

$$\begin{aligned} \mathbf{V}_h \cdot \nu &= \{(\phi_1, \phi_2) \in L^2(\Gamma) \times L^2(\Gamma) : \phi_1|_{\Gamma_{i,j}} \in \mathbf{V}_{h,i} \cdot \nu_i \text{ and} \\ &\quad \phi_2|_{\Gamma_{i,j}} \in \mathbf{V}_{h,j} \cdot \nu_j \quad \forall 1 \leq i < j \leq n\} \end{aligned}$$

and

$$\mathbf{V}_{h,0} \cdot \nu = \{(\phi_1, \phi_2) \in L^2(\Gamma) \times L^2(\Gamma) : \exists \mathbf{v} \in \mathbf{V}_{h,0} \text{ such that} \\ \phi_1|_{\Gamma_{i,j}} = \mathbf{v}|_{\Omega_i} \cdot \nu_i \text{ and } \phi_2|_{\Gamma_{i,j}} = \mathbf{v}|_{\Omega_j} \cdot \nu_j \quad \forall \quad 1 \leq i < j \leq n\}.$$

Henceforth, for any $\phi \in (L^2(\Gamma))^2$, we write $\phi|_{\Gamma_{i,j}} = (\phi_i, \phi_j)$, $1 \leq i < j \leq n$. Define a projection $\mathcal{Q}_{h,0} : (L^2(\Gamma))^2 \rightarrow \mathbf{V}_{h,0} \cdot \nu$ such that, for any $\phi \in (L^2(\Gamma))^2$,

$$\sum_{i=1}^n \langle \phi_i - (\mathcal{Q}_{h,0}\phi)_i, \xi_i \rangle_{\Gamma_i} = 0, \quad \xi \in \mathbf{V}_{h,0} \cdot \nu. \quad (3.12)$$

LEMMA 3.1. *Assume that (2.11) holds. For any $\phi \in (L^2(\Gamma))^2$, there exists $\lambda_h \in \Lambda_h$ such that on $\Gamma_{i,j}$, $1 \leq i < j \leq n$,*

$$\mathcal{Q}_{h,i}\lambda_h = \mathcal{Q}_{h,i}\phi_i - (\mathcal{Q}_{h,0}\phi)_i, \quad (3.13)$$

$$\mathcal{Q}_{h,j}\lambda_h = \mathcal{Q}_{h,j}\phi_j - (\mathcal{Q}_{h,0}\phi)_j, \quad (3.14)$$

$$\langle \lambda_h, 1 \rangle_{\Gamma_{i,j}} = \frac{1}{2} \langle \phi_i + \phi_j, 1 \rangle_{\Gamma_{i,j}}. \quad (3.15)$$

Proof. Consider the following auxiliary problem. Given $\phi \in (L^2(\Gamma))^2$, find $\psi_h \in \mathbf{V}_h \cdot \nu$ and $\lambda_h \in \Lambda_h$ such that

$$\sum_{i=1}^n \langle \phi_i - \psi_{h,i} - \lambda_h, \xi_i \rangle_{\Gamma_i} = 0, \quad \xi \in \mathbf{V}_h \cdot \nu, \quad (3.16)$$

$$\sum_{i=1}^n \langle \psi_{h,i}, \mu \rangle_{\Gamma_i} = 0, \quad \mu \in \Lambda_h. \quad (3.17)$$

To show existence and uniqueness of a solution of (3.16)–(3.17), take $\phi = 0$, $\xi = \psi_h$, and $\mu = \lambda_h$ to conclude that $\psi_h = 0$. Now (3.16) and (2.11) imply that $\lambda_h = 0$.

With $\xi \in \mathbf{V}_{h,0} \cdot \nu$ in (3.16) we have

$$\sum_{i=1}^n \langle \phi_i - \psi_{h,i}, \xi_i \rangle_{\Gamma_i} = 0.$$

Also, from (3.17), $\psi_h \in \mathbf{V}_{h,0} \cdot \nu$. Therefore $\psi_h = \mathcal{Q}_{h,0}\phi$. Equation (3.16) now implies (3.13) and (3.14). Since any constant function is in $\mathbf{V}_{h,i} \cdot \nu_i$, $\mathbf{V}_{h,j} \cdot \nu_j$, and $\Lambda_{h,i,j}$, we have

$$\begin{aligned} 2\langle \lambda_h, 1 \rangle_{\Gamma_{i,j}} &= \langle \mathcal{Q}_{h,i}\lambda_h, 1 \rangle_{\Gamma_{i,j}} + \langle \mathcal{Q}_{h,j}\lambda_h, 1 \rangle_{\Gamma_{i,j}} \\ &= \langle \mathcal{Q}_{h,i}\phi_i - (\mathcal{Q}_{h,0}\phi)_i, 1 \rangle_{\Gamma_{i,j}} + \langle \mathcal{Q}_{h,j}\phi_j - (\mathcal{Q}_{h,0}\phi)_j, 1 \rangle_{\Gamma_{i,j}} \\ &= \langle \mathcal{Q}_{h,i}\phi_i, 1 \rangle_{\Gamma_{i,j}} + \langle \mathcal{Q}_{h,j}\phi_j, 1 \rangle_{\Gamma_{i,j}} \\ &= \langle \phi_i + \phi_j, 1 \rangle_{\Gamma_{i,j}}, \end{aligned}$$

and (3.15) follows. \square

The next lemma shows that, under a relatively mild assumption on the mortar space Λ_h , $\mathcal{Q}_{h,0}$ has optimal approximation properties for normal traces:

$$\phi = (\mathbf{u} \cdot \nu_i, \mathbf{u} \cdot \nu_j) = (\mathbf{u} \cdot \nu_i, -\mathbf{u} \cdot \nu_i).$$

LEMMA 3.2. Assume that there exists a constant C , independent of h , such that

$$\|\mu\|_{0,\Gamma_{i,j}} \leq C(\|\mathcal{Q}_{h,i}\mu\|_{0,\Gamma_{i,j}} + \|\mathcal{Q}_{h,j}\mu\|_{0,\Gamma_{i,j}}), \quad \forall \mu \in \Lambda_h, \quad 1 \leq i < j \leq n. \quad (3.18)$$

Then, for any ϕ such that $\phi|_{\Gamma_{i,j}} = (\phi_i, -\phi_i)$, there exists a constant C , independent of h , such that

$$\left(\sum_{i=1}^n \|\mathcal{Q}_{h,i}\phi_i - (\mathcal{Q}_{h,0}\phi)_i\|_{-s,\Gamma_i}^2 \right)^{1/2} \leq C \sum_{i=1}^n \|\phi_i\|_{r,\Gamma_i} h^{r+s}, \quad 0 \leq r \leq k+1, \quad 0 \leq s \leq k+1. \quad (3.19)$$

Proof. By Lemma 3.1, there is a $\lambda_h \in \Lambda_h$ such that

$$\mathcal{Q}_{h,i}\lambda_h = \mathcal{Q}_{h,i}\phi_i - (\mathcal{Q}_{h,0}\phi)_i. \quad (3.20)$$

Since $\sum_{i=1}^n \langle (\mathcal{Q}_{h,0}\phi)_i, \lambda_h \rangle_{\Gamma_i} = \sum_{i=1}^n \langle \phi_i, \lambda_h \rangle_{\Gamma_i} = 0$,

$$\begin{aligned} \sum_{i=1}^n \|\mathcal{Q}_{h,i}\lambda_h\|_{0,\Gamma_i}^2 &= \sum_{i=1}^n \langle \mathcal{Q}_{h,i}\lambda_h, \lambda_h \rangle_{\Gamma_i} \\ &= \sum_{i=1}^n \langle \mathcal{Q}_{h,i}\phi_i - \phi_i, \lambda_h \rangle_{\Gamma_i} \\ &\leq \left(\sum_{i=1}^n \|\mathcal{Q}_{h,i}\phi_i - \phi_i\|_{0,\Gamma_i}^2 \right)^{1/2} \left(\sum_{i=1}^n \|\lambda_h\|_{0,\Gamma_i}^2 \right)^{1/2} \\ &\leq C \left(\sum_{i=1}^n \|\mathcal{Q}_{h,i}\phi_i - \phi_i\|_{0,\Gamma_i}^2 \right)^{1/2} \left(\sum_{i=1}^n \|\mathcal{Q}_{h,i}\lambda_h\|_{0,\Gamma_i}^2 \right)^{1/2}, \end{aligned}$$

by (3.18), and (3.19) with $s = 0$ follows from (3.20) and (3.5).

On any interface $\Gamma_{i,j}$ take any $\varphi \in H^s(\Gamma_{i,j})$, $0 \leq s \leq k+1$, and write

$$\begin{aligned} \langle \mathcal{Q}_{h,i}\lambda_h, \varphi \rangle_{\Gamma_{i,j}} &= \langle \lambda_h, \mathcal{Q}_{h,i}\varphi - \varphi \rangle_{\Gamma_{i,j}} + \langle \lambda_h, \varphi \rangle_{\Gamma_{i,j}} \\ &\leq C \|\lambda_h\|_{0,\Gamma_{i,j}} h^s \|\varphi\|_{s,\Gamma_{i,j}} + \langle \lambda_h, \varphi \rangle_{\Gamma_{i,j}}. \end{aligned} \quad (3.21)$$

The last term is

$$\begin{aligned} \langle \lambda_h, \varphi \rangle_{\Gamma_{i,j}} &= \langle \lambda_h, \varphi - \frac{1}{2}(\mathcal{Q}_{h,i}\varphi + \mathcal{Q}_{h,j}\varphi) \rangle_{\Gamma_{i,j}} \\ &\quad + \frac{1}{2} \langle \lambda_h, \mathcal{Q}_{h,i}\varphi + \mathcal{Q}_{h,j}\varphi \rangle_{\Gamma_{i,j}} \\ &\leq C \|\lambda_h\|_{0,\Gamma_{i,j}} h^s \|\varphi\|_{s,\Gamma_{i,j}} + \frac{1}{2} \langle \lambda_h, \mathcal{Q}_{h,i}\varphi + \mathcal{Q}_{h,j}\varphi \rangle_{\Gamma_{i,j}}. \end{aligned} \quad (3.22)$$

Using Lemma 3.1, for the last term in (3.22) we have

$$\begin{aligned} &\langle \lambda_h, \mathcal{Q}_{h,i}\varphi + \mathcal{Q}_{h,j}\varphi \rangle_{\Gamma_{i,j}} \\ &= \langle \mathcal{Q}_{h,i}\lambda_h, \mathcal{Q}_{h,i}\varphi \rangle_{\Gamma_{i,j}} + \langle \mathcal{Q}_{h,j}\lambda_h, \mathcal{Q}_{h,j}\varphi \rangle_{\Gamma_{i,j}} \\ &= \langle \phi_i - (\mathcal{Q}_{h,0}\phi)_i, \mathcal{Q}_{h,i}\varphi \rangle_{\Gamma_{i,j}} + \langle \phi_j - (\mathcal{Q}_{h,0}\phi)_j, \mathcal{Q}_{h,j}\varphi \rangle_{\Gamma_{i,j}} \\ &= \langle \phi_i - (\mathcal{Q}_{h,0}\phi)_i, \mathcal{Q}_{h,i}\varphi - P_h\varphi \rangle_{\Gamma_{i,j}} + \langle \phi_j - (\mathcal{Q}_{h,0}\phi)_j, \mathcal{Q}_{h,j}\varphi - P_h\varphi \rangle_{\Gamma_{i,j}} \\ &\leq Ch^r \|\phi_i\|_{r,\Gamma_{i,j}} h^s \|\varphi\|_{s,\Gamma_{i,j}}, \quad 0 \leq r \leq k+1. \end{aligned} \quad (3.23)$$

Combining (3.21)–(3.23) with (3.18), we obtain (3.19). \square

We are now ready to construct our projection. For any $\mathbf{q} \in (H^{1/2+\varepsilon}(\Omega_i))^d \cap \mathbf{V}_i$ define

$$\Pi_0 \mathbf{q}|_{\Omega_i} = \Pi_i(\mathbf{q} + \delta \mathbf{q}_i),$$

where $\delta \mathbf{q}_i$ solves

$$\delta \mathbf{q}_i = -\nabla \pi_i \quad \text{in } \Omega_i, \quad (3.24)$$

$$\nabla \cdot \delta \mathbf{q}_i = 0 \quad \text{in } \Omega_i, \quad (3.25)$$

$$\delta \mathbf{q}_i \cdot \nu_i = -\mathcal{Q}_{h,i} \mathbf{q} \cdot \nu_i + (\mathcal{Q}_{h,0} \mathbf{q} \cdot \nu)_i \quad \text{on } \Gamma_i, \quad (3.26)$$

$$\delta \mathbf{q}_i \cdot \nu_i = 0 \quad \text{on } \partial \Omega_i \cap \partial \Omega, \quad (3.27)$$

wherein, on any $\Gamma_{i,j}$, $\mathbf{q} \cdot \nu|_{\Gamma_{i,j}} = (\mathbf{q} \cdot \nu_i, \mathbf{q} \cdot \nu_j)$. Note that the Neumann problems (3.25)–(3.27) are well posed, since (3.15) and (3.13) imply that

$$\langle \mathcal{Q}_{h,i} \mathbf{q} \cdot \nu_i - (\mathcal{Q}_{h,0} \mathbf{q} \cdot \nu)_i, 1 \rangle_{\Gamma_i} = 0.$$

Also, note that the piece-wise constant Neumann data is in $H^{1/2-\varepsilon}(\partial \Omega_i)$, so $\delta \mathbf{q}_i \in (H^{1-\varepsilon}(\Omega_i))^d$ and Π_i can be applied to $\delta \mathbf{q}_i$.

We first notice that by (3.8),

$$\sum_{i=1}^n \langle (\Pi_0 \mathbf{q}) \cdot \nu_i, \mu \rangle_{\Gamma_i} = \sum_{i=1}^n \langle (\mathcal{Q}_{h,0} \mathbf{q} \cdot \nu)_i, \mu \rangle_{\Gamma_i} = 0, \quad \forall \mu \in \Lambda_h;$$

therefore $\Pi_0 \mathbf{q} \in \mathbf{V}_{h,0}$. Also, by (3.7),

$$(\nabla \cdot \Pi_0 \mathbf{q}, w)_{\Omega_i} = (\nabla \cdot \Pi_i \mathbf{q}, w)_{\Omega_i} + (\nabla \cdot \Pi_i \delta \mathbf{q}_i, w)_{\Omega_i} = (\nabla \cdot \mathbf{q}, w)_{\Omega_i}, \quad \forall w \in W_{h,i}.$$

It remains to estimate the approximability of Π_0 . Since on Ω_i

$$\Pi_0 \mathbf{q} - \mathbf{q} = \Pi_i \mathbf{q} - \mathbf{q} + \Pi_i \delta \mathbf{q}_i,$$

with (3.3) we need only bound the correction $\Pi_i \delta \mathbf{q}_i$. By elliptic regularity [18, 19], for any $s \geq 0$

$$\|\delta \mathbf{q}_i\|_{1/2-s, \Omega_i} \leq C \|\mathcal{Q}_{h,i} \mathbf{q} \cdot \nu_i - (\mathcal{Q}_{h,0} \mathbf{q} \cdot \nu)_i\|_{-s, \Gamma_i}. \quad (3.28)$$

We now have

$$\begin{aligned} \|\Pi_i \delta \mathbf{q}_i\|_{0, \Omega_i} &\leq \|\Pi_i \delta \mathbf{q}_i - \delta \mathbf{q}_i\|_{0, \Omega_i} + \|\delta \mathbf{q}_i\|_{0, \Omega_i} \\ &\leq Ch^{1/2} \|\delta \mathbf{q}_i\|_{1/2, \Omega_i} + \|\delta \mathbf{q}_i\|_{0, \Omega_i} \\ &\leq C \{ \|\mathcal{Q}_{h,i} \mathbf{q} \cdot \nu_i - (\mathcal{Q}_{h,0} \mathbf{q} \cdot \nu)_i\|_{0, \Gamma_i} h^{1/2} \\ &\quad + \|\mathcal{Q}_{h,i} \mathbf{q}_i \cdot \nu_i - (\mathcal{Q}_{h,0} \mathbf{q} \cdot \nu)_i\|_{-1/2, \Gamma_i} \}, \end{aligned} \quad (3.29)$$

using an estimate by Mathew [20] for any divergence free vector ψ

$$\|\Pi_i \psi - \psi\|_{0, \Omega_i} \leq Ch^\varepsilon \|\psi\|_{\varepsilon, \Omega_i}, \quad 0 < \varepsilon < 1.$$

Note that the result in [20] is for Raviart-Thomas spaces, but can be trivially extended to any of the mixed spaces under consideration. Together with Lemma 3.2, (3.29) gives

$$\|\Pi_0 \mathbf{q} - \Pi \mathbf{q}\|_0 \leq C \sum_{i=1}^n \|\mathbf{q}\|_{r+1/2, \Omega_i} h^{r+1/2}, \quad 0 \leq r \leq k+1, \quad (3.30)$$

and, with (3.3),

$$\|\Pi_0 \mathbf{q} - \mathbf{q}\|_0 \leq C \sum_{i=1}^n \|\mathbf{q}\|_{r, \Omega_i} h^r, \quad 1 \leq r \leq k+1. \quad (3.31)$$

4. Error estimates for the velocity. We start this section with a lemma needed later in the analysis.

LEMMA 4.1. *For any function $\mathbf{v} \in \mathbf{V}_{h,i}$,*

$$\|\mathbf{v} \cdot \nu\|_{0, \partial \Omega_i} \leq C h^{-1/2} \|\mathbf{v}\|_{0, \Omega_i}.$$

Proof. All spaces under consideration admit nodal bases that include the degrees of freedom of the normal traces on the element boundaries. Since for any element E and any of its faces (edges) e , $|e| \leq C h^{-1} |E|$, the lemma follows. \square

4.1. Optimal convergence. Subtracting (3.9)–(3.10) from (2.3)–(2.4) gives the error equations

$$(K^{-1}(\mathbf{u} - \mathbf{u}_h), \mathbf{v}) = \sum_{i=1}^n ((p - p_h, \nabla \cdot \mathbf{v})_{\Omega_i} - \langle p, \mathbf{v} \cdot \nu_i \rangle_{\Gamma_i}) \quad \mathbf{v} \in \mathbf{V}_{h,0}, \quad (4.1)$$

$$\sum_{i=1}^n (\nabla \cdot (\mathbf{u} - \mathbf{u}_h), w)_{\Omega_i} = 0, \quad w \in W_h. \quad (4.2)$$

We first notice that (4.2) implies that

$$\nabla \cdot (\Pi_0 \mathbf{u} - \mathbf{u}_h) = \nabla \cdot (\Pi \mathbf{u} - \mathbf{u}_h) = 0. \quad (4.3)$$

We now take $\mathbf{v} = \Pi_0 \mathbf{u} - \mathbf{u}_h$ to get

$$\begin{aligned} & (K^{-1}(\Pi_0 \mathbf{u} - \mathbf{u}_h), \Pi_0 \mathbf{u} - \mathbf{u}_h) \\ &= \sum_{i=1}^n \langle P_h p - p, (\Pi_0 \mathbf{u} - \mathbf{u}_h) \cdot \nu_i \rangle_{\Gamma_i} + (K^{-1}(\Pi_0 \mathbf{u} - \mathbf{u}), \Pi_0 \mathbf{u} - \mathbf{u}_h) \\ &\leq \sum_{i=1}^n \|P_h p - p\|_{0, \Gamma_i} \|(\Pi_0 \mathbf{u} - \mathbf{u}_h) \cdot \nu_i\|_{0, \Gamma_i} + (K^{-1}(\Pi_0 \mathbf{u} - \mathbf{u}), \Pi_0 \mathbf{u} - \mathbf{u}_h) \\ &\leq C \left(\sum_{i=1}^n \|p\|_{r+1, \Omega_i} h^{r+1/2} \|\Pi_0 \mathbf{u} - \mathbf{u}_h\|_{0, \Omega_i} h^{-1/2} + \sum_{i=1}^n \|\mathbf{u}\|_{r, \Omega_i} h^r \|\Pi_0 \mathbf{u} - \mathbf{u}_h\|_0 \right), \\ &\quad 1 \leq r \leq k+1, \end{aligned} \quad (4.4)$$

where we used (3.1), Lemma 4.1, and (3.31) for the last inequality. With (4.3)–(4.4), (3.4), and (3.31) we have shown the following theorem.

THEOREM 4.2. *For the velocity \mathbf{u}_h of the mixed method (2.7)–(2.9), if (2.11) holds, then there exists a positive constant C independent of h such that*

$$\|\nabla \cdot (\mathbf{u} - \mathbf{u}_h)\|_0 \leq C \sum_{i=1}^n \|\nabla \cdot \mathbf{u}\|_{r, \Omega_i} h^r, \quad 1 \leq r \leq l+1.$$

Moreover, if (3.18) holds,

$$\|\mathbf{u} - \mathbf{u}_h\|_0 \leq C \sum_{i=1}^n (\|p\|_{r+1, \Omega_i} + \|\mathbf{u}\|_{r, \Omega_i}) h^r, \quad 1 \leq r \leq k+1.$$

4.2. Superconvergence. In this subsection we restrict to the case of diagonal tensor K and RTN spaces on rectangular type grids. In this case superconvergence of the velocity is attained at certain discrete points. To show this we modify the last inequality in (4.4). In particular, (3.1) gives, for $1 \leq r \leq k+1$,

$$\sum_{i=1}^n \|P_h p - p\|_{0,\Gamma_i} \|(\Pi_0 \mathbf{u} - \mathbf{u}_h) \cdot \nu_i\|_{0,\Gamma_i} \leq C \sum_{i=1}^n \|p\|_{r+3/2,\Omega_i} h^{r+1} \|\Pi_0 \mathbf{u} - \mathbf{u}_h\|_{0,\Omega_i} h^{-1/2},$$

and (3.30) gives, for $1 \leq r \leq k+1$,

$$(K^{-1}(\Pi_0 \mathbf{u} - \Pi \mathbf{u}), \Pi_0 \mathbf{u} - \mathbf{u}_h) \leq C \sum_{i=1}^n \|\mathbf{u}\|_{r+1/2,\Omega_i} h^{r+1/2} \|\Pi_0 \mathbf{u} - \mathbf{u}_h\|_0,$$

which, combined with and an estimate by Durán ([13], Theorem 3.1)

$$(K^{-1}(\Pi_i \mathbf{u} - \mathbf{u}), \Pi_0 \mathbf{u} - \mathbf{u}_h)_{\Omega_i} \leq C \|\mathbf{u}\|_{r+1,\Omega_i} h^{r+1} \|\Pi_0 \mathbf{u} - \mathbf{u}_h\|_{0,\Omega_i}, \quad 0 \leq r \leq k+1,$$

implies

$$\|\Pi_0 \mathbf{u} - \mathbf{u}_h\|_0 \leq C \sum_{i=1}^n (\|p\|_{r+3/2,\Omega_i} + \|\mathbf{u}\|_{r+1/2,\Omega_i}) h^{r+1/2}, \quad 1 \leq r \leq k+1. \quad (4.5)$$

This estimate implies superconvergence along the Gaussian lines. Consider (for $d = 3$) an element $E = [a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]$. Denote by g_1^i, \dots, g_{k+1}^i , $i = 1, 2, 3$, the Gaussian points on $[a_i, b_i]$, i.e., the roots of the Legendre polynomials of degree $k+1$ on $[a_i, b_i]$. As in [15, 13], for a vector $\mathbf{q} = (q_1, q_2, q_3)$ define

$$\begin{aligned} |||q_1|||_{1,E}^2 &= \sum_{j_2=1}^{k+1} A_{j_2} (b_2 - a_2) \sum_{j_3=1}^{k+1} A_{j_3} (b_3 - a_3) \int_{a_1}^{b_1} |q_1(x_1, g_{j_2}^2, g_{j_3}^3)|^2 dx_1, \\ |||q_2|||_{2,E}^2 &= \sum_{j_1=1}^{k+1} A_{j_1} (b_1 - a_1) \sum_{j_3=1}^{k+1} A_{j_3} (b_3 - a_3) \int_{a_2}^{b_2} |q_2(g_{j_1}^1, x_2, g_{j_3}^3)|^2 dx_2, \\ |||q_3|||_{3,E}^2 &= \sum_{j_1=1}^{k+1} A_{j_1} (b_1 - a_1) \sum_{j_2=1}^{k+1} A_{j_2} (b_2 - a_2) \int_{a_3}^{b_3} |q_3(g_{j_1}^1, g_{j_2}^2, x_3)|^2 dx_3, \end{aligned}$$

where A_{j_i} , $j_i = 1, \dots, k+1$ are the coefficients of Gaussian quadrature in $[-1, 1]$. Define

$$|||\mathbf{q}|||^2 = \sum_{i=1}^3 \sum_{E \in \mathcal{T}_h} |||q_i|||_{i,E}^2.$$

Note that, for $\mathbf{q} \in \mathbf{V}_h$, $|||\mathbf{q}|||$ is equal to the L^2 -norm of \mathbf{q} .

THEOREM 4.3. *Assume that the tensor K is diagonal and the mixed finite element spaces are RTN on rectangular type grids. For the velocity \mathbf{u}_h of the mixed method (2.7)–(2.9), if (3.18) holds, then there exists a positive constant C independent of h such that*

$$|||\mathbf{u} - \mathbf{u}_h||| \leq C \sum_{i=1}^n (\|p\|_{r+3/2,\Omega_i} + \|\mathbf{u}\|_{r+1/2,\Omega_i}) h^{r+1/2}, \quad 1 \leq r \leq k+1.$$

Proof. By the triangle inequality,

$$|||\mathbf{u} - \mathbf{u}_h||| \leq |||\mathbf{u} - \Pi\mathbf{u}||| + |||\Pi\mathbf{u} - \Pi_0\mathbf{u}||| + |||\Pi_0\mathbf{u} - \mathbf{u}_h|||.$$

The theorem follows from (3.30), (4.5), and the bound (see [13])

$$|||\mathbf{u} - \Pi\mathbf{u}|||_{\Omega_i} \leq C\|\mathbf{u}\|_{r+1,\Omega_i}h^{r+1}, \quad 1 \leq r \leq k+1.$$

□

5. Error estimates for the pressure. In this section we use a duality argument to derive a superconvergence estimate for $\hat{p} - p_h$. Let φ be the solution of

$$\begin{aligned} -\nabla \cdot K\nabla\varphi &= -(\hat{p} - p_h) && \text{in } \Omega, \\ \varphi &= 0 && \text{on } \partial\Omega. \end{aligned}$$

By elliptic regularity,

$$\|\varphi\|_2 \leq C\|\hat{p} - p_h\|_0. \quad (5.1)$$

Take $\mathbf{v} = \Pi_0 K\nabla\varphi$ in (4.1) to get

$$\begin{aligned} \|\hat{p} - p_h\|_0^2 &= \sum_{i=1}^n (\hat{p} - p_h, \nabla \cdot \Pi_0 K\nabla\varphi)_{\Omega_i} \\ &= \sum_{i=1}^n ((K^{-1}(\mathbf{u} - \mathbf{u}_h), \Pi_0 K\nabla\varphi)_{\Omega_i} + \langle p - \mathcal{P}_h p, \Pi_0 K\nabla\varphi \cdot \nu_i \rangle_{\Gamma_i}). \end{aligned} \quad (5.2)$$

The first term on the right can be manipulated as

$$\begin{aligned} &\sum_{i=1}^n (K^{-1}(\mathbf{u} - \mathbf{u}_h), \Pi_0 K\nabla\varphi)_{\Omega_i} \\ &= \sum_{i=1}^n ((K^{-1}(\mathbf{u} - \mathbf{u}_h), \Pi_0 K\nabla\varphi - K\nabla\varphi)_{\Omega_i} + (\mathbf{u} - \mathbf{u}_h, \nabla\varphi)_{\Omega_i}) \\ &= \sum_{i=1}^n ((K^{-1}(\mathbf{u} - \mathbf{u}_h), \Pi_0 K\nabla\varphi - K\nabla\varphi)_{\Omega_i} \\ &\quad + (\nabla \cdot (\mathbf{u} - \mathbf{u}_h), \varphi - \hat{\varphi})_{\Omega_i} - \langle (\mathbf{u} - \mathbf{u}_h) \cdot \nu_i, \varphi - \mathcal{P}_h \varphi \rangle_{\Gamma_i}) \\ &\leq C \sum_{i=1}^n (\|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega_i} h + \|\nabla \cdot (\mathbf{u} - \mathbf{u}_h)\|_{0,\Omega_i} h + \|(\mathbf{u} - \mathbf{u}_h) \cdot \nu_i\|_{0,\Gamma_i} h^{3/2}) \|\varphi\|_2 \end{aligned} \quad (5.3)$$

using (3.31), (3.2), and (3.1) for the last inequality with $C = C(\max_i \|K\|_{1,\infty,\Omega_i})$. The last term on the right is

$$\begin{aligned} &\|(\mathbf{u} - \mathbf{u}_h) \cdot \nu_i\|_{0,\Gamma_i} h^{3/2} \\ &\leq (\|(\mathbf{u} - \Pi_i \mathbf{u}) \cdot \nu_i\|_{0,\Gamma_i} + \|(\Pi_i \mathbf{u} - \mathbf{u}_h) \cdot \nu_i\|_{0,\Gamma_i}) h^{3/2} \\ &\leq C(h^r \|\mathbf{u}\|_{r,\Gamma_i} + h^{-1/2} \|\Pi_i \mathbf{u} - \mathbf{u}_h\|_{0,\Omega_i}) h^{3/2} \\ &= C(\|\mathbf{u}\|_{r,\Gamma_i} h^{r+3/2} + \|\Pi_i \mathbf{u} - \mathbf{u}_h\|_{0,\Omega_i} h), \quad 0 \leq r \leq k+1, \end{aligned} \quad (5.4)$$

using (3.6) and Lemma 4.1.

For the second term on the right in (5.2) we have

$$\begin{aligned}
& \langle p - \mathcal{P}_h p, \Pi_0 K \nabla \varphi \cdot \nu_i \rangle_{\Gamma_i} \\
&= \langle p - \mathcal{P}_h p, (\Pi_0 K \nabla \varphi - \Pi_i K \nabla \varphi) \cdot \nu_i + (\Pi_i K \nabla \varphi - K \nabla \varphi) \cdot \nu_i + K \nabla \varphi \cdot \nu_i \rangle_{\Gamma_i} \\
&\leq \|p - \mathcal{P}_h p\|_{0,\Gamma_i} (\|\delta(K \nabla \varphi)_i \cdot \nu_i\|_{0,\Gamma_i} + \|(\Pi_i K \nabla \varphi - K \nabla \varphi) \cdot \nu_i\|_{0,\Gamma_i}) \\
&\quad + \|p - \mathcal{P}_h p\|_{-1/2,\Gamma_i} \|K \nabla \varphi \cdot \nu_i\|_{1/2,\Gamma_i}.
\end{aligned}$$

With (3.1), (3.26), (3.19), and (3.6) we have

$$\begin{aligned}
\|p - \mathcal{P}_h p\|_{0,\Gamma_i} &\leq C \|p\|_{r+1/2,\Gamma_i} h^{r+1/2}, \quad 0 \leq r \leq k+1, \\
\|p - \mathcal{P}_h p\|_{-1/2,\Gamma_i} &\leq C \|p\|_{r+1/2,\Gamma_i} h^{r+1}, \quad 0 \leq r \leq k+1, \\
\|\delta(K \nabla \varphi)_i \cdot \nu_i\|_{0,\Gamma_i} &\leq C \|\varphi\|_{2,\Omega_i} h^{1/2}, \\
\|(\Pi_i K \nabla \varphi - K \nabla \varphi) \cdot \nu_i\|_{0,\Gamma_i} &\leq C \|\varphi\|_{2,\Omega_i} h^{1/2};
\end{aligned}$$

therefore,

$$\langle p - \mathcal{P}_h p, \Pi_0 K \nabla \varphi \cdot \nu_i \rangle_{\Gamma_i} \leq C h^{r+1} \|p\|_{r+1,\Gamma_i} \|\varphi\|_{2,\Omega_i}, \quad 0 \leq r \leq k+1. \quad (5.5)$$

A combination of (5.1)–(5.5), Theorem 4.2, and (3.3) gives the following theorem.

THEOREM 5.1. *For the pressure p_h of the mixed method (2.7)–(2.9), if (3.18) holds, then there exists a positive constant C , independent of h , such that*

$$\begin{aligned}
\|\hat{p} - p_h\|_0 &\leq C \sum_{i=1}^n (\|p\|_{r+1,\Omega_i} + \|\mathbf{u}\|_{r,\Omega_i} + \|\nabla \cdot \mathbf{u}\|_{r,\Omega_i}) h^{r+1}, \\
\|p - p_h\|_0 &\leq C \sum_{i=1}^n (\|p\|_{r+1,\Omega_i} + \|\mathbf{u}\|_{r,\Omega_i} + \|\nabla \cdot \mathbf{u}\|_{r,\Omega_i}) h^r,
\end{aligned}$$

where $1 \leq r \leq \min(k+1, l+1)$.

6. An interface operator. In this section we introduce a reduced problem involving only the mortar pressure. This reduced problem arose naturally in the work of Glowinski and Wheeler [17] on substructuring domain decomposition methods for mixed finite elements and is closely related to the inter-element multiplier formulation of Arnold and Brezzi [3]. The reason to consider the interface operator is twofold. First, we use it to derive a bound on the error in the mortar space. Second, it is the basis for our parallel domain decomposition implementation.

6.1. The reduced problem. Define a bilinear form $d_h : L^2(\Gamma) \times L^2(\Gamma) \rightarrow \mathbf{R}$ by

$$d_h(\lambda, \mu) = \sum_{i=1}^n d_{h,i}(\lambda, \mu) = - \sum_{i=1}^n \langle \mathbf{u}_h^*(\lambda) \cdot \nu_i, \mu \rangle_{\Gamma_i},$$

where for $\lambda \in L^2(\Gamma)$, $(\mathbf{u}_h^*(\lambda), p_h^*(\lambda)) \in \mathbf{V}_h \times W_h$ solve, for $1 \leq i \leq n$,

$$(K^{-1} \mathbf{u}_h^*(\lambda), \mathbf{v})_{\Omega_i} = (p_h^*(\lambda), \nabla \cdot \mathbf{v})_{\Omega_i} - \langle \lambda, \mathbf{v} \cdot \nu_i \rangle_{\Gamma_i}, \quad \mathbf{v} \in \mathbf{V}_{h,i}, \quad (6.1)$$

$$(\nabla \cdot \mathbf{u}_h^*(\lambda), w)_{\Omega_i} = 0, \quad w \in W_{h,i}. \quad (6.2)$$

Define a linear functional $g_h : L^2(\Gamma) \rightarrow \mathbf{R}$ by

$$g_h(\mu) = \sum_{i=1}^n g_{h,i}(\mu) = \sum_{i=1}^n \langle \bar{\mathbf{u}}_h \cdot \nu_i, \mu \rangle_{\Gamma_i},$$

where $(\bar{\mathbf{u}}_h, \bar{p}_h) \in \mathbf{V}_h \times W_h$ solve, for $1 \leq i \leq n$,

$$(K^{-1} \bar{\mathbf{u}}_h, \mathbf{v})_{\Omega_i} = (\bar{p}_h, \nabla \cdot \mathbf{v})_{\Omega_i} - \langle g, \mathbf{v} \cdot \nu_i \rangle_{\partial\Omega_i \setminus \Gamma}, \quad \mathbf{v} \in \mathbf{V}_{h,i}, \quad (6.3)$$

$$(\nabla \cdot \bar{\mathbf{u}}_h, w)_{\Omega_i} = (f, w)_{\Omega_i}, \quad w \in W_{h,i}. \quad (6.4)$$

It is straightforward to show (see [17]) that the solution $(\mathbf{u}_h, p_h, \lambda_h)$ of (2.7)–(2.9) satisfies

$$d_h(\lambda_h, \mu) = g_h(\mu), \quad \mu \in \Lambda_h, \quad (6.5)$$

with

$$\mathbf{u}_h = \mathbf{u}_h^*(\lambda_h) + \bar{\mathbf{u}}_h, \quad p_h = p_h^*(\lambda_h) + \bar{p}_h. \quad (6.6)$$

LEMMA 6.1. *The interface bilinear form $d_h(\cdot, \cdot)$ is symmetric and positive semi-definite on $L^2(\Gamma)$. If (2.11) holds, then $d_h(\cdot, \cdot)$ is positive definite on Λ_h .*

Proof. With $\mathbf{v} = \mathbf{u}_h^*(\mu)$ in (6.1) for some $\mu \in L^2(\Gamma)$, we have

$$d_{h,i}(\mu, \lambda) = -\langle \lambda, \mathbf{u}_h^*(\mu) \cdot \nu_i \rangle_{\Gamma_i} = (K^{-1} \mathbf{u}_h^*(\lambda), \mathbf{u}_h^*(\mu))_{\Omega_i} = d_{h,i}(\lambda, \mu), \quad (6.7)$$

which shows that $d_h(\cdot, \cdot)$ is symmetric and

$$d_{h,i}(\mu, \mu) = (K^{-1} \mathbf{u}_h^*(\mu), \mathbf{u}_h^*(\mu))_{\Omega_i} \geq 0. \quad (6.8)$$

For $\mu \in \Lambda_h$, if (2.11) holds, the argument from Lemma 2.1 shows that $d_h(\mu, \mu) = 0$ implies $\mu = 0$. \square

6.2. Error estimates for the mortar pressure. Denote by $\|\cdot\|_{d_h}$ the semi-norm induced by $d_h(\cdot, \cdot)$ on $L^2(\Gamma)$, i.e.,

$$\|\mu\|_{d_h} = d_h(\mu, \mu)^{1/2}, \quad \mu \in L^2(\Gamma).$$

THEOREM 6.2. *For the mortar pressure λ_h of the mixed method (2.7)–(2.9), if (3.18) holds, then there exists a positive constant C , independent of h , such that*

$$\|p - \lambda_h\|_{d_h} \leq C \sum_{i=1}^n (\|p\|_{r+1, \Omega_i} + \|\mathbf{u}\|_{r, \Omega_i}) h^r, \quad 1 \leq r \leq k+1, \quad (6.9)$$

$$\|\mathcal{P}_h p - \lambda_h\|_{d_h} \leq C \sum_{i=1}^n (\|p\|_{r+1, \Omega_i} + \|\mathbf{u}\|_{r, \Omega_i}) h^r, \quad 1 \leq r \leq k+1. \quad (6.10)$$

In the case of diagonal tensor K and RTN spaces on rectangular type grids,

$$\|p - \lambda_h\|_{d_h} \leq C \sum_{i=1}^n (\|p\|_{r+3/2, \Omega_i} + \|\mathbf{u}\|_{r+1/2, \Omega_i}) h^{r+1/2}, \quad 1 \leq r \leq k+1, \quad (6.11)$$

$$\|\mathcal{P}_h p - \lambda_h\|_{d_h} \leq C \sum_{i=1}^n (\|p\|_{r+3/2, \Omega_i} + \|\mathbf{u}\|_{r+1/2, \Omega_i}) h^{r+1/2}, \quad 1 \leq r \leq k+1. \quad (6.12)$$

Proof. With (6.8) we have

$$\|p - \lambda_h\|_{d_h} \leq C \|\mathbf{u}_h^*(p) - \mathbf{u}_h^*(\lambda_h)\|_0, \quad (6.13)$$

using that $\mathbf{u}_h^*(\cdot)$ depends linearly on its argument. Define, for $\mu \in L^2(\Gamma)$,

$$\mathbf{u}_h(\mu) = \mathbf{u}_h^*(\mu) + \bar{\mathbf{u}}_h, \quad p_h(\mu) = p_h^*(\mu) + \bar{p}_h,$$

and note that $(\mathbf{u}_h(\mu), p_h(\mu)) \in \mathbf{V}_h \times W_h$ satisfy, for $1 \leq i \leq n$,

$$\begin{aligned} (K^{-1} \mathbf{u}_h(\mu), \mathbf{v})_{\Omega_i} &= (p_h(\mu), \nabla \cdot \mathbf{v})_{\Omega_i} - \langle \mu, \mathbf{v} \cdot \nu \rangle_{\Gamma_i} \\ &\quad - \langle g, \mathbf{v} \cdot \nu \rangle_{\partial \Omega_i \setminus \Gamma}, \quad \mathbf{v} \in \mathbf{V}_{h,i}, \end{aligned} \quad (6.14)$$

$$(\nabla \cdot \mathbf{u}_h(\mu), w)_{\Omega_i} = (f, w)_{\Omega_i}, \quad w \in W_{h,i}. \quad (6.15)$$

We now have

$$\begin{aligned} \|\mathbf{u}_h^*(p) - \mathbf{u}_h^*(\lambda_h)\|_0 &= \|\mathbf{u}_h(p) - \mathbf{u}_h(\lambda_h)\|_0 \\ &= \|\mathbf{u}_h(p) - \mathbf{u}_h\|_0 \\ &\leq \|\mathbf{u}_h(p) - \mathbf{u}\|_0 + \|\mathbf{u} - \mathbf{u}_h\|_0 \end{aligned} \quad (6.16)$$

Bound (6.9) now follows from (6.13), (6.16), Theorem 4.2, and the standard mixed method estimate for (2.3)–(2.4) and (6.14)–(6.15) [25, 23, 12]

$$\|\mathbf{u}_h(p) - \mathbf{u}\|_{0,\Omega_i} \leq C(\|p\|_{r+1,\Omega_i} + \|\mathbf{u}\|_{r,\Omega_i})h^r, \quad 1 \leq r \leq k+1.$$

To show (6.11), we modify (6.16) as

$$\|\mathbf{u}_h(p) - \mathbf{u}_h\|_0 \leq \|\mathbf{u}_h(p) - \Pi \mathbf{u}\|_0 + \|\Pi \mathbf{u} - \Pi_0 \mathbf{u}\|_0 + \|\Pi_0 \mathbf{u} - \mathbf{u}_h\|_0. \quad (6.17)$$

Bound (6.11) now follows from (6.13), (6.17), (3.30), (4.5), and a superconvergence estimate for the standard mixed method [13] (see also [21, 15])

$$\|\mathbf{u}_h(p) - \Pi_i \mathbf{u}\|_{0,\Omega_i} \leq C \|\mathbf{u}\|_{r+1,\Omega_i} h^{r+1}, \quad 1 \leq r \leq k+1.$$

To prove (6.10) and (6.12), note that, by (6.1),

$$\begin{aligned} (K^{-1} \mathbf{u}_h^*(\mathcal{P}_h p - p), \mathbf{u}_h^*(\mathcal{P}_h p - p))_{\Omega_i} &= -\langle \mathcal{P}_h p - p, \mathbf{u}_h^*(\mathcal{P}_h p - p) \cdot \nu \rangle_{\Gamma_i} \\ &\leq \|\mathcal{P}_h p - p\|_{0,\Gamma_i} \|\mathbf{u}_h^*(\mathcal{P}_h p - p) \cdot \nu\|_{0,\Gamma_i} \\ &\leq C \|p\|_{r,\Gamma_i} h^r \|\mathbf{u}_h^*(\mathcal{P}_h p - p)\|_{0,\Omega_i} h^{-1/2}, \quad 0 \leq r \leq k+2, \end{aligned}$$

using (3.1) and Lemma 4.1 for the last inequality. Therefore, with (6.8),

$$\|\mathcal{P}_h p - p\|_{d_h} \leq C \sum_{i=1}^n \|p\|_{r+3/2,\Omega_i} h^{r+1/2}, \quad 0 \leq r \leq k+1. \quad (6.18)$$

Bounds (6.10) and (6.12) follow from (6.9) and (6.11), respectively, using the triangle inequality and (6.18). \square

REMARK 6.1. In the case of the lowest order RTN spaces, it is proven in [10] that, for any $\phi \in \Lambda_h$, $d_{h,i}(\phi, \phi)$ is equivalent to $|\mathcal{I}^{\partial \Omega_i} \mathcal{Q}_{h,i} \phi|_{1/2,\partial \Omega_i}$, where $\mathcal{I}^{\partial \Omega_i}$ is an interpolation operator onto the space of continuous piece-wise linears on $\partial \Omega_i$. Therefore $\|\cdot\|_{d_h}$ can be characterized as a certain discrete $H^{1/2}$ -norm on Γ (see [27]). This is also in accordance with the numerically observed $O(h^2)$ convergence for the mortars in a discrete L^2 -norm (see Section 8).

7. A substructuring domain decomposition algorithm. In this section we discuss the implementation of a parallel domain decomposition algorithm for solving the resulting linear system. We apply the substructuring algorithm by Glowinski and Wheeler [17] to the lowest order RTN discretization on non-matching multiblock rectangular type grids.

The original method for matching grids solves an interface problem in the space consisting of the normal traces of the velocity. In our case we solve an interface problem in the space of the mortar pressures. We use the conjugate gradient method to solve the interface problem (6.5). Note that Lemma 6.1 guarantees convergence of the iterative procedure in Λ_h .

Every iteration of the conjugate gradient requires an evaluation of the bilinear form $d_h(\cdot, \cdot)$, and therefore, solving subdomain problems (6.1)–(6.2) with a given Dirichlet data in the mortar space Λ_h . Because of the property

$$d_{h,i}(\lambda, \mu) = d_{h,i}(\mathcal{Q}_{h,i}\lambda, \mathcal{Q}_{h,i}\mu),$$

the subdomain solves only use projections of the mortar data onto the local spaces. Therefore, no change in the local solvers is needed for the implementation. Moreover, the conjugate gradient is performed in the space

$$\{(\phi_1, \phi_2) \in (L^2(\Gamma))^2 : \phi_1|_{\Gamma_{i,j}} \in \mathcal{Q}_{h,i}\Lambda_h \text{ and } \phi_2|_{\Gamma_{i,j}} \in \mathcal{Q}_{h,j}\Lambda_h, \ 1 \leq i < j \leq n\}.$$

The conjugate gradient residual is the jump in the fluxes across subdomain boundaries. The jump is computed after projecting the local boundary fluxes onto the mortar space, as indicated by

$$d_h(\lambda, \mu) = - \sum_{i=1}^n \langle \mathbf{u}_h^*(\lambda) \cdot \nu_i, \mu \rangle_{\Gamma_i} = - \sum_{i=1}^n \langle \mathcal{P}_h \mathbf{u}_h^*(\lambda) \cdot \nu_i, \mu \rangle_{\Gamma_i}, \quad \lambda, \mu \in \Lambda_h.$$

Therefore the only additional computational cost compared to the case of matching grids is computing the projections $\mathcal{Q}_{h,i} : \Lambda_h \rightarrow \mathbf{V}_{h,i} \cdot \nu_i$ and $\mathcal{P}_h : \mathbf{V}_{h,i} \cdot \nu_i \rightarrow \Lambda_h$.

8. Numerical results. In this section we present some numerical tests confirming the theoretical convergence rates. All examples are on the unit square and consider only the lowest order RTN spaces on rectangles. In the first example we solve a problem with known analytic solution

$$p(x, y) = x^3 y^4 + x^2 + \sin(xy) \cos(y)$$

and tensor coefficient

$$K = \begin{pmatrix} (x+1)^2 + y^2 & 0 \\ 0 & (x+1)^2 \end{pmatrix}.$$

The boundary conditions are Dirichlet on the left and right edge and Neumann on the rest of the boundary. The domain is divided into four subdomains with interfaces along the $x = 1/2$ and $y = 1/2$ lines. The initial non-matching grids are shown in Figure 8.1. We test both continuous and discontinuous mortars. The initial mortar grids on all interfaces have 4 elements with 5 degrees of freedom in the continuous case and 2 elements with 4 degrees of freedom in the discontinuous case, therefore satisfying the solvability condition (2.11). Convergence rates for the test case are given in Table 8.1. Here $\|\cdot\|_M$ is the discrete L^2 -norm induced by the midpoint rule

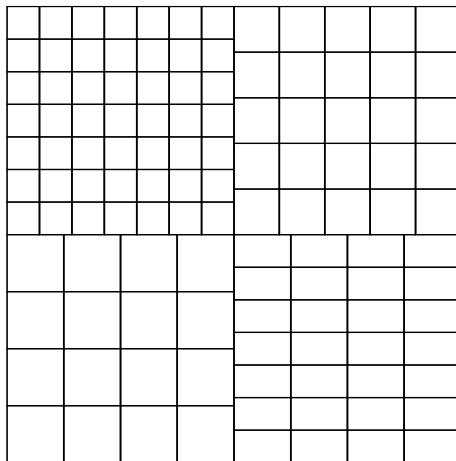


FIG. 8.1. Initial non-matching grids.

$1/h$	Continuous mortars			Discontinuous mortars		
	$E(p)$	$E(\mathbf{u})$	$E(\lambda)$	$E(p)$	$E(\mathbf{u})$	$E(\lambda)$
8	9.62E-03	2.95E-02	1.31E-02	9.52E-03	4.12E-02	1.36E-02
16	2.41E-03	8.54E-03	3.31E-03	2.40E-03	1.36E-02	3.45E-03
32	6.04E-04	2.42E-03	8.30E-04	6.03E-04	4.55E-03	8.68E-04
64	1.51E-04	6.66E-04	2.08E-04	1.51E-04	1.54E-03	2.19E-04
128	3.91E-05	1.88E-04	5.39E-05	3.75E-05	5.29E-04	5.35E-05
rate	$O(h^{1.99})$	$O(h^{1.83})$	$O(h^{1.99})$	$O(h^{2.00})$	$O(h^{1.57})$	$O(h^{2.00})$

TABLE 8.1

Discrete norm errors and convergence rates, where $E(p) = \|p - p_h\|_M$, $E(\mathbf{u}) = \|\mathbf{u} - \mathbf{u}_h\|$, and $E(\lambda) = \|\lambda - \mathcal{Q}_h \lambda_h\|_M$.

on \mathcal{T}_h (or the trace of \mathcal{T}_h on Γ). The rates were established by running the test case and 4 levels of grid refinement (we halve the element diameters for each refinement) and computing a least squares fit to the error. We observe numerically convergence rates corresponding to those predicted by the theory. The computed pressure and velocity with continuous and discontinuous mortars on the first level of refinement are shown in Figure 8.2. Although both solutions look the same, Table 8.1 indicates that they differ. This can also be seen in Figure 8.3, where the magnified numerical error is shown. The error in the continuous mortar case is concentrated at the cross-points, where the only discontinuities in the mortar space occur. The error in the discontinuous mortar case is distributed along the interfaces and is somewhat larger. We have to point out however, that the discontinuous mortars provide flux continuity in a more local sense, as indicated by the flux matching condition (2.9).

In the next example we test a problem with a discontinuous coefficient. We choose $K = I$ for $0 \leq x < 1/2$ and $K = 10 * I$ for $1/2 < x \leq 1$. The solution

$$p(x, y) = \begin{cases} x^2 y^3 + \cos(xy), & 0 \leq x \leq 1/2, \\ \left(\frac{2x+9}{20}\right)^2 y^3 + \cos\left(\frac{2x+9}{20}y\right), & 1/2 \leq x \leq 1, \end{cases}$$

is chosen to be continuous and have continuous normal flux at $x = 1/2$. The domain is divided into two subdomains with an interface along $x = 1/2$. The initial grids are 4×8 on the left and 4×11 on the right. Continuous mortars on a grid of 7 elements with 8 degrees of freedom or discontinuous mortars on a grid of 4 elements with 8

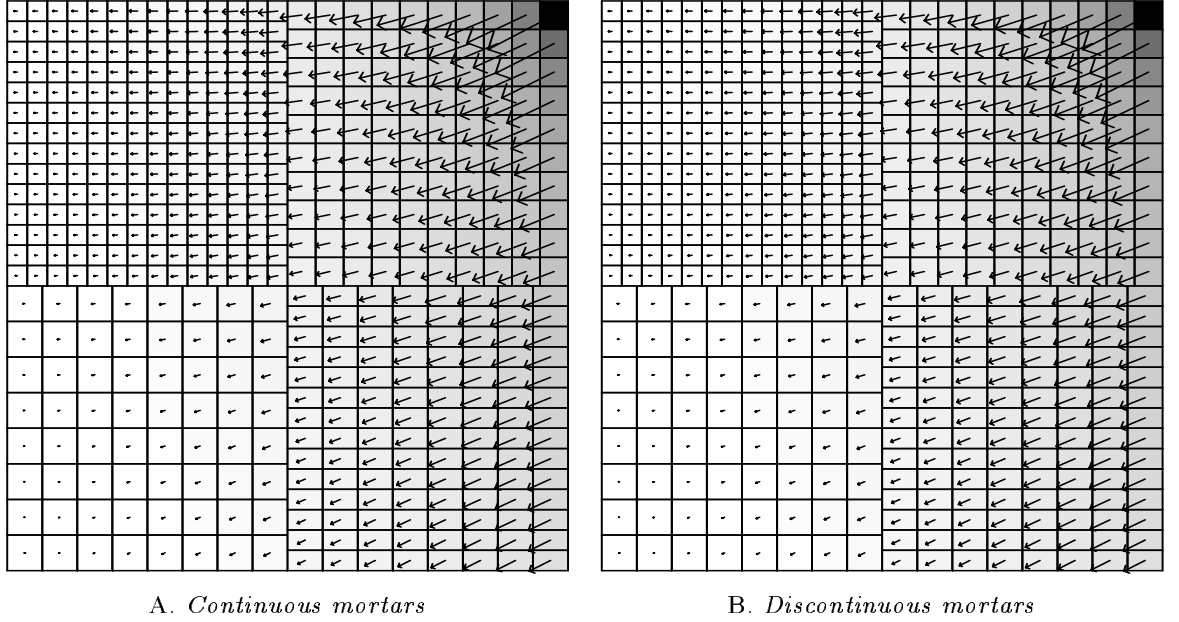


FIG. 8.2. Computed pressure (shade) and velocity (arrows).

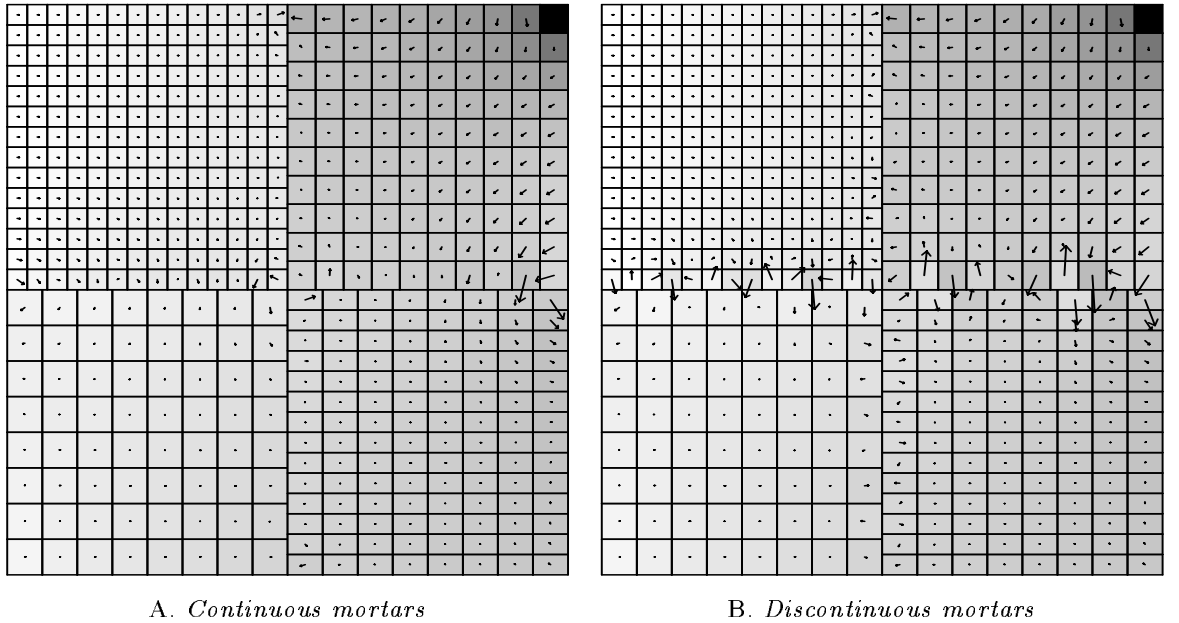


FIG. 8.3. Pressure and velocity error.

degrees of freedom are introduced on the interface. Convergence rates for the test case are given in Table 8.2.

In the last example we would like to compare the mortar element mixed method on locally refined grids to the more common “slave” or “worker” nodes local refinement technique [14, 16]. In the latter, the fine grid interface fluxes within a coarse cell are forced to be equal to the coarse grid flux. We note that this scheme can be recovered as a special case of the mortar element method with discontinuous mortars, if the trace of the fine grid is a refinement by two of the interface grid. Indeed, in this case the flux matching condition (2.9) becomes a local condition over two (four if $d = 3$) fine grid boundary elements and forces all fine grid fluxes to be equal to the coarse grid flux. Our theory also recovers the convergence and superconvergence results

$1/h$	Continuous mortars			Discontinuous mortars		
	$E(p)$	$E(\mathbf{u})$	$E(\lambda)$	$E(p)$	$E(\mathbf{u})$	$E(\lambda)$
8	3.20E-04	1.60E-02	3.32E-04	3.38E-04	3.27E-02	1.22E-03
16	8.38E-05	4.27E-03	8.18E-05	8.55E-05	1.13E-02	3.17E-04
32	2.12E-05	1.18E-03	2.01E-05	2.14E-05	3.93E-03	8.01E-05
64	5.35E-06	3.41E-04	4.89E-06	5.34E-06	1.37E-03	2.01E-05
128	1.38E-06	1.05E-04	1.15E-06	1.35E-06	4.82E-04	5.02E-06
rate	$O(h^{1.97})$	$O(h^{1.81})$	$O(h^{2.04})$	$O(h^{1.99})$	$O(h^{1.52})$	$O(h^{1.98})$

TABLE 8.2

Discrete norm errors and convergence rates for the example with a discontinuous coefficient, where $E(p) = \|p - p_h\|_M$, $E(\mathbf{u}) = \|\mathbf{u} - \mathbf{u}_h\|$, and $E(\lambda) = \|\lambda - \mathcal{Q}_h \lambda_h\|_M$.

derived by Ewing and Wang [16]. In the mortar method however, the flux continuity condition can be relaxed by choosing a coarser mortar space. In this case the fine grid fluxes are not forced to be equal and approximate the solution better. Our numerical experience shows that this may reduce the flux error on the interface by up to a factor of two.

We solve a problem with a solution

$$p(x, y) = x^3 y^2 + \sin(xy)$$

and a coefficient

$$K = \begin{pmatrix} 10 + 5\cos(xy) & 0 \\ 0 & 1 \end{pmatrix}.$$

on locally refined grids. The domain is divided into four subdomains with interfaces along the $x = 1/2$ and $y = 1/2$ lines. The domains are numbered starting from the lower left corner and first increasing x . The initial grids are 4×4 on Ω_1 – Ω_3 and 16×16 on Ω_4 . We use discontinuous piece-wise linear mortars on the non-matching interface. We report the numerical error on the grid and three levels of refinement for two cases. If the coarse grid is $n \times n$, we take a mortar grid with $n - 1$ elements in the first case and $2n$ elements in the second case, which is equivalent to the “slave” nodes method. The results are summarized in Table 8.3. The pressure and velocity error on the first level of refinement are shown in Figure 8.4.

$1/h$	Discontinuous mortars			“Slave” nodes		
	$E(p)$	$E(\mathbf{u})$	$E(\lambda)$	$E(p)$	$E(\mathbf{u})$	$E(\lambda)$
8	1.12E-3	6.70E-2	3.80E-3	1.30E-3	1.45E-1	5.74E-3
16	2.67E-4	2.48E-2	1.03E-3	2.90E-4	5.00E-2	1.39E-3
32	6.57E-5	9.77E-3	2.72E-4	6.86E-5	1.74E-2	3.41E-4
64	1.64E-5	3.62E-3	6.93E-5	1.66E-5	6.09E-3	8.42E-5
rate	$O(h^{2.03})$	$O(h^{1.40})$	$O(h^{1.93})$	$O(h^{2.09})$	$O(h^{1.52})$	$O(h^{2.03})$

TABLE 8.3

Discrete norm errors and convergence rates on locally refined grids, where $E(p) = \|p - p_h\|_M$, $E(\mathbf{u}) = \|\mathbf{u} - \mathbf{u}_h\|$, and $E(\lambda) = \|\lambda - \mathcal{Q}_h \lambda_h\|_M$.

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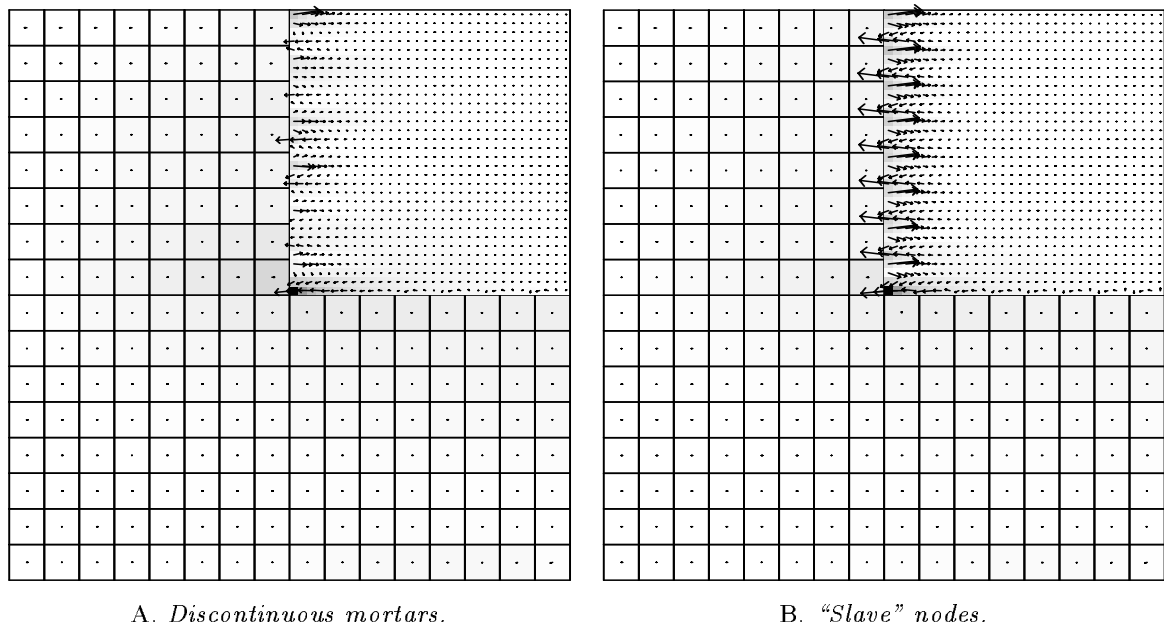


FIG. 8.4. Pressure and velocity error on locally refined grids.

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