A Two-grid Finite Difference Scheme for Non-linear Parabolic Equations

C. N. Dawson
M. F. Wheeler
C. S. Woodward

CRPC-TR96669
November 1996

Center for Research on Parallel Computation
Rice University
6100 South Main Street
CRPC - MS 41
Houston, TX 77005

A TWO–GRID FINITE DIFFERENCE SCHEME FOR NONLINEAR PARABOLIC EQUATIONS *

CLINT N. DAWSON †, CAROL S. WOODWARD ‡ AND MARY F. WHEELER †

Abstract. We present a two level finite difference scheme for the approximation of nonlinear parabolic equations. Discrete inner products and the lowest order Raviart-Thomas approximating space are used in the expanded mixed method in order to develop the finite difference scheme. Analysis of the scheme is given assuming an implicit time discretization. In this two level scheme, the full nonlinear problem is solved on a "coarse" grid of size H. The nonlinearities are expanded about the coarse grid solution and an appropriate interpolation operator is used to provide values of the coarse grid solution on the fine grid in terms of superconvergent node points. The resulting linear but nonsymmetric system is solved on a "fine" grid of size h. Some *a priori* error estimates are derived which show that the discrete $L^\infty(L^2)$ and $L^2(H^1)$ errors are $O(h^2 + H^{4-d/2} + \Delta t)$, where $d \geq 1$ is the spatial dimension.

Keywords: error estmates, finite differences, mixed finite elements, nonlinear, superconvergent
AMS(MOS) subject classification: 65M06, 65M12, 65M15, 65M55, 65M60, 35K55

1. Introduction. In this paper, we consider a finite difference scheme for the solution of the nonlinear parabolic differential equation

$$\frac{\partial p}{\partial t} - \nabla \cdot (K(\mathbf{x}, p)\nabla p) = f(t, \mathbf{x}) \text{ in } (0, T] \times \Omega, \tag{1}$$

$$p(0, \mathbf{x}) = p^0(\mathbf{x}) \text{ in } \Omega, \tag{2}$$

$$-(K(p)\nabla p) \cdot \nu = g \text{ on } (0, T] \times \Gamma, \tag{3}$$

where Ω is a rectangular domain in $\mathbb{R}^d$ ($d = 1$, 2 or 3) with boundary Γ, ν is the outward, unit, normal vector on Γ, and $K : \Omega \times \mathbb{R} \to \mathbb{R}^{d\times d}$ is a symmetric, positive definite second order diagonal tensor; that is, $K = \text{diag}(K_{ll})$, $l = 1, \ldots, d$.

Equation (1) arises in many applications. Our particular interest is to view (1) as a simplification of Richard's equation, a nonlinear parabolic equation arising in the modeling of flow through porous media.

To avoid time-step constraints, it is often preferable to solve (1) implicitly in time. However, for fine meshes, the resulting large systems of nonlinear equations can be costly to solve. In order to decrease the amount of work necessary to solve (1), we consider a two level method where the nonlinear problem is solved only on a coarse grid of diameter H and a linear problem is solved on a fine grid of diameter $h << H$. On the fine grid, we approximate $K(p)$ by a first order Taylor expansion about the solution from the coarse grid. Thus, instead of solving a large nonlinear problem on the fine grid, we solve a small nonlinear problem on the coarse grid and a large linearized problem on the fine grid.

This work is motivated by the work of Xu [11, 12] for Galerkin procedures applied to nonlinear elliptic equations and the work of Dawson and Wheeler [5] for the expanded mixed finite element method applied to nonlinear parabolic equations. Xu was the first to analyze two level methods applied to nonlinear differential equations. He showed optimal estimates in both H^1 and L^2 norms for both grids in the case of Galerkin finite element methods.

* This work was supported by the State of Texas, the United States Department of Energy and, in part, performed under the auspices of the U.S. Department of Energy by Lawrence Livermore National Laboratory under contract number W-7405-Eng-48.

† Texas Institute for Computational and Applied Mathematics, University of Texas, Austin, TX 78712
‡ Center for Applied Scientific Computing, Lawrence Livermore National Laboratory, Livermore, CA 94551

Dawson and Wheeler showed optimal H^1 and L^2 estimates for the coarse and fine grids, and for the case of the lowest order Raviart-Thomas space, they showed superconvergence results for the coarse grid in both norms. In this paper, we demonstrate superconvergence on the coarse grid for the lowest order Raviart-Thomas space with special quadrature rules, which give rise to a nonlinear cell-centered finite difference method on the coarse grid. Using these superconvergence estimates along with interpolation operators and quadrature, we are able to show superconvergence of the pressure and flux in discrete L^2 and H^1 norms also for the fine grid scheme. The fine grid scheme is a linear finite difference method. We consider finite differences in this paper since these schemes are of practical use in implementation.

Before analyzing the convergence of our two level method, we develop estimates of the expanded mixed finite element method with quadrature applied to a linear parabolic equation. Estimates for this method have been shown for linear elliptic equations in [1]. The estimates we derive show superconvergence of fluxes and pressures in discrete norms. We also bound the error in the time derivative of the solution. For the case of a diagonal, positive definite tensor, the scheme we analyze reduces to a cell-centered finite difference method, and our results are equivalent to those for the difference scheme [10]. This is the case we analyze here in the context of expanded mixed methods. However, our method does not require the inverse of the tensor K, and thus can be defined for the case of semi-definite K.

This paper is organized into five sections. We establish notation and some basic approximation results in Section 2. The coarse grid scheme is presented in Section 3 along with optimal order error estimates. The fine grid scheme with error estimates is presented in Section 4. In Section 5 we give conclusions, remarks and extensions of this work.

2. Notation and Approximation Results. In this section we define some notation. Let $0 = t^0 < t^1 < \cdots < t^N = T$ be a given sequence, $\Delta t^n = t^n - t^{n-1}$, $\Delta t = \max_n \Delta t^n$, and for $\phi = \phi(t,.)$, let $\phi^n = \phi(t^n,.)$ and

$$d_t\phi^n = \frac{\phi^n - \phi^{n-1}}{\Delta t^n}.$$

Let $L^p(\Omega)$ be the standard Banach space with norm,

$$\|w\|_{L^p(\Omega)} = \left(\int_\Omega |w|^p d\Omega\right)^{1/p}.$$

For simplicity, let $(.,.)$ denote the $L^2(\Omega)$ inner product, scalar and vector. Let $W_p^k(\Omega)$ be the standard Sobolev space

$$W_p^k(\Omega) = \{f \ : \ \|f\|_{W_p^k(\Omega)} < \infty\},$$

where,

$$\|f\|_{W_p^k(\Omega)} = \left(\sum_{|\alpha|\le k} \|D^\alpha f\|^p_{L^p(\Omega)}\right)^{1/p}.$$

Let $H_s(\Omega)$ for s a positive integer be the Sobolev space, $W_2^s(\Omega)$. Denote the inner product for the H_s Sobolev space as,

$$(f,g)_s = \sum_{|\alpha|\le s} \int_\Omega D^\alpha f \cdot D^\alpha g \; d\Omega,$$

where $f, g \in H_s(\Omega)$. Finally, we denote by $\mathcal{C}^{p,1}(\Omega)$ the space of functions whose p-th spatial derivative is Lipschitz continuous.

For $\mathcal{X}(\Omega)$ any of the above spaces and $[a,b] \subset [0,T]$, denote by $||\cdot||_{W^k_\infty(a,b;\mathcal{X})}$ the norm of $\mathcal{X}$-valued functions f with the map $t \to ||f(\cdot,t)||_{\mathcal{X}}$ belonging to $W^k_\infty(a,b)$.

Let $V = H(\Omega, div) = \{\mathbf{v} \in (L^2(\Omega))^d : \nabla \cdot \mathbf{v} \in L^2(\Omega)\}$ and $W = L^2(\Omega)$. We denote the subspaces of V containing functions with normal traces weakly equal to 0 and g^n as V^0 and V^n, respectively.

We will consider two quasi-uniform triangulations of Ω, a coarse triangulation with mesh size H denoted by $\mathcal{T}_H$, and a refinement of this triangulation with mesh size h denoted by $\mathcal{T}_h$. Both of these triangulations will consist of rectangles in two dimensions or bricks in three dimensions. We consider the lowest order Raviart-Thomas-Nedelec space on rectangles, [9, 7]. Thus, on an element $E \in \mathcal{T}_k, k = h$ or H, we have

$$
\begin{aligned}
V_k(E) = \{(\alpha_1 x_1 + \beta_1, \alpha_2 x_2 + \beta_2, \alpha_3 x_3 + \beta_3)^T : \alpha_i, \beta_i \in \mathrm{IR}\}, \\
W_k(E) = \{\alpha : \alpha \in \mathrm{IR}\},
\end{aligned}
$$

where the last component in V_k should be deleted in two dimensions. Define $V_k^0 = V^0 \cap V_k$ and $V_k^n = V^n \cap V_k$.

We use the standard nodal basis, where for V_k the nodes are at the midpoints of edges or faces of the elements, and for W_k the nodes are at the centers of the elements. We denote the grid points of the fine grid by

$$
(x_{i+1/2}, y_{j+1/2}), i = 0, \ldots, N_x, j = 0, \ldots, N_y,
$$

and define

$$
\begin{aligned}
x_i &= \frac{1}{2}(x_{i+1/2} + x_{i-1/2}), i = 1, \ldots, N_x, \\
y_j &= \frac{1}{2}(y_{j+1/2} + y_{j-1/2}), j = 1, \ldots, N_y, \\
h^x_{i+1/2} &= x_{i+1} - x_i, i = 1, \ldots, N_x - 1, \\
h^y_{j+1/2} &= y_{j+1} - y_j, j = 1, \ldots, N_y - 1, \\
h^x_i &= x_{i+1/2} - x_{i-1/2}, i = 1, \ldots, N_x, \\
h^y_j &= y_{j+1/2} - y_{j-1/2}, j = 1, \ldots, N_y, \\
h &= \max_{i,j}(h^x_i, h^y_j),
\end{aligned}
$$

with corresponding notation for a third dimension. For the coarse grid, similar quantities are defined, but the number of points in each direction are notated as $\hat{N}_x$ and $\hat{N}_y$.

We define discrete inner products corresponding to applications of the midpoint (M), trapezoidal (T) and midpoint by trapezoidal (TM) quadrature rules by

$$
\begin{aligned}
(r, s)_{\mathrm{M}} &= \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} h^x_i h^y_j r_{ij} s_{ij}, \\
(\mathbf{v}, \mathbf{q})_{\mathrm{TM}} &= \sum_{i=0}^{N_x} \sum_{j=1}^{N_y} h^x_{i+1/2} h^y_j v^x_{i+1/2 j} q^x_{i+1/2 j} + \sum_{i=1}^{N_x} \sum_{j=0}^{N_y} h^x_i h^y_{j+1/2} v^y_{ij+1/2} q^y_{ij+1/2},
\end{aligned}
$$

$$(\mathbf{v},\mathbf{q})_{\mathrm{T}} = \sum_{i=0}^{N_x}\sum_{j=1}^{N_y} h^x_{i+1/2}h^y_j \frac{1}{2}(v^x_{i+1/2j-1/2}q^x_{i+1/2j-1/2} + v^x_{i+1/2j+1/2}q^x_{i+1/2j+1/2})$$
$$+\sum_{i=1}^{N_x}\sum_{j=0}^{N_y} h^x_i h^y_{j+1/2}\frac{1}{2}(v^y_{i-1/2j+1/2}q^y_{i-1/2j+1/2} + v^y_{i+1/2j+1/2}q^y_{i+1/2j+1/2}),$$

where we add a third sum in each for the case of three dimensions. We denote the associated norms by $\|.\|_{\mathrm{R}}$, where R $=$ M, T or TM and by $E_R(\mathbf{q},\mathbf{r})$, the error in approximating an integral by the given rule, i.e. $E_T(\mathbf{q},\mathbf{r}) = (\mathbf{q},\mathbf{r}) - (\mathbf{q},\mathbf{r})_T$. The error in approximating an integral by either the trapezoidal or the trapezoidal by midpoint rule is [4],

$$|E_{\mathrm{Q}}(\mathbf{q},\mathbf{v})| \le C\sum_{E\in\mathcal{T}_h}\sum_{|\alpha|=2}\|\frac{\partial^\alpha}{\partial\mathbf{x}^\alpha}(\mathbf{q}\cdot\mathbf{v})\|_{L^1(E)}h^2. \tag{4}$$

For any $\phi\in L^2(\Omega)$ we let $\hat{\phi}_k$ denote the L^2 projection of ϕ onto W_k, i.e.

$$(\phi,w) = (\hat{\phi}_k,w),\ \forall w\in W_k. \tag{5}$$

This L^2 projection operator has the following approximation property for $\phi\in H^j(\Omega)$,

$$\|\hat{\phi}_k-\phi\| \le C\|\phi\|_j k^j,\ 0\le j\le 1, \tag{6}$$

for $k=h$ or H.

Associated with the RTN mixed finite element spaces is the projection operator Π : $(H^1(\Omega))^d\to V_k$, defined by,

$$(\nabla\cdot\Pi\mathbf{q},w) = (\nabla\cdot\mathbf{q},w),\ \forall w\in W_k, \tag{7}$$

with approximation properties,

$$\|\mathbf{q}-\Pi\mathbf{q}\| \le C\|\mathbf{q}\|_1 k, \tag{8}$$
$$\|\nabla\cdot(\mathbf{q}-\Pi\mathbf{q})\| \le C\|\nabla\cdot\mathbf{q}\|_1 k. \tag{9}$$

Furthermore, by the definition of $\Pi\mathbf{q}$ and the midpoint rule of integration, we have that the error in the first component of the projection evaluated at the center of a gridblock side is given by,

$$|(\Pi q)^x - q^x|(x_{i+1/2},y_j) \le Ch^2\|q^x\|_{W_2^\infty(\Omega)}. \tag{10}$$

Using this estimate, we can bound the L^∞ norm of the projection by,

$$\|\Pi\mathbf{q}-\mathbf{q}\|_{L^\infty(\Omega)} \le Ch\|\mathbf{q}\|_{W_2^\infty(\Omega)}. \tag{11}$$

In the expanded mixed formulation of (1), we define the variables $\tilde{\mathbf{u}} = -\nabla p$ and $\mathbf{u} = K(p)\tilde{\mathbf{u}}$. The following analysis will use the estimate [6]

$$\|\Pi\mathbf{u}^n-\mathbf{u}^n\|_{\mathrm{TM}} + \|\Pi\tilde{\mathbf{u}}^n-\tilde{\mathbf{u}}^n\|_{\mathrm{TM}} \le Ck^2\|\tilde{\mathbf{u}}\|_2, \tag{12}$$

where, again, we let $k=h$ or H.

We will also make use of the following lemma proven in Arbogast, Wheeler and Yotov [1].

LEMMA 2.1. *For the lowest order RTN spaces on rectangles, for any* $\mathbf{q} = (q^x, q^y) \in H^1(\Omega)$ *and* $E \in \mathcal{T}_k$,

$$\left\| \frac{\partial}{\partial x}(\Pi \mathbf{q})^x \right\|_{L^2(E)} \leq \left\| \frac{\partial q^x}{\partial x} \right\|_{L^2(E)}, \tag{13}$$

$$\left\| \frac{\partial}{\partial y}(\Pi \mathbf{q})^y \right\|_{L^2(E)} \leq \left\| \frac{\partial q^y}{\partial y} \right\|_{L^2(E)}. \tag{14}$$

In the following arguments, C will represent a generic constant independent of H, h and Δt. We will use the standard inequality,

$$ab \leq \frac{\delta}{2}a^2 + \frac{1}{2\delta}b^2 \; a, b, \delta \in \mathbb{R}, \; \delta > 0. \tag{15}$$

3. A Coarse Grid Nonlinear Finite Difference Scheme. In this section we develop and give convergence estimates for a nonlinear cell-centered finite difference scheme on the coarse grid. For simplicity we consider two dimensions and note that extensions to three dimensions are straightforward.

3.1. Definition of the Scheme. The variational formulation of (1) at time t^n uses the auxiliary variables, $\tilde{\mathbf{u}}^n$ and $\mathbf{u}^n$ defined by,

$$\tilde{\mathbf{u}}^n \equiv -\nabla p^n,$$
$$\mathbf{u}^n \equiv K(p^n)\tilde{\mathbf{u}}^n.$$

Thus, our problem is to find $(p^n, \tilde{\mathbf{u}}^n, \mathbf{u}^n) \in (W \times V \times V^n)$ satisfying

$$(p_t^n, w) + (\nabla \cdot \mathbf{u}^n, w) = (f^n, w), \; \forall w \in W, \tag{16}$$

$$(\tilde{\mathbf{u}}^n, \mathbf{v}) = (p^n, \nabla \cdot \mathbf{v}), \; \forall \mathbf{v} \in V^0, \tag{17}$$

$$(\mathbf{u}^n, \mathbf{v}) = (K(p^n)\tilde{\mathbf{u}}^n, \mathbf{v}), \; \forall \mathbf{v} \in V, \tag{18}$$

We choose cell-centered finite difference approximations $P_H^n \in W_H, \tilde{\mathbf{U}}_H^n \in V_H$ and $\mathbf{U}_H^n \in V_H^n$ to the functions $p(t^n, .), \tilde{\mathbf{u}}(t^n, .)$ and $\mathbf{u}(t^n, .)$, respectively, for each $n = 1, \ldots, N$, satisfying

$$(d_t P_H^n, w) + (\nabla \cdot \mathbf{U}_H^n, w) = (f^n, w), \; \forall w \in W_H, \tag{19}$$

$$(\tilde{\mathbf{U}}_H^n, \mathbf{v})_{\mathrm{TM}} = (P_H^n, \nabla \cdot \mathbf{v}), \; \forall \mathbf{v} \in V_H^0, \tag{20}$$

$$(\mathbf{U}_H^n, \mathbf{v})_{\mathrm{TM}} = (K(\mathcal{P}_H(P_H^n))\tilde{\mathbf{U}}_H^n, \mathbf{v})_{\mathrm{T}}, \; \forall \mathbf{v} \in V_H, \tag{21}$$

and we take $P_H^0 = \hat{p}_H(t^0, .)$. This scheme is based on an expansion of the standard mixed finite element method that was formulated for linear elliptic problems in [1].

We define $\mathcal{P}_H(p)$ from the values of p_{ij} for $i = 1, \ldots, \hat{N}_x$ and $j = 1, \ldots, \hat{N}_y$ as follows. For points (x, y) such that $x_i \leq x \leq x_{i+1}, i \in \{1, \ldots, \hat{N}_x\}$ and $y_j \leq y \leq y_{j+1}, j \in \{1, \ldots, \hat{N}_y\}$, we take $\mathcal{P}_H(p)(x, y)$ to be the bilinear interpolant,

$$\begin{aligned}\mathcal{P}_H(p)(x, y) = &(p_{ij}(\frac{x_{i+1} - x}{x_{i+1} - x_i}) + p_{i+1j}(\frac{x - x_i}{x_{i+1} - x_i}))(\frac{y_{j+1} - y}{y_{j+1} - y_j}) \\ &+(p_{ij+1}(\frac{x_{i+1} - x}{x_{i+1} - x_i}) + p_{i+1j+1}(\frac{x - x_i}{x_{i+1} - x_i}))(\frac{y - y_j}{y_{j+1} - y_j}).\end{aligned}$$

For $i = 1, \ldots, \hat{N}_x - 1$, we set

$$\mathcal{P}_H(p)(x_i, y_{1/2}) = \frac{(2H_1^y + H_2^y)p_{i1} - H_1^y p_{i2}}{H_1^y + H_2^y}.$$

This is a two point extrapolation, and by Taylor's theorem we have $|(\mathcal{P}_H(p) - p)(x_i, y_{1/2})| = CH^2$. For points (x, y) such that $x_i \leq x \leq x_{i+1}$ and $y_{1/2} \leq y \leq y_1$, we define $\mathcal{P}_H(p)$ as the bilinear interpolant between $p_{i,1}, p_{i+1,1}, \mathcal{P}_H(p)(x_i, y_{1/2})$ and $\mathcal{P}_H(p)(x_{i+1}, y_{1/2})$. By interpolation theory $|\mathcal{P}_H(p) - p| = CH^2$ for these points. In a similar way we can define $\mathcal{P}_H(p)$ for (x, y) such that $x_i \leq x \leq x_{i+1}$ and $y_{\hat{N}_y} \leq y \leq y_{\hat{N}_y+1/2}$ as well as for points (x, y) where $x_{1/2} \leq x \leq x_1$ or $x_{\hat{N}_x} \leq x \leq x_{\hat{N}_x+1/2}$ and $y_j \leq y \leq y_{j+1}$ for j such that $1 \leq j \leq \hat{N}_y$. Lastly, we define $\mathcal{P}_H(p)$ at the corners of the domain. Here, we use three point extrapolation,

$$\begin{aligned}\mathcal{P}_H(p)(x_{1/2}, y_{1/2}) &= \mathcal{P}_H(p)_{1,1/2} + \mathcal{P}_H(p)_{1/2,1} - p_{1,1} \\ &= p_{1,1/2} + p_{1/2,1} - p_{1,1} + O(H^2).\end{aligned}$$

By Taylor's theorem, $|(\mathcal{P}_H(p)-p)(x_{1/2}, y_{1/2})| \leq CH^2$. For points (x, y) such that $x_{1/2} \leq x \leq x_1$ and $y_{1/2} \leq y \leq y_1$, we define $\mathcal{P}_H(p)(x, y)$ as the bilinear interpolant of $\mathcal{P}_H(p)(x_{1/2}, y_{1/2})$, $\mathcal{P}_H(p)(x_{1/2}, y_1)$, $\mathcal{P}_H(p)(x_1, y_{1/2})$ and $p_{1,1}$ which is an $O(H^2)$ approximation to $p(x, y)$ within this "corner region". Similarly, we can define $\mathcal{P}_H(p)$ as an $O(H^2)$ approximation to p in the other three "corner" regions.

We summarize the above in the following lemma.

Lemma 3.1. *If p is twice differentiable in space, then for $\mathcal{P}_H(p)$ defined above,*

$$\|\mathcal{P}_H(p) - p\|_\infty \leq CH^2.$$

If a uniform mesh is used and K is a diagonal tensor, equations (19)-(21) reduce to a standard nonlinear finite difference procedure. Denoting P_H^n by P^n, we have in the interior of Ω:

$$\begin{aligned}\int_{\Omega_{ij}} f^n dx + P_{ij}^{n-1}\frac{H^2}{\Delta t} = \frac{1}{2}[&(K_{11}(\mathcal{P}_H(P^n))_{i+1/2j+1/2} + K_{11}(\mathcal{P}_H(P^n))_{i+1/2j-1/2})(P_{ij}^n - P_{i+1j}^n) \\ &+(K_{11}(\mathcal{P}_H(P^n))_{i-1/2j+1/2} + K_{11}(\mathcal{P}_H(P^n))_{i-1/2j-1/2})(P_{ij}^n - P_{i-1j}^n) \\ &+(K_{22}(\mathcal{P}_H(P^n))_{i+1/2j+1/2} + K_{22}(\mathcal{P}_H(P^n))_{i-1/2j+1/2})(P_{ij}^n - P_{ij+1}^n) \\ &+(K_{22}(\mathcal{P}_H(P^n))_{i+1/2j-1/2} + K_{22}(\mathcal{P}_H(P^n))_{i-1/2j-1/2})(P_{ij}^n - P_{ij-1}^n)] \\ &+\frac{H^2}{\Delta t}P_{ij}^n.\end{aligned}$$

Existence and uniqueness of a solution to this discrete nonlinear problem is given in the following theorem.

Theorem 3.2. *Assume $f^n \in L^2(\Omega)$ for each n and K is continuously differentiable in its arguments. Then, for Δt sufficiently small, there exists a unique solution to equations (19)-(21).*

Proof. We are seeking a unique solution to the nonlinear equation $F(P^n) = 0$, where $F(P^n) = b^n + P^n + \frac{\Delta t}{H^2}A(P^n)P^n$. Here, b^n is a vector whose entry corresponding to grid cell (x_i, y_j) is $-\frac{\Delta t}{H^2}\int_{\Omega_{ij}} f_{ij}^n - P_{ij}^{n-1}$, P is a vector whose ijth entry corresponds to the value of the

scalar variable P_{ij}^n and A is a matrix function of P^n given by the stencil above. By Theorem 5.4.5 of Ortega and Reinbolt [8], if F is continuously differentiable and uniformly monotone on $\mathbb{R}^n$, then a unique solution to $F(P^n)=0$ exists. It is easily verified that the F defined above is continuously differentiable. In order to prove that F is uniformly monotonic we note that uniform monotoncity is equivalent to positive definiteness of the Jacobian, $J=F'$, and that a real matrix J is positive definite if and only if its symmetric part, $(J+J^T)/2$, is positive definite [2, Lemma 3.1]. Furthermore, we know that if a matrix is stricly diagonal dominant with positive diagonal entries, then the eigenvalues of the matrix have positive real parts [2, Theorem 4.9]. Now, $J=I+\frac{\Delta t}{H^2}A(P^n)+\frac{\Delta t}{H^2}A'(P^n)P^n$. Thus, with $\frac{\Delta t}{H^2}$ sufficiently small, we have that the symmetric part of J has positive real eigenvalues and, hence, is positive definite, making J positive definite and F uniformly monotonic. $\square$

3.2. Preliminary Estimates. Before we show convergence estimates for this finite difference scheme, we show convergence for a related linear scheme. The arguments given below closely follow those of Arbogast, Wheeler and Yotov [1] except that we extend their work to time differenced time dependent problems. In order to derive these estimates we make the following smoothness assumptions:

(S1) $f\in W^1_\infty(0,T;L^2(\Omega))$,

(S2) $K_{ll}(\mathbf{x},p)\in\mathcal{C}^1(\bar\Omega\times\mathbb{R})\cap W^2_\infty(\Omega\times\mathbb{R})$, $l=1,\ldots,d$, K_{ll} and $\frac{\partial K_{ll}}{\partial p}$ are uniformly Lipschitz functions of p.

(S3) There exist positive constants K_* and K^* such that for $z\in\mathbb{R}^d$,

$$K_*\|z\|^2\le z^tK(\mathbf{x},p)z\le K^*\|z\|^2,\ \text{for } \mathbf{x}\in\Omega,\ p\in\mathbb{R}. \tag{22}$$

(S4) $p\in W^2_\infty(0,T;\mathcal{C}^{3,1}(\Omega))$,

(S5) $u,\tilde u\in W^2_\infty(0,T;\mathcal{C}^1(\bar\Omega))^d\cap W^2_\infty(0,T;W^2_\infty(\Omega))^d$.

THEOREM 3.3. *For each $n=1,\ldots,N$, let $(\underline{P}_H^n,\underline{\tilde{\mathbf{U}}}_H^n,\underline{\mathbf{U}}_H^n)\in(W_H\times V_H\times V_H^n)$ satisfy*

$$(\nabla\cdot\underline{\mathbf{U}}_H^n,w)=(b^n,w),\ \forall w\in W_H, \tag{23}$$

$$(\underline{\tilde{\mathbf{U}}}_H^n,\mathbf{v})_{\mathrm{TM}}=(\underline{P}_H^n,\nabla\cdot\mathbf{v}),\ \forall\mathbf{v}\in V_H^0, \tag{24}$$

$$(\underline{\mathbf{U}}_H^n,\mathbf{v})_{\mathrm{TM}}=(K(\mathcal{P}_H(p^n))\underline{\tilde{\mathbf{U}}}_H^n,\mathbf{v})_{\mathrm{T}},\ \forall\mathbf{v}\in V_H, \tag{25}$$

with $b^n=f^n-p_t^n$ and $\underline{P}_H^0=\hat p_H^0$. Then, under the assumptions (S1)-(S5),

$$\|\underline{\mathbf{U}}_H^n-\mathbf{u}^n\|_{\mathrm{TM}}+\|\underline{\tilde{\mathbf{U}}}_H^n-\tilde{\mathbf{u}}^n\|_{\mathrm{TM}}\le CH^2, \tag{26}$$

$$\|\underline{P}_H^n-p^n\|_{\mathrm{M}}\le CH^2, \tag{27}$$

$$\|d_t\underline{P}_H^n-d_tp^n\|_{\mathrm{M}}\le C(H^2+\Delta t). \tag{28}$$

In order to prove this theorem, we will first prove two preliminary lemmas.

LEMMA 3.4. *There exist $\tilde{\mathbf{U}}^{*,n}\in V_H, P^{*,n}\in W_H, \mathbf{Z}^{*,n}\in V_H^n, \tilde{\mathbf{Z}}^{*,n}\in V_H$ and $W^{*,n}\in W_H$ such that*

$$(\tilde{\mathbf{U}}^{*,n},\mathbf{v})_{\mathrm{TM}}=(P^{*,n},\nabla\cdot\mathbf{v}),\ \forall\mathbf{v}\in V_H^0, \tag{29}$$

$$(\tilde{\mathbf{Z}}^{*,n},\mathbf{v})_{\mathrm{TM}}=(W^{*,n},\nabla\cdot\mathbf{v}),\ \forall\mathbf{v}\in V_H^0, \tag{30}$$

$$(\mathbf{Z}^{*,n},\mathbf{v})_{\mathrm{TM}}=(K(p^n)\tilde{\mathbf{Z}}^{*,n},\mathbf{v})_{\mathrm{T}}+(K(p^n)_t\tilde{\mathbf{U}}^{*,n},\mathbf{v})_{\mathrm{T}},\ \forall\mathbf{v}\in V_H, \tag{31}$$

and

$$|P^{*,n}_{i,j} - p^n_{i,j}| \le CH^2, \tag{32}$$

$$|W^{*,n}_{i,j} - p^n_{t,i,j}| \le CH^2, \tag{33}$$

$$|\tilde{U}^{*,n}_{x,i+1/2j} - \tilde{u}^n_{x,i+1/2j}| + |\tilde{U}^{*,n}_{y,ij+1/2} - \tilde{u}^n_{y,ij+1/2}| \le CH^2, \tag{34}$$

$$|\tilde{Z}^{*,n}_{x,i+1/2j} - \tilde{u}^n_{t,x,i+1/2j}| + |\tilde{Z}^{*,n}_{y,ij+1/2} - \tilde{u}^n_{t,y,ij+1/2}| \le CH^2, \tag{35}$$

$$|Z^{*,n}_{x,i+1/2j} - u^n_{t,x,i+1/2j}| + |Z^{*,n}_{y,ij+1/2} - u^n_{t,y,ij+1/2}| \le CH^2. \tag{36}$$

Here the subscript t denotes time differentiation, and the non-bold x and y subscripted variables denote x and y vector components.

Proof. Arbogast, Wheeler and Yotov [1] present a lemma which gives the desired $P^{*,n}$ and $\tilde{\mathbf{U}}^{*,n}$ above. In order to derive (33) and (35), we apply a lemma due to Weiser and Wheeler [10] to the solution pair $(\tilde{\mathbf{u}}^n_t, p^n_t)$ satisfying the elliptic problem

$$\begin{aligned} \nabla \cdot \tilde{\mathbf{u}}^n_t &= F^n, \text{in } \Omega, \\ \tilde{\mathbf{u}}^n_t &= -\nabla p^n_t, \text{in } \Omega, \\ p^n_t = G^n &\equiv \frac{\partial p^n}{\partial t}|_{\partial\Omega}, \text{on } \partial\Omega, \end{aligned}$$

where $F^n = f^n_t + p^n_{tt}$. This result gives a $W^{*,n}$ satisfying (33) and through (30), $\tilde{\mathbf{Z}}^{*,n}$ satisfies (35) in the interior of Ω. Define $\tilde{\mathbf{Z}}$ on Γ by,

$$\begin{aligned} \tilde{Z}^{*,n}_{x,i+1/2j} &= \tilde{u}^n_{t,x,i+1/2j}, \\ \tilde{Z}^{*,n}_{y,ij+1/2} &= \tilde{u}^n_{t,y,ij+1/2}. \end{aligned}$$

Then, (35) clearly holds on Γ.

Choosing $\mathbf{v}$ in (31) to be the basis function associated with node $(x_{i+1/2}, y_j)$, we have for $i = 1, \ldots, \hat{N}_x - 1$,

$$\begin{aligned} \mathbf{Z}^{*,n}_{x,i+1/2j} = {} & \frac{1}{2}[K_{11}(p^n)_{i+1/2j+1/2} + K_{11}(p^n)_{i+1/2j-1/2}]\tilde{\mathbf{Z}}^{*,n}_{x,i+1/2j} \\ & + \frac{1}{2}[K_{11}(p^n)_{t,i+1/2j+1/2} + K_{11}(p^n)_{t,i+1/2j-1/2}]\tilde{\mathbf{U}}^{*,n}_{x,i+1/2j}. \end{aligned}$$

Since $\mathbf{u}^n_t = K(p^n)\tilde{\mathbf{u}}^n_t + K(p^n)_t\tilde{\mathbf{u}}^n$, Taylor's theorem gives for $i = 1, \ldots, \hat{N}_x - 1$,

$$\begin{aligned} \mathbf{u}^n_{t,x,i+1/2j} = {} & \frac{1}{2}[K_{11}(p^n)_{i+1/2j+1/2} + K_{11}(p^n)_{i+1/2j-1/2}]\tilde{\mathbf{u}}^n_{t,x,i+1/2j} \\ & + \frac{1}{2}[K_{11}(p^n)_{t,i+1/2j+1/2} + K_{11}(p^n)_{t,i+1/2j-1/2}]\tilde{\mathbf{u}}^n_{x,i+1/2j} + O(H^2). \end{aligned}$$

Therefore,

$$|\mathbf{Z}^{*,n}_{x,i+1/2j} - \mathbf{u}^n_{t,x,i+1/2j}| \le C|\tilde{\mathbf{Z}}^{*,n}_{x,i+1/2j} - \tilde{\mathbf{u}}^n_{t,x,i+1/2j}| + O(H^2).$$

In a similar manner we can bound $|\mathbf{Z}^{*,n}_{y,ij+1/2} - \mathbf{u}^n_{t,y,ij+1/2}|$, and (36) follows. $\square$

We can now extend a corollary from Arbogast, Wheeler and Yotov [1] to give: For the $\tilde{\mathbf{U}}^{*,n}$, $P^{*,n}$, $\mathbf{Z}^{*,n}$, $\tilde{\mathbf{Z}}^{*,n}$ and $W^{*,n}$ in Lemma 3.4, there exists a constant C, independent of H, such that

$$\begin{aligned}\|\tilde{\mathbf{U}}^{*,n} - \tilde{\mathbf{u}}^n\|_{\mathrm{TM}} &\leq CH^2,\\ \|\tilde{\mathbf{Z}}^{*,n} - \tilde{\mathbf{u}}_t^n\|_{\mathrm{TM}} &\leq CH^2,\\ \|\mathbf{Z}^{*,n} - \mathbf{u}_t^n\|_{\mathrm{TM}} &\leq CH^2.\end{aligned}$$

LEMMA 3.5. *There exists a constant C independent of H and Δt such that*

$$\|\nabla\cdot(d_t\mathbf{u}^n - d_t\underline{\mathbf{U}}_H^n)\| \leq CH, \tag{37}$$

$$\|d_t\tilde{\mathbf{u}}^n - d_t\underline{\tilde{\mathbf{U}}}_H^n\|_{\mathrm{TM}} + \|d_t\mathbf{u}^n - d_t\underline{\mathbf{U}}_H^n\|_{\mathrm{TM}} \leq C(H^2+\Delta t). \tag{38}$$

Proof. To prove this lemma, we consider the time difference of (16)-(18),

$$(\nabla\cdot d_t\mathbf{u}^n, w) = (d_tb^n, w),\ \forall w\in W, \tag{39}$$

$$(d_t\tilde{\mathbf{u}}^n, \mathbf{v}) = (d_tp^n, \nabla\cdot\mathbf{v}),\ \forall \mathbf{v}\in V^0, \tag{40}$$

$$(d_t\mathbf{u}^n, \mathbf{v}) = (d_t(K(p^n))\tilde{\mathbf{u}}^n, \mathbf{v}) + (K(p^{n-1})d_t\tilde{\mathbf{u}}^n, \mathbf{v}),\ \forall\mathbf{v}\in V, \tag{41}$$

and the time difference of (23)-(25),

$$(\nabla\cdot d_t\underline{\mathbf{U}}_H^n, w) = (d_tb^n, w),\ \forall w\in W_H, \tag{42}$$

$$(d_t\underline{\tilde{\mathbf{U}}}_H^n, \mathbf{v})_{\mathrm{TM}} = (d_t\underline{P}_H^n, \nabla\cdot\mathbf{v}),\ \forall\mathbf{v}\in V_H^0, \tag{43}$$

$$(d_t\underline{\mathbf{U}}_H^n, \mathbf{v})_{\mathrm{TM}} = (d_t(K(\mathcal{P}_H(p^n)))\underline{\tilde{\mathbf{U}}}_H^n, \mathbf{v})_{\mathrm{T}} + (K(\mathcal{P}_H(p^{n-1}))d_t\underline{\tilde{\mathbf{U}}}_H^n, \mathbf{v})_{\mathrm{T}},\ \forall\mathbf{v}\in V_H. \tag{44}$$

We subtract (42) from (39), and we subtract (43) and (44) from (30) and (31) to give

$$(\nabla\cdot(d_t\mathbf{u}^n - d_t\underline{\mathbf{U}}_H^n), w) = 0,\ \forall w\in W_H, \tag{45}$$

$$(\tilde{\mathbf{Z}}^{*,n} - d_t\underline{\tilde{\mathbf{U}}}_H^n, \mathbf{v})_{\mathrm{TM}} = (W^{*,n} - d_t\underline{P}_H^n, \nabla\cdot\mathbf{v}),\ \forall\mathbf{v}\in V_H^0, \tag{46}$$

$$\begin{aligned}(\mathbf{Z}^{*,n} - d_t\underline{\mathbf{U}}_H^n, \mathbf{v})_{\mathrm{TM}} &= (K(p^n)\tilde{\mathbf{Z}}^{*,n} - K(\mathcal{P}_H(p^{n-1}))d_t\underline{\tilde{\mathbf{U}}}_H^n, \mathbf{v})_{\mathrm{T}}\\ &\quad + (K(p^n)_t\tilde{\mathbf{U}}^{*,n} - d_t(K(\mathcal{P}_H(p^n)))\underline{\tilde{\mathbf{U}}}_H^n, \mathbf{v})_{\mathrm{T}},\ \forall\mathbf{v}\in V_H.\end{aligned} \tag{47}$$

Using (45) and applying the Cauchy-Schwarz inequality we have,

$$\begin{aligned}\|\nabla\cdot(d_t\mathbf{u}^n - d_t\underline{\mathbf{U}}_H^n)\|^2 &= (\nabla\cdot(d_t\mathbf{u}^n - d_t\underline{\mathbf{U}}_H^n), \nabla\cdot(d_t\mathbf{u}^n - \Pi d_t\mathbf{u}^n))\\ &\leq \|\nabla\cdot(d_t\mathbf{u}^n - d_t\underline{\mathbf{U}}_H^n)\|\|\nabla\cdot(d_t\mathbf{u}^n - \Pi d_t\mathbf{u}^n)\|.\end{aligned}$$

Thus, by (9) the first part of the lemma is obtained.

Now, let $\mathbf{v} = \Pi d_t\mathbf{u}^n - d_t\underline{\mathbf{U}}_H^n$ in (46) and $\mathbf{v} = \tilde{\mathbf{Z}}^{*,n} - d_t\underline{\tilde{\mathbf{U}}}_H^n$ in (47), use (45) and combine to get

$$\begin{aligned}&(K(p^n)\tilde{\mathbf{Z}}^{*,n} - K(\mathcal{P}_H(p^{n-1}))d_t\underline{\tilde{\mathbf{U}}}_H^n, \tilde{\mathbf{Z}}^{*,n} - d_t\underline{\tilde{\mathbf{U}}}_H^n)_{\mathrm{T}}\\ &\qquad = -(\tilde{\mathbf{Z}}^{*,n} - d_t\underline{\tilde{\mathbf{U}}}_H^n, \Pi d_t\mathbf{u}^n - d_t\underline{\mathbf{U}}_H^n)_{\mathrm{TM}} + (\mathbf{Z}^{*,n} - d_t\underline{\mathbf{U}}_H^n, \tilde{\mathbf{Z}}^{*,n} - d_t\underline{\tilde{\mathbf{U}}}_H^n)_{\mathrm{TM}}\\ &\qquad\quad - (\partial_tK(p^n)\tilde{\mathbf{U}}^{*,n} - d_tK(\mathcal{P}_H(p^n))\underline{\tilde{\mathbf{U}}}_H^n, \tilde{\mathbf{Z}}^{*,n} - d_t\underline{\tilde{\mathbf{U}}}_H^n)_{\mathrm{T}}.\end{aligned} \tag{48}$$

Adding $(K(\mathcal{P}_H(p^{n-1}))\tilde{\mathbf{Z}}^{*,n}, \tilde{\mathbf{Z}}^{*,n} - d_t\tilde{\underline{\mathbf{U}}}_H^n)_{\mathrm{T}}$ to both sides of (48), using the boundedness assumption on K, Taylor's Theorem, the Cauchy-Schwarz inequality and (15) we have

$$\begin{aligned}\|\tilde{\mathbf{Z}}^{*,n} - d_t\tilde{\underline{\mathbf{U}}}_H^n\|_{\mathrm{TM}} \le C(&\|\Pi d_t\mathbf{u}^n - \mathbf{Z}^{*,n}\|_{\mathrm{TM}} + \Delta t\|\tilde{\mathbf{U}}^{*,n}\|_{\mathrm{TM}}\\ &+\|d_t(K(p^n) - K(\mathcal{P}_H(p^n)))\tilde{\mathbf{U}}^{*,n}\|_{\mathrm{T}}\\ &+\|d_tK(\mathcal{P}_H(p^n))(\tilde{\mathbf{U}}^{*,n} - \tilde{\underline{\mathbf{U}}}_H^n)\|_{\mathrm{T}} + C(H^2+\Delta t)\|\tilde{\mathbf{Z}}^{*,n}\|_{\mathrm{T}}).\end{aligned}$$

Taylor's theorem, the estimate (12) and Lemma 3.4 imply that $\|\Pi d_t\mathbf{u}^n - \mathbf{Z}^{*,n}\|_{\mathrm{TM}} \le C(H^2 + \Delta t)$. By Lemma 3.4 $\|\tilde{\mathbf{U}}^{*,n}\|_{\mathrm{TM}}$ and $\|\tilde{\mathbf{Z}}^{*,n}\|_{\mathrm{T}}$ are bounded. Thus, by Taylor's theorem, the smoothness assumptions, and approximation properties of $\mathcal{P}_H$,

$$\|\tilde{\mathbf{Z}}^{*,n} - d_t\tilde{\underline{\mathbf{U}}}_H^n\|_{\mathrm{TM}} \le C(H^2 + \Delta t + \|\tilde{\mathbf{U}}^{*,n} - \tilde{\underline{\mathbf{U}}}_H^n\|_{\mathrm{T}}). \tag{49}$$

By results from Arbogast, Wheeler and Yotov [1], $\|\tilde{\mathbf{U}}^{*,n} - \tilde{\underline{\mathbf{U}}}_H^n\|_{\mathrm{T}} \le CH^2$. Hence, by the triangle inequality and Lemma 3.4,

$$\|d_t\tilde{\mathbf{u}}^n - d_t\tilde{\underline{\mathbf{U}}}_H^n\|_{\mathrm{TM}} \le C(H^2 + \Delta t).$$

Now, let $\mathbf{v} = \mathbf{Z}^{*,n} - d_t\underline{\mathbf{U}}_H^n$ in (47) and use the Cauchy-Schwarz inequality to get

$$\begin{aligned}\|\mathbf{Z}^{*,n} - d_t\underline{\mathbf{U}}_H^n\|_{\mathrm{TM}} \le &\|K(p^n)\tilde{\mathbf{Z}}^{*,n} - K(\mathcal{P}_H(p^{n-1}))d_t\tilde{\underline{\mathbf{U}}}_H^n\|_{\mathrm{T}}\\ &+\|K(p^n)_t\tilde{\mathbf{U}}^{*,n} - d_t(K(\mathcal{P}_H(p^n)))\tilde{\underline{\mathbf{U}}}_H^n\|_{\mathrm{T}}.\end{aligned}$$

By Lipschitz continuity of K and K_p, Taylor's theorem, the approximation properties of $\mathcal{P}_H$ and the boundedness of p_t,

$$\begin{aligned}\|\mathbf{Z}^{*,n} - d_t\underline{\mathbf{U}}_H^n\|_{\mathrm{TM}} \le &\|(K(p^n) - K(\mathcal{P}_H(p^{n-1})))\tilde{\mathbf{Z}}^{*,n}\|_{\mathrm{T}} + \|K(\mathcal{P}_H(p^{n-1}))(\tilde{\mathbf{Z}}^{*,n} - d_t\tilde{\underline{\mathbf{U}}}_H^n)\|_{\mathrm{T}}\\ &+\|(K(p^n)_t - d_tK(\mathcal{P}_H(p^n)))\tilde{\mathbf{U}}^{*,n}\|_{\mathrm{T}} + \|d_tK(\mathcal{P}_H(p^n))(\tilde{\mathbf{U}}^{*,n} - \tilde{\underline{\mathbf{U}}}_H^n)\|_{\mathrm{T}}\\ \le &\, C(H^2+\Delta t).\end{aligned}$$

The triangle inequality and Lemma 3.4 result in,

$$\|d_t\mathbf{u}^n - d_t\underline{\mathbf{U}}_H^n\|_{\mathrm{TM}} \le C(H^2 + \Delta t).$$

□

REMARK 3.1. *By the inverse assumption, equivalence of norms on V_H, and (12) we have*

$$\begin{aligned}\|\tilde{\underline{\mathbf{U}}}_H^n\|_\infty &\le \|\tilde{\underline{\mathbf{U}}}_H^n - \Pi\tilde{\mathbf{u}}^n\|_\infty + \|\Pi\tilde{\mathbf{u}}^n - \tilde{\mathbf{u}}^n\|_\infty + \|\tilde{\mathbf{u}}^n\|_\infty\\ &\le CH^{-d/2}\|\tilde{\underline{\mathbf{U}}}_H^n - \Pi\tilde{\mathbf{u}}^n\|_{\mathrm{TM}} + \|\Pi\tilde{\mathbf{u}}^n - \tilde{\mathbf{u}}^n\|_\infty + \|\tilde{\mathbf{u}}^n\|_\infty\\ &\le C(H^{-d/2}H^2 + H + 1),\end{aligned}$$

Thus, $\|\tilde{\underline{\mathbf{U}}}_H^n\|_\infty$ is bounded.

Proof. (Of Theorem 3.3) Results (26) and (27) have been proven by Arbogast, Wheeler and Yotov [1].

In order to derive (28), we subtract (43)-(44) from (40)-(41) and use the definition of the L^2 projection to give

$$(d_t\tilde{\mathbf{u}}^n - d_t\tilde{\underline{\mathbf{U}}}_H^n, \mathbf{v}) + E_{\mathrm{TM}}(d_t\tilde{\underline{\mathbf{U}}}_H^n, \mathbf{v}) = (d_t\hat{p}_H^n - d_t\underline{P}_H^n, \nabla\cdot\mathbf{v}),\ \mathbf{v}\in V_H^0, \tag{50}$$

$$\begin{aligned}(d_t\mathbf{u}^n - d_t\underline{\mathbf{U}}_H^n, \mathbf{v}) + E_{\mathrm{TM}}(d_t\underline{\mathbf{U}}_H^n, \mathbf{v}) = &(d_t(K(p^n))\tilde{\mathbf{u}}^n - d_t(K(\mathcal{P}_H(p^n)))\tilde{\underline{\mathbf{U}}}_H^n, \mathbf{v})\\ &+(K(p^{n-1})d_t\tilde{\mathbf{u}}^n - K(\mathcal{P}_H(p^{n-1}))d_t\tilde{\underline{\mathbf{U}}}_H^n, \mathbf{v})\\ &+E_{\mathrm{T}}(d_t(K(\mathcal{P}_H(p^n)))\tilde{\underline{\mathbf{U}}}_H^n, \mathbf{v})\\ &+E_{\mathrm{T}}(K(\mathcal{P}_H(p^{n-1}))d_t\tilde{\underline{\mathbf{U}}}_H^n, \mathbf{v}),\ \mathbf{v}\in V_H,\end{aligned} \tag{51}$$

Let ϕ satisfy the auxilliary problem with $\rho^n \in L^2(\Omega)$

$$-\nabla \cdot K(\mathcal{P}_H(p^{n-1}))\nabla \phi^n = \rho^n,\ \Omega, \tag{52}$$

$$-K(\mathcal{P}_H(p^{n-1}))\nabla \phi^n \cdot \nu = 0,\ \Gamma. \tag{53}$$

Elliptic regularity implies that

$$\|\phi^n\|_2 \le C\|\rho^n\|. \tag{54}$$

By equations (52) and (50) and the definition of Π,

$$\begin{aligned}
&(d_t\hat{p}_H^n - d_t\underline{P}_H^n, \rho^n)\\
&= -(d_t\hat{p}_H^n - d_t\underline{P}_H^n, \nabla\cdot \Pi K(\mathcal{P}_H(p^{n-1}))\nabla\phi^n)\\
&= -(d_t\tilde{\mathbf{u}}^n - d_t\tilde{\underline{\mathbf{U}}}_H^n, \Pi K(\mathcal{P}_H(p^{n-1}))\nabla\phi^n)\\
&\quad -E_{\mathrm{TM}}(d_t\tilde{\underline{\mathbf{U}}}_H^n, \Pi K(\mathcal{P}_H(p^{n-1}))\nabla\phi^n)\\
&= -(d_t\tilde{\mathbf{u}}^n - d_t\tilde{\underline{\mathbf{U}}}_H^n, \Pi K(\mathcal{P}_H(p^{n-1}))\nabla\phi^n - K(\mathcal{P}_H(p^{n-1}))\nabla\phi^n)\\
&\quad -(K(\mathcal{P}_H(p^{n-1}))(d_t\tilde{\mathbf{u}}^n - d_t\tilde{\underline{\mathbf{U}}}_H^n), \nabla\phi^n - \Pi\nabla\phi^n)\\
&\quad -(K(\mathcal{P}_H(p^{n-1}))(d_t\tilde{\mathbf{u}}^n - d_t\tilde{\underline{\mathbf{U}}}_H^n), \Pi\nabla\phi^n)\\
&\quad -E_{\mathrm{TM}}(d_t\tilde{\underline{\mathbf{U}}}_H^n, \Pi K(\mathcal{P}_H(p^{n-1}))\nabla\phi^n).
\end{aligned} \tag{55}$$

By (51)

$$\begin{aligned}
-(K(\mathcal{P}_H(p^{n-1}))(d_t\tilde{\mathbf{u}}^n - d_t\tilde{\underline{\mathbf{U}}}_H^n), \Pi\nabla\phi^n) &= ((K(p^{n-1}) - K(\mathcal{P}_H(p^{n-1})))d_t\tilde{\mathbf{u}}^n, \Pi\nabla\phi^n)\\
&\quad -(d_t\mathbf{u}^n - d_t\underline{\mathbf{U}}_H^n, \Pi\nabla\phi^n)\\
&\quad +(d_t(K(p^n))\tilde{\mathbf{u}}^n - d_t(K(\mathcal{P}_H(p^n)))\tilde{\underline{\mathbf{U}}}_H^n, \Pi\nabla\phi^n)\\
&\quad -E_{\mathrm{TM}}(d_t\underline{\mathbf{U}}_H^n, \Pi\nabla\phi^n)\\
&\quad +E_{\mathrm{T}}(d_t(K(\mathcal{P}_H(p^n)))\tilde{\underline{\mathbf{U}}}_H^n, \Pi\nabla\phi^n)\\
&\quad +E_{\mathrm{T}}(K(\mathcal{P}_H(p^{n-1}))d_t\tilde{\underline{\mathbf{U}}}_H^n, \Pi\nabla\phi^n).
\end{aligned} \tag{56}$$

We also have by integration by parts, (45) and (47)

$$\begin{aligned}
&-(d_t\mathbf{u}^n - d_t\underline{\mathbf{U}}_H^n, \Pi\nabla\phi^n)\\
&= -(d_t\mathbf{u}^n - d_t\underline{\mathbf{U}}_H^n, \Pi\nabla\phi^n - \nabla\phi^n) - (d_t\mathbf{u}^n - d_t\underline{\mathbf{U}}_H^n, \nabla\phi^n)\\
&= -(d_t\mathbf{u}^n - d_t\underline{\mathbf{U}}_H^n, \Pi\nabla\phi^n - \nabla\phi^n) + (\nabla\cdot(d_t\mathbf{u}^n - d_t\underline{\mathbf{U}}_H^n), \phi^n - \hat{\phi}_H^n).
\end{aligned} \tag{57}$$

Furthermore, we can write

$$\begin{aligned}
(d_tK(p^n)\tilde{\mathbf{u}}^n - d_tK(\mathcal{P}_H(p^n))\tilde{\underline{\mathbf{U}}}_H^n, \Pi\nabla\phi^n) &= ((d_tK(p^n) - d_tK(\mathcal{P}_H(p^n)))\tilde{\mathbf{u}}^n, \Pi\nabla\phi^n)\\
&\quad +(d_tK(\mathcal{P}_H(p^n))(\tilde{\mathbf{u}}^n - \tilde{\underline{\mathbf{U}}}_H^n), \Pi\nabla\phi^n).
\end{aligned} \tag{58}$$

Using (4) gives

$$\begin{aligned}
|E_{\mathrm{TM}}(d_t\tilde{\underline{\mathbf{U}}}_H^n, \Pi K(p^n)\nabla\phi^n)| &\le C\sum_E \sum_{|\alpha|=2} \left\| \frac{\partial^\alpha}{\partial \mathbf{x}^\alpha}(d_t\tilde{\underline{\mathbf{U}}}_H^n \cdot \Pi K(p^n)\nabla\phi^n)\right\|_{L^1(E)} H^2\\
&\le C\sum_E \Big(\left\|\frac{\partial d_t\tilde{\underline{\mathbf{U}}}_{H,x}^n}{\partial x}\right\|_{L^2(E)} \left\|\frac{\partial}{\partial x}(\Pi K(p^n)\nabla\phi^n)^x\right\|_{L^2(E)}\\
&\quad + \left\|\frac{\partial d_t\tilde{\underline{\mathbf{U}}}_{H,y}^n}{\partial y}\right\|_{L^2(E)} \left\|\frac{\partial}{\partial y}(\Pi K(p^n)\nabla\phi^n)^y\right\|_{L^2(E)}\Big) H^2.
\end{aligned} \tag{59}$$

By Lemma 2.1 and the inverse inequality we have

$$
\begin{aligned}
\left\|\frac{\partial d_t\underline{\tilde{\mathbf{U}}}^n_{H,x}}{\partial x}\right\|_{L^2(E)} &\leq \left\|\frac{\partial}{\partial x}(d_t\underline{\tilde{\mathbf{U}}}^n_{H,x}-\Pi d_t\tilde{\mathbf{u}}^n_x)\right\|_{L^2(E)}+\left\|\frac{\partial}{\partial x}\Pi d_t\tilde{\mathbf{u}}^n_x\right\|_{L^2(E)} \\
&\leq C\|d_t\underline{\tilde{\mathbf{U}}}^n_{H,x}-\Pi d_t\tilde{\mathbf{u}}^n_x\|_{L^2(E)}H^{-1}+\left\|\frac{\partial}{\partial x}d_t\tilde{\mathbf{u}}^n_x\right\|_{L^2(E)} \\
&\leq C\|d_t\underline{\tilde{\mathbf{U}}}^n_{H}-\Pi d_t\tilde{\mathbf{u}}^n\|_{L^2(E)}H^{-1}+\|d_t\tilde{\mathbf{u}}^n\|_{1,E}. \qquad (60)
\end{aligned}
$$

Since $d_t\underline{\tilde{\mathbf{U}}}^n_H$ and $\Pi K(p^n)\nabla\phi^n$ are in V_H, by (59) and the Cauchy-Schwarz inequality,

$$
\begin{aligned}
|E_{\mathrm{TM}}(d_t\underline{\tilde{\mathbf{U}}}^n_H,\Pi K(p^n)\nabla\phi^n)| &\leq C\sum_E(\|d_t\underline{\tilde{\mathbf{U}}}^n_H-\Pi d_t\tilde{\mathbf{u}}^n\|_{L^2(E)}+\|d_t\tilde{\mathbf{u}}^n\|_{1,E}H)\|\phi^n\|_{2,E}H \\
&\leq C(\|d_t\underline{\tilde{\mathbf{U}}}^n_H-\Pi d_t\tilde{\mathbf{u}}^n\|_0+\|d_t\tilde{\mathbf{u}}^n\|_1H)\|\phi^n\|_2H. \qquad (61)
\end{aligned}
$$

As done above and noting that K has bounded second derivatives,

$$
|E_{\mathrm{TM}}(d_t\underline{\mathbf{U}}^n_H,\Pi\nabla\phi^n)| \leq C(\|d_t\underline{\mathbf{U}}^n_H-\Pi d_t\mathbf{u}^n\|_0+\|d_t\mathbf{u}^n\|_1H)\|\phi^n\|_2H, \qquad (62)
$$

$$
|E_{\mathrm{T}}(d_t(K(\mathcal{P}_H(p^n)))\underline{\tilde{\mathbf{U}}}^n_H,\Pi\nabla\phi^n)| \leq (C+\Delta t)(\|\underline{\tilde{\mathbf{U}}}^n_H-\Pi\tilde{\mathbf{u}}^n\|_0+\|\tilde{\mathbf{u}}^n\|_1H)\|\phi^n\|_2H, \qquad (63)
$$

$$
|E_{\mathrm{T}}(K(\mathcal{P}_H(p^{n-1}))d_t\underline{\tilde{\mathbf{U}}}^n_H,\Pi\nabla\phi^n)| \leq C(\|d_t\underline{\tilde{\mathbf{U}}}^n_H-\Pi d_t\tilde{\mathbf{u}}^n\|_0+\|d_t\tilde{\mathbf{u}}^n\|_1H)\|\phi^n\|_2H, \qquad (64)
$$

where C in the second and third inequalities depends on K^*.

Combining (55) with (56)-(64), applying approximation properties of the L^2 and Π projections, using Lemma 3.5, and equations (26) and (12) gives

$$
\begin{aligned}
&(d_t\hat{p}^n_H-d_t\underline{P}^n_H,\rho^n) \\
&\leq \|d_t\tilde{\mathbf{u}}^n-d_t\underline{\tilde{\mathbf{U}}}^n_H\|\|\Pi K(p^n)\nabla\phi^n-K(p^n)\nabla\phi^n\| \\
&\quad+\|K(\mathcal{P}_H(p^{n-1}))(d_t\tilde{\mathbf{u}}^n-d_t\underline{\tilde{\mathbf{U}}}^n_H)\|\|\nabla\phi^n-\Pi\nabla\phi^n\| \\
&\quad+\|(d_tK(p^n)-d_tK(\mathcal{P}_H(p^n)))\tilde{\mathbf{u}}^n\|\|\Pi\nabla\phi^n\| \\
&\quad+\|d_tK(\mathcal{P}_H(p^n))(\tilde{\mathbf{u}}^n-\underline{\tilde{\mathbf{U}}}^n)\|\|\Pi\nabla\phi^n\| \\
&\quad+\|(K(p^{n-1})-K(\mathcal{P}_H(p^{n-1})))d_t\tilde{\mathbf{u}}^n\|\|\Pi\nabla\phi^n\| \\
&\quad+\|d_t\mathbf{u}^n-d_t\underline{\mathbf{U}}^n_H\|\|\Pi\nabla\phi^n-\nabla\phi^n\| \\
&\quad+\|\nabla\cdot(d_t\mathbf{u}^n-d_t\underline{\mathbf{U}}^n_H)\|\|\phi^n-\hat{\phi}^n_H\| \\
&\quad+\|d_t(K(p^n))\tilde{\mathbf{u}}^n-d_t(K(\mathcal{P}_H(p^n)))\underline{\tilde{\mathbf{U}}}^n_H\|\|\Pi\nabla\phi^n\| \\
&\quad+|E_{\mathrm{TM}}(d_t\underline{\tilde{\mathbf{U}}}^n_H,\Pi K(p^n)\nabla\phi^n)|+|E_{\mathrm{TM}}(d_t\underline{\mathbf{U}}^n_H,\Pi\nabla\phi^n)| \\
&\quad+|E_{\mathrm{T}}(d_t(K(\mathcal{P}_H(p^n)))\underline{\tilde{\mathbf{U}}}^n_H,\Pi\nabla\phi^n)|+|E_{\mathrm{T}}(K(\mathcal{P}_H(p^{n-1}))d_t\underline{\tilde{\mathbf{U}}}^n_H,\Pi\nabla\phi^n)| \\
&\leq C(H^2+\Delta t)\|\phi^n\|_2.
\end{aligned}
$$

Applying equation (54) and taking $\rho^n=d_t\hat{p}^n-d_t\underline{P}^n_H$ we have

$$
\|d_t\hat{p}^n_H-d_t\underline{P}^n_H\| \leq C(H^2+\Delta t). \qquad (65)
$$

□

3.3. Convergence Estimate of the Nonlinear Scheme. We now prove the following theorem about the convergence of the above finite difference scheme:

THEOREM 3.6. *Let $P_H^n, \tilde{\mathbf{U}}_H^n$ and $\mathbf{U}_H^n, n = 1, \ldots, N$ be defined as in (19) - (21) with initial values $P_H^0 = \hat{p}_H(t^0, .)$. Assume (S1)-(S4) hold. Then, there exists a positive constant C, independent of H and Δt such that*

$$\|P_H^N - p^N\|_{\mathrm{M}} + \{\Delta t \sum_{n=1}^{N} K_* \|\tilde{\mathbf{U}}_H^n - \tilde{\mathbf{u}}^n\|_{\mathrm{T}}^2\}^{1/2} \le C(H^2 + \Delta t). \tag{66}$$

Proof. Let $\gamma^n = P_H^n - \underline{P}_H^n, \boldsymbol{\eta}^n = \tilde{\mathbf{U}}_H^n - \underline{\tilde{\mathbf{U}}}_H^n, \boldsymbol{\xi}^n = \mathbf{U}_H^n - \underline{\mathbf{U}}_H^n$ and $\alpha^n = \underline{P}_H^n - p^n$. Subtracting $(d_t \underline{P}_H^n, w) + (\nabla \cdot \underline{\mathbf{U}}_H^n, w)$ from both sides of (19) and using equations (23) and (16) we have

$$\begin{aligned}(d_t\gamma^n, w) + (\nabla \cdot \boldsymbol{\xi}^n, w) &= (f^n, w) - (\nabla \cdot \underline{\mathbf{U}}_H^n, w) - (d_t \underline{P}_H^n, w)\\ &= (p_t^n, w) - (d_t p^n, w) - (d_t \alpha^n, w)\\ &= (\epsilon^n, w) - (d_t \alpha^n, w),\ w \in W_H,\end{aligned} \tag{67}$$

where ϵ is a time truncation term. Subtracting (24) from (20) results in

$$(\boldsymbol{\eta}^n, \mathbf{v})_{\mathrm{TM}} = (\gamma^n, \nabla \cdot \mathbf{v}),\ \mathbf{v} \in V_H^0, \tag{68}$$

and subtracting (25) from (21) gives

$$\begin{aligned}(\boldsymbol{\xi}^n, \mathbf{v})_{\mathrm{TM}} &= (K(\mathcal{P}_H(P^n))\tilde{\mathbf{U}}_H^n, \mathbf{v})_{\mathrm{T}} - (K(\mathcal{P}_H(p^n))\underline{\tilde{\mathbf{U}}}_H^n, \mathbf{v})_{\mathrm{T}}\\ &= (K(\mathcal{P}_H(P_H^n))\boldsymbol{\eta}^n, \mathbf{v})_{\mathrm{T}} + ((K(\mathcal{P}_H(P_H^n)) - K(\mathcal{P}_H(\underline{P}_H^n)))\underline{\tilde{\mathbf{U}}}_H^n, \mathbf{v})_{\mathrm{T}}\\ &\quad -((K(\mathcal{P}_H(p^n)) - K(\mathcal{P}_H(\underline{P}_H^n)))\underline{\tilde{\mathbf{U}}}_H^n, \mathbf{v})_{\mathrm{T}},\ \mathbf{v} \in V_H.\end{aligned} \tag{69}$$

Letting $w = \gamma^n$ in (67), $\mathbf{v} = \boldsymbol{\xi}^n$ in (68) and $\mathbf{v} = \boldsymbol{\eta}^n$ in (69) gives

$$(d_t\gamma^n, \gamma^n) = -(\nabla \cdot \boldsymbol{\xi}^n, \gamma^n) + (\epsilon^n, \gamma^n)_{\mathrm{M}} - (d_t\alpha^n, \gamma^n), \tag{70}$$

$$(\boldsymbol{\eta}^n, \boldsymbol{\xi}^n)_{\mathrm{TM}} = (\gamma^n, \nabla \cdot \boldsymbol{\xi}^n), \tag{71}$$

$$\begin{aligned}(\boldsymbol{\xi}^n, \boldsymbol{\eta}^n)_{\mathrm{TM}} &= (K(\mathcal{P}_H(P_H^n))\boldsymbol{\eta}^n, \boldsymbol{\eta}^n)_{\mathrm{T}} + ((K(\mathcal{P}_H(P_H^n)) - K(\mathcal{P}_H(\underline{P}_H^n)))\underline{\tilde{\mathbf{U}}}_H^n, \boldsymbol{\eta}^n)_{\mathrm{T}}\\ &\quad -((K(\mathcal{P}_H(p^n)) - K(\mathcal{P}_H(\underline{P}_H^n)))\underline{\tilde{\mathbf{U}}}_H^n, \boldsymbol{\eta}^n)_{\mathrm{T}}.\end{aligned} \tag{72}$$

Combining equations (70)-(72), applying the Cauchy-Schwarz inequality and (15) we have

$$\begin{aligned}&\frac{1}{2\Delta t}[\|\gamma^n\|_{\mathrm{M}}^2 - \|\gamma^{n-1}\|_{\mathrm{M}}^2] + \|K(\mathcal{P}_H(P_H^n))^{1/2}\boldsymbol{\eta}^n\|_{\mathrm{T}}^2\\ &\le (d_t\gamma^n, \gamma^n)_{\mathrm{M}} + \|K(\mathcal{P}_H(P_H^n))^{1/2}\boldsymbol{\eta}^n\|_{\mathrm{T}}^2\\ &\le (\epsilon^n, \gamma^n)_{\mathrm{M}} - (d_t\alpha^n, \gamma^n)_{\mathrm{M}} + ((K(\mathcal{P}_H(p^n)) - K(\mathcal{P}_H(\underline{P}_H^n)))\underline{\tilde{\mathbf{U}}}_H^n, \boldsymbol{\eta}^n)_{\mathrm{T}}\\ &\quad -((K(\mathcal{P}_H(P_H^n)) - K(\mathcal{P}_H(\underline{P}_H^n)))\underline{\tilde{\mathbf{U}}}_H^n, \boldsymbol{\eta}^n)_{\mathrm{T}}\\ &\le \frac{1}{2}\|\epsilon^n\|_{\mathrm{M}}^2 + \|\gamma^n\|_{\mathrm{M}}^2 + \frac{1}{2}\|d_t\alpha^n\|_{\mathrm{M}}^2 + C\|(K(\mathcal{P}_H(p^n)) - K(\mathcal{P}_H(\underline{P}_H^n)))\underline{\tilde{\mathbf{U}}}_H^n\|_{\mathrm{T}}^2\\ &\quad +C\|(K(\mathcal{P}_H(P_H^n)) - K(\mathcal{P}_H(\underline{P}_H^n)))\underline{\tilde{\mathbf{U}}}_H^n\|_{\mathrm{T}}^2 + \delta\|\boldsymbol{\eta}^n\|_{\mathrm{TM}}^2,\end{aligned}$$

where $\delta \le K_*/2$.

Now,

$$|\epsilon_{ij}^n| = \frac{1}{\Delta t}|\int_{t^{n-1}}^{t^n} p_{tt}(x_i,y_j,t)(t-t^n)dt| \le \|p_{tt}(x_i,y_j,.)\|_{L^2(t^{n-1},t^n)}(\Delta t)^{\frac{1}{2}}.$$

So,

$$\|\epsilon^n\|_{\mathrm{M}}^2 \le \Delta t \sum_{ij} H_i^x H_j^y \|p_{tt}(x_i,y_j,.)\|^2_{L^2(t^{n-1},t^n)}. \tag{73}$$

By the triangle inequality and Theorem 3.3,

$$\|d_t\alpha^n\|_{\mathrm{M}}^2 \le C(H^4+\Delta t^2).$$

By (S2), the definition of $\mathcal{P}_H$ and Theorem 3.3

$$\|(K(\mathcal{P}_H(p^n)) - K(\mathcal{P}_H(\underline{P}_H^n)))\tilde{\mathbf{\underline{U}}}_H^n\|_{\mathrm{T}}^2 \le C\|p^n - \underline{P}_H^n\|_{\mathrm{M}}^2 \le CH^4, \tag{74}$$

$$\|(K(\mathcal{P}_H(P_H^n)) - K(\mathcal{P}_H(\underline{P}_H^n)))\tilde{\mathbf{\underline{U}}}_H^n\|_{\mathrm{T}}^2 \le C\|\gamma^n\|_{\mathrm{M}}^2, \tag{75}$$

where we have used the boundedness of $\|\tilde{\mathbf{\underline{U}}}_H^n\|_\infty$ as per Remark 3.1.

Multiplying by $2\Delta t$, bringing the $\delta\|\boldsymbol{\eta}^n\|_{\mathrm{TM}}^2$ term to the left-hand side, summing on $n, n=1,\ldots,N$, using (73)-(75) and applying Gronwall's Lemma gives,

$$\begin{aligned}
&\|\gamma^N\|_{\mathrm{M}}^2 - \|\gamma^0\|_{\mathrm{M}}^2 + \Delta t\sum_{n=1}^N \|K(\mathcal{P}_H(P_H^n))^{1/2}\boldsymbol{\eta}^n\|_{\mathrm{T}}^2 \\
&\le C\Delta t\sum_{n=1}^N(\|\epsilon^n\|_{\mathrm{M}}^2 + \|d_t\alpha^n\|_{\mathrm{M}}^2 + \|\alpha^n\|_{\mathrm{M}}^2) + CH^4 + C\Delta t\sum_{n=1}^N\|\gamma^n\|_{\mathrm{M}}^2 \\
&\le C(\Delta t^2 + H^4).
\end{aligned}$$

The proof is completed by applying the initial conditions on P_H^0 and $\underline{P}_H^0$, Theorem 3.3 and the triangle inequality. □

4. Fine Grid Linear Scheme. We now consider a linear cell-centered finite difference scheme on the fine grid where we make use of the nonlinear solution on the coarse grid.

We solve the following problem for $P_h^n \in W_h, \tilde{\mathbf{U}}_h^n \in V_h$ and $\mathbf{U}_h^n \in V_H^n$ at each $n = 1,\ldots,N$,

$$(d_tP_h^n, w) = -(\nabla\cdot\mathbf{U}_h^n, w) + (f^n, w), w\in W_h, \tag{76}$$

$$(\tilde{\mathbf{U}}_h^n, \mathbf{v})_{\mathrm{TM}} = (P_h^n, \nabla\cdot\mathbf{v}), \mathbf{v}\in V_h^0, \tag{77}$$

$$\begin{aligned}
(\mathbf{U}_h^n,\mathbf{v})_{\mathrm{TM}} &= (K(\mathcal{P}_H(P_H^n))\tilde{\mathbf{U}}_h^n,\mathbf{v})_{\mathrm{T}} \\
&\quad +(K_p(\mathcal{P}_H(P_H^n))\mathcal{Q}_H(\tilde{\mathbf{U}}_H^n)(\mathcal{P}_h(P_h^n) - \mathcal{P}_H(P_H^n)),\mathbf{v})_{\mathrm{T}}, \mathbf{v}\in V_h.
\end{aligned} \tag{78}$$

We define $\mathcal{Q}_H(\tilde{\mathbf{u}})$ as a vector quantity with entries $\mathcal{Q}_H^x(\tilde{u}^x)$ and $\mathcal{Q}_H^y(\tilde{u}^y)$. The entry $\mathcal{Q}_H^x(\tilde{u}^x)$ is defined from the values of $\tilde{u}_{i+1/2j}^x$ for $i=0,\ldots,\hat{N}_x$ and $j=1,\ldots,\hat{N}_y$ as follows. For points (x,y) such that $x_{i-1/2}\le x\le x_{i+1/2}, i\in\{1,\ldots,\hat{N}_x\}$ and $y_j\le y\le y_{j+1}, j\in\{1,\ldots,\hat{N}_y\}$, we take $\mathcal{Q}_H^x(\tilde{u}^x)$ to be the bilinear interpolant of $\tilde{u}_{i-1/2,j}^x, \tilde{u}_{i+1/2,j}^x, \tilde{u}_{i-1/2,j+1}^x$ and $\tilde{u}_{i+1/2,j+1}^x$. This leaves a strip half a cell in height along the top and bottom of the domain. We will consider the bottom strip. For $i=0,\ldots,\hat{N}_x$, we set

$$\mathcal{Q}_H^x(\tilde{u}^x)(x_{i+1/2},y_{1/2}) = \frac{(2H_1^y+H_2^y)\tilde{u}_{i+1/2,1}^x - H_1^y\tilde{u}_{i+1/2,2}^x}{H_1^y+H_2^y}.$$

Now, for points (x, y) such that $x_{i-1/2} \leq x \leq x_{i+1/2}, i \in \{1, \ldots, \hat{N}_x\}$ and $y_{1/2} \leq y \leq y_1$, we let $\mathcal{Q}_H^x(\tilde{u}^x)(x, y)$ be the bilinear interpolant of $\mathcal{Q}_H^x(\tilde{u}^x)(x_{i-1/2}, y_{1/2})$, $\mathcal{Q}_H^x(\tilde{u}^x)(x_{i+1/2}, y_{1/2})$, $\tilde{u}^x_{i-1/2,1}$ and $\tilde{u}^x_{i+1/2,1}$. An analogous defintion is made along the top strip of the domain. The definition of $\mathcal{Q}_H^y(\tilde{u}^y)$ is similar to the above, except that the strips are along the left and right sides of the domain.

We have the following lemma for the approximation error of $\mathcal{Q}_H$,

LEMMA 4.1. *If each component of $\tilde{\mathbf{u}}$ is twice differentiable, then for $\mathcal{Q}_H(\tilde{\mathbf{u}})$ defined above,*

$$\|\mathcal{Q}_H(\tilde{\mathbf{u}}) - \tilde{\mathbf{u}}\|_\infty \leq CH^2.$$

Proof. By Taylor's theorem we have that the two point extrapolation for the boundary points described above is $O(H^2)$ accurate. Thus, since bilinear interpolation is also $O(H^2)$ accurate, the lemma is proven. □

We now prove the following theorem about the convergence of the above linear finite difference scheme,

THEOREM 4.2. *Let $P_h^n, \tilde{\mathbf{U}}_h^n$ and $\mathbf{U}_h^n, n = 1, \ldots, N$ be defined as in (76) - (78) with initial values $P_h^0 = \hat{p}_h(t^0, .)$. Assume (S1)-(S4) hold and that H and $\Delta t H^{-d/2}$ are sufficiently small. Then, there exists a positive constant C, independent of h, H and Δt such that*

$$\|P_h^N - p^N\|_{\mathrm{M}} + \{\Delta t \sum_{n=1}^{N} K_* \|\tilde{\mathbf{U}}_h^n - \tilde{\mathbf{u}}^n\|_{\mathrm{T}}^2\}^{1/2} \leq C(H^{4-d/2} + h^2 + \Delta t). \tag{79}$$

Proof. By Section 3 we can define $\underline{P}_h^n \in W_h, \underline{\tilde{\mathbf{U}}}_h^n \in V_h$ and $\underline{\mathbf{U}}_h^n \in V_h^n$ at each $n = 1, \ldots, N$ satisfying equations (23)-(25) and Theorem 3.3 on the fine grid.

Let $\gamma^n = P_h^n - \underline{P}_h^n, \boldsymbol{\eta}^n = \tilde{\mathbf{U}}_h^n - \underline{\tilde{\mathbf{U}}}_h^n, \boldsymbol{\xi}^n = \mathbf{U}_h^n - \underline{\mathbf{U}}_h^n$ and $\alpha^n = \underline{P}_h^n - p^n$. As done in Theorem 3.6, we subtract $(d_t\underline{P}_h^n, w) + (\nabla \cdot \underline{\mathbf{U}}_h^n, w)$ from both sides of equation (76) and combine with equation (23) applied to the fine grid. We also subtract (24) and (25) from (77) and (78) to give the error equations,

$$(d_t\gamma^n, w) = -(\nabla \cdot \boldsymbol{\xi}^n, w) + (\epsilon^n, w) - (d_t\alpha^n, w), \tag{80}$$

$$(\boldsymbol{\eta}^n, \mathbf{v})_{\mathrm{TM}} = (\gamma^n, \nabla \cdot \mathbf{v}), \tag{81}$$

$$\begin{aligned}(\boldsymbol{\xi}^n, \mathbf{v})_{\mathrm{TM}} &= (K(\mathcal{P}_H(P_H^n))\tilde{\mathbf{U}}_h^n, \mathbf{v})_{\mathrm{T}} - (K(\mathcal{P}_h(p^n))\underline{\tilde{\mathbf{U}}}_h^n, \mathbf{v})_{\mathrm{T}} \\ &\quad + (K_p(\mathcal{P}_H(P_H^n))\mathcal{Q}_H(\tilde{\mathbf{U}}_H^n)(\mathcal{P}_h(P_h^n) - \mathcal{P}_H(P_H^n)), \mathbf{v})_{\mathrm{T}}.\end{aligned} \tag{82}$$

Using Taylor's Theorem, $K(\mathcal{P}_h(p^n))$ can be written as

$$\begin{aligned}K(\mathcal{P}_h(p^n)) &= K(\mathcal{P}_h(P_H^n)) + K_p(\mathcal{P}_H(P_H^n))(\mathcal{P}_h(p^n) - \mathcal{P}_H(P_H^n)) \\ &\quad + \frac{K_{pp}(\theta^n)}{2}(\mathcal{P}_h(p^n) - \mathcal{P}_H(P_H^n))^2,\end{aligned}$$

where θ^n is between $\mathcal{P}_h(p^n)$ and $\mathcal{P}_H(P_H^n)$.

Using this expression in (82), adding and subtracting $(K_p(\mathcal{P}_H(P_H^n))\mathcal{Q}_H(\tilde{\mathbf{U}}_H^n)\mathcal{P}_h(p^n), \mathbf{v})_{\mathrm{T}}$ we have

$$\begin{aligned}(\boldsymbol{\xi}^n, \mathbf{v})_{\mathrm{TM}} &= (K(\mathcal{P}_H(P_H^n))\boldsymbol{\eta}^n, \mathbf{v})_{\mathrm{T}} \\ &\quad + (K_p(\mathcal{P}_H(P_H^n))(\mathcal{Q}_H(\tilde{\mathbf{U}}_H^n) - \underline{\tilde{\mathbf{U}}}_h^n)(\mathcal{P}_h(p^n) - \mathcal{P}_H(P_H^n)), \mathbf{v})_{\mathrm{T}} \\ &\quad + (K_p(\mathcal{P}_H(P_H^n))\mathcal{Q}_H(\tilde{\mathbf{U}}_H^n)(\mathcal{P}_h(P_h^n) - \mathcal{P}_h(p^n)), \mathbf{v})_{\mathrm{T}} \\ &\quad + (\frac{K_{pp}(\theta^n)}{2}(\mathcal{P}_h(p^n) - \mathcal{P}_H(P_H^n))^2\underline{\tilde{\mathbf{U}}}_h^n, \mathbf{v})_{\mathrm{T}}.\end{aligned}$$

Let $w = \gamma^n, \mathbf{v} = \boldsymbol{\xi}^n$ and $\mathbf{v} = \boldsymbol{\eta}^n$ in (80), (81) and (83), respectively, and combine to give

$$\begin{aligned}
&\frac{1}{2\Delta t}[\|\gamma^n\|^2 - \|\gamma^{n-1}\|^2] + K_*\|\boldsymbol{\eta}^n\|_{\mathrm{T}}^2 \\
&\le (d_t\gamma^n, \gamma^n) + \|K(\mathcal{P}_H(P_H^n))^{1/2}\boldsymbol{\eta}^n\|_{\mathrm{T}}^2 \\
&\le \frac{1}{2}\|\epsilon^n\|^2 + \|\gamma^n\|^2 + \frac{1}{2}\|d_t\alpha^n\|^2 + \delta\|\boldsymbol{\eta}^n\|^2 \\
&\quad + C\|K_p(\mathcal{P}_H(P_H^n))(\mathcal{Q}_H(\tilde{\mathbf{U}}_H^n) - \underline{\tilde{\mathbf{U}}}_h^n)(\mathcal{P}_h(p^n) - \mathcal{P}_H(P_H^n))\|_{\mathrm{T}}^2 \\
&\quad + C\|K_p(\mathcal{P}_H(P_H^n))\mathcal{Q}_H(\tilde{\mathbf{U}}_H^n)(\mathcal{P}_h(P_h^n) - \mathcal{P}_h(p^n))\|_{\mathrm{T}}^2 \\
&\quad + C\|\frac{K_{pp}(\theta^n)}{2}(\mathcal{P}_h(p^n) - \mathcal{P}_H(P_H^n))^2\underline{\tilde{\mathbf{U}}}_h^n\|_{\mathrm{T}}^2, \qquad (83)
\end{aligned}$$

where $\delta \le K_*/2$.

We will consider the last three terms of (83). The first of these can be bounded as follows.

$$\begin{aligned}
&\|K_p(\mathcal{P}_H(P_H^n))(\mathcal{Q}_H(\tilde{\mathbf{U}}_H^n) - \underline{\tilde{\mathbf{U}}}_h^n)(\mathcal{P}_h(p^n) - \mathcal{P}_H(P_H^n))\|_{\mathrm{T}}^2 \\
&\le C\|\mathcal{Q}_H(\tilde{\mathbf{U}}_H^n) - \underline{\tilde{\mathbf{U}}}_h^n\|_{\mathrm{TM}}^2\|\mathcal{P}_h(p^n) - \mathcal{P}_H(P_H^n)\|_\infty^2, \qquad (84)
\end{aligned}$$

where we can write,

$$\begin{aligned}
\|\mathcal{Q}_H(\tilde{\mathbf{U}}_H^n) - \underline{\tilde{\mathbf{U}}}_h^n\|_{\mathrm{TM}}^2 \le\ & \|\mathcal{Q}_H(\tilde{\mathbf{U}}_H^n) - \mathcal{Q}_H(\tilde{\mathbf{u}}^n)\|_{\mathrm{TM}}^2 + \|\mathcal{Q}_H(\tilde{\mathbf{u}}^n) - \tilde{\mathbf{u}}^n\|_{\mathrm{TM}}^2 \\
& + \|\tilde{\mathbf{u}}^n - \underline{\tilde{\mathbf{U}}}_h^n\|_{\mathrm{TM}}^2.
\end{aligned}$$

Since $\mathcal{Q}_H(\tilde{\mathbf{U}}_H^n)$ is a bilinear interpolant of terms that can be expressed in terms of nodal values of $\tilde{\mathbf{U}}_H^n$ on the coarse grid, it can be shown that,

$$\|\mathcal{Q}_H(\tilde{\mathbf{U}}_H^n) - \mathcal{Q}_H(\tilde{\mathbf{u}}^n)\|_{\mathrm{TM}}^2 \le C\|\tilde{\mathbf{U}}_H^n - \tilde{\mathbf{u}}^n\|_{\mathrm{TM},H}^2,$$

where $\|\cdot\|_{\mathrm{TM},H}$ denotes the midpoint by trapezoidal norm on the coarse grid. We also have $\|\mathcal{Q}_H(\tilde{\mathbf{u}}^n) - \tilde{\mathbf{u}}^n\|_{\mathrm{TM}}^2 \le CH^4$ by Lemma 4.1. In order to bound the second term in (84) we write it as,

$$\begin{aligned}
\|\mathcal{P}_h(p^n) - \mathcal{P}_H(P_H^n)\|_\infty^2 \le\ & \|\mathcal{P}_H(P_H^n) - \mathcal{P}_H(p^n)\|_\infty^2 + \|\mathcal{P}_H(p^n) - p^n\|_\infty^2 \\
& + \|p^n - \mathcal{P}_h(p^n)\|_\infty^2.
\end{aligned}$$

By the definition of $\mathcal{P}_H$, the equivalence of norms on the space W_H, and Theorem 3.6, we have

$$\begin{aligned}
\|\mathcal{P}_H(P_H^n) - \mathcal{P}_H(p^n)\|_\infty^2 &\le \frac{C}{H^d}\|P_H^n - p^n\|_{M,H}^2 \\
&\le CH^{-d}(H^4 + \Delta t^2),
\end{aligned}$$

where d is the space dimension. By Lemma 3.1, we have $\|\mathcal{P}_H(p^n) - p^n\|_\infty^2 \le CH^4$ and $\|\mathcal{P}_h(p^n) - p^n\|_\infty^2 \le Ch^4$. Thus,

$$\begin{aligned}
&\|K_p(\mathcal{P}_H(P_H^n))(\mathcal{Q}_H(\tilde{\mathbf{U}}_H^n) - \underline{\tilde{\mathbf{U}}}_h^n)(\mathcal{P}_h(p^n) - \mathcal{P}_H(P_H^n))\|_{\mathrm{T}}^2 \\
&\le C(\|\tilde{\mathbf{U}}_H^n - \tilde{\mathbf{u}}^n\|_{\mathrm{TM},H}^2 + \|\tilde{\mathbf{u}}^n - \underline{\tilde{\mathbf{U}}}_h^n\|_{\mathrm{TM}}^2 + H^4)(H^{4-d} + H^{-d}\Delta t^2 + h^4). \qquad (85)
\end{aligned}$$

The second to last term in (83) can be bounded by,

$$\begin{aligned}
&\|K_p(\mathcal{P}_H(P_H^n))\mathcal{Q}_H(\tilde{\mathbf{U}}_H^n)(\mathcal{P}_h(P_h^n) - \mathcal{P}_h(p^n))\|_{\mathrm{T}}^2 \\
&\leq C\|\tilde{\mathbf{U}}_H^n\|_\infty^2 \|\mathcal{P}_h(P_h^n) - \mathcal{P}_h(p^n)\|_{\mathrm{T}}^2 \\
&\leq C(H^{-d}\|\tilde{\mathbf{U}}_H^n - \tilde{\mathbf{u}}^n\|_{\mathrm{TM}}^2 + \|\tilde{\mathbf{u}}^n\|_\infty^2)(\|P_h^n - \underline{P}_h^n\|_{\mathrm{M}}^2 + \|\underline{P}_h^n - p^n\|_{\mathrm{M}}^2) \\
&\leq C(H^{-d}\|\tilde{\mathbf{U}}_H^n - \tilde{\mathbf{u}}^n\|_{\mathrm{TM}}^2 + C)(\|\gamma^n\|^2 + h^4). \qquad (86)
\end{aligned}$$

The last term in (83) can be bounded by,

$$\begin{aligned}
&\|\frac{K_{pp}(\theta^n)}{2}(\mathcal{P}_h(p^n) - \mathcal{P}_H(P_H^n))^2 \underline{\tilde{\mathbf{U}}}_h^n\|_{\mathrm{T}}^2 \\
&\leq C\|\mathcal{P}_h(p^n) - \mathcal{P}_H(P_H^n)\|_\infty^2 \|\mathcal{P}_h(p^n) - \mathcal{P}_H(P_H^n)\|_{\mathrm{T}}^2 \\
&\leq C(H^{4-d} + H^{-d}\Delta t^2 + h^4)(h^4 + H^4 + \|p^n - P_H^n\|_{\mathrm{M},H}^2) \\
&\leq C(H^{4-d} + H^{-d}\Delta t^2 + h^4)(h^4 + H^4 + \Delta t^2) \\
&\leq C(H^{8-d} + h^4 + \Delta t^2 + H^{-d}h^4\Delta t^2 + H^{-d}\Delta t^4). \qquad (87)
\end{aligned}$$

Combining equation (83) with equations (85)-(87), taking the $\delta\|\boldsymbol{\eta}^n\|^2$ term to the left side, multiplying by $2\Delta t$ and summing over $n, n = 1, \ldots, N^*$ where N^* is the time step at which $\|\gamma^n\|$ achieves its maximum value gives,

$$\begin{aligned}
&\|\gamma^{N^*}\|^2 - \|\gamma^0\| + \Delta t \sum_{n=1}^{N^*} K_* \|\boldsymbol{\eta}^n\|_{\mathrm{TM}}^2 \\
&\leq \Delta t \sum_{n=1}^{N^*} (\|\epsilon^n\|^2 + \|d_t \alpha^n\|^2) + C\Delta t \sum_{n=1}^{N^*} \|\gamma^n\|^2 \\
&\quad + C(H^{4-d} + H^{-d}\Delta t^2 + h^4)\Delta t \sum_{n=1}^{N^*} (\|\tilde{\mathbf{U}}_H^n - \tilde{\mathbf{u}}^n\|_{\mathrm{TM},H}^2 + \|\tilde{\mathbf{u}}^n - \underline{\tilde{\mathbf{U}}}_h^n\|_{\mathrm{TM}}^2 + H^4) \\
&\quad + C(h^4 + \|\gamma^{N^*}\|^2)\Delta t \sum_{n=1}^{N^*} H^{-d}\|\tilde{\mathbf{U}}_H^n - \tilde{\mathbf{u}}^n\|^2 \\
&\quad + C(H^{8-d} + h^4 + \Delta t^2 + H^{-d}h^4\Delta t^2 + H^{-d}\Delta t^4).
\end{aligned}$$

Recalling the bound on ϵ^n, using Theorem 3.3 and recalling the initial conditions on $\underline{P}_h^0$ and P_h^0 gives,

$$\begin{aligned}
\|\gamma^{N^*}\|^2 + \Delta t \sum_{n=1}^{N^*} K_* \|\boldsymbol{\eta}^n\|_{\mathrm{TM}}^2 &\leq C(H^{8-d} + h^4 + \Delta t^2 + H^{-d}h^4\Delta t^2 + H^{-d}\Delta t^4) \\
&\quad + C\Delta t \sum_{n=1}^{N^*} \|\gamma^n\|^2 + \tilde{C}\|\gamma^{N^*}\|^2 (H^{4-d} + H^{-d}\Delta t^2).
\end{aligned}$$

We can choose H and Δt such that $H^{4-d} + H^{-d}\Delta t^2 \leq \frac{1}{2\tilde{C}}$, and the last term can be moved to the left-hand side. Applying Gronwall's Lemma gives

$$\|\gamma^{N^*}\|^2 + \Delta t \sum_{n=1}^{N^*} K_* \|\boldsymbol{\eta}^n\|_{\mathrm{TM}}^2 \leq C(H^{8-d} + h^4 + \Delta t^2).$$

Applying Theorem 3.3 and the triangle inequality give the desired result. □

5. Conclusions. We have presented and derived error estimates for a two level finite difference scheme for nonlinear parabolic equations. Through the use of the $\mathcal{P}_h, \mathcal{P}_H$ and $\mathcal{Q}_H$ operators, we have taken advantage of superconvergent node points and have shown optimal order convergence in both H^1 and L^2 for the coarse and fine grids.

We remark that we have only considered the case of Neumann boundary conditons and a diagonal tensor K. The expanded mixed method employed here was developed in order to handle a full symmetric tensor for K. However, in the case of a full tensor, convergence may be lost on the boundary. In this case we can show a coarse grid estimate of $H^r + \Delta t$ and a fine grid estimate of $h^r + H^{2r-d/2} + \Delta t$, where $r = 2$ if K is diagonal and no Dirichlet conditions are enforced, $r = 3/2$ if K is diagonal or the grids are generated by a C^2 map, and $r = 1$ otherwise for 2 dimensions and no convergence otherwise for 3 dimensions. These estimates for a full tensor use the inverse estimate and may be improved through the derivation of explicit L^∞ estimates. The loss of convergence due to the boundary is shown for the expanded mixed method in [1].

The estimates derived in this paper use the inverse estimate to bound L^∞ norms in terms of L^2 norms. As a result, our estimates may not be as sharp as possible. However, no better L^∞ estimates exist at this time for the expanded mixed finite element method.

The two level scheme described above could be extended by adding more levels and expanding about the next coarser solution in the nonlinear term at each new level. This corresponds to adding more Newton-like iterations with each iteration taking place on the next finer grid. This possibility is under investigation. We are currently implementing these two-level methods for equations of interest to flow in porous media. Computational results for this work will be contained in later papers.

REFERENCES

[1] T. Arbogast, M. F. Wheeler, and I. Yotov, *Mixed finite elements for elliptic problems with tensor coefficients as cell-centered finite differences*, Dept. Comp. Appl. Math. TR95-06, Rice University, Houston, TX 77251, Mar. 1995. To appear SIAM J. Numer. Anal., 1997, vol. 34.

[2] O. Axelsson, *Iterative Solution Methods*, Cambridge University Press, New York, 1994.

[3] S. C. Brenner and L. R. Scott, *The Mathematical Theory of Finite Element Methods*, Springer-Verlag, New York, 1994.

[4] P. J. Davis, *Interpolation and Approximation*, Dover Publications, New York, 1975.

[5] C. N. Dawson and M. F. Wheeler, *Two-grid methods for mixed finite element approximations of nonlinear parabolic equations*, Contemporary Mathematics, 180 (1994), pp. 191–203.

[6] R. G. Durán, *Superconvergence for rectangular mixed finite elements*, Numer. Math., 58 (1990), pp. 287–298.

[7] J. Nedelec, *Mixed finite elements in* $\mathbb{R}^3$, Numer. Math., 35 (1980), pp. 315–341.

[8] J. M. Ortega and W. C. Reinbolt, *Iterative Solution of Nonlinear Equations in Several Variables*, Academic Press, San Diego, 1970.

[9] P. A. Raviart and J. M. Thomas, *A mixed finite element method for second order elliptic problems*, in Mathematical Aspects of Finite Element Methods: Lecture Notes in Mathematics 606, I. Galligani and E. Magenes, eds., Berlin, 1977, Springer-Verlag, pp. 292–315.

[10] A. Weiser and M. Wheeler, *On convergence of block-centered finite differences for elliptic problems*, SIAM J. Numer. Anal., 25 (1988), pp. 351–357.

[11] J. Xu, *Two-grid finite element discretizations for nonlinear elliptic equations*, Dept. of Mathematics AM105, Pennsylvania State University, University Park, Pennsylvania, July 1992. To appear in SIAM J Numer. Anal.

[12] ———, *A novel two-grid method for semilinear elliptic equations*, SIAM J. Sci. Comput., 15 (1994), pp. 231–237.