

**A Two-Grid Method for Nonlinear
Parabolic Equations**

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A TWO-GRID FINITE DIFFERENCE SCHEME FOR NONLINEAR PARABOLIC EQUATIONS *

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Abstract. We present a two level finite difference scheme for the approximation of nonlinear parabolic equations. Discrete inner products and the lowest order Raviart-Thomas approximating space are used in the expanded mixed method in order to develop the finite difference scheme. Analysis of the scheme is given assuming an implicit time discretization. In this two level scheme, the full nonlinear problem is solved on a “coarse” grid of size H . The nonlinearities are expanded about the coarse grid solution and an appropriate interpolation operator is used to provide values of the coarse grid solution on the fine grid in terms of superconvergent node points. The resulting linear but nonsymmetric system is solved on a “fine” grid of size h . Some *a priori* error estimates are derived which show that the discrete $L^\infty(L^2)$ and $L^2(H^1)$ errors are $O(h^2 + H^{4-d/2} + \Delta t)$, where $d \geq 1$ is the spatial dimension.

Keywords: error estimates, finite differences, mixed finite elements, nonlinear, superconvergent

AMS(MOS) subject classification: 65N06, 65N12, 65N15, 65N22, 65N30

1. Introduction. In this paper, we consider a finite difference approximation scheme for the solution of the nonlinear parabolic differential equation

$$(1) \quad \frac{\partial p}{\partial t} - \nabla \cdot (K(p) \nabla p) = f(t, \mathbf{x}) \text{ in } (0, T] \times \Omega,$$

$$(2) \quad p(0, \mathbf{x}) = p^0(\mathbf{x}) \text{ in } \Omega,$$

$$(3) \quad -(K(p) \nabla p) \cdot \nu = g \text{ on } (0, T] \times \Gamma,$$

where Ω is a rectangular domain in \mathbb{R}^d ($d = 2$ or 3) with boundary Γ , ν is the outward, unit, normal vector on Γ , and $K : \Omega \times \mathbb{R} \rightarrow \mathbb{R}^{d \times d}$ is a symmetric, positive definite second order diagonal tensor. We also assume that there exist positive constants K_* and K^* such that for $z \in \mathbb{R}^d$,

$$(4) \quad K_* \|z\|^2 \leq z^t K(\mathbf{x}, s) z \leq K^* \|z\|^2, \text{ for } \mathbf{x} \in \Omega \text{ and } s \in \mathbb{R},$$

and that each element of K is twice continuously differentiable in space and time with derivatives up to second order bounded above by K^* .

Equation (1) arises in many applications. Our particular interest is to view (1) as a simplification of Richard’s equation, a nonlinear parabolic equation arising in the modeling of flow through porous media.

Both the solution of (1) on a fine grid of diameter h at each time step and a two level procedure involving the solution of the nonlinear problem only on a coarse grid of diameter $H \gg h$ will be discussed. In the two level scheme for the fine grid, we approximate $K(p)$ by a first order Taylor expansion about the solution from the coarse grid. This work is motivated by the work of Xu [10, 11] for Galerkin procedures applied to nonlinear elliptic equations and the work of Dawson and Wheeler [4] for the expanded mixed method applied to nonlinear parabolic equations. Xu showed optimal H^1 estimates for both the coarse and fine grids, but his results were not optimal in L^2 . Dawson and Wheeler showed optimal H^1 and L^2 estimates for the coarse and fine grids, and for the case of the lowest order Raviart-Thomas space, they showed superconvergence results for the coarse grid in both norms. By

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using a finite difference scheme taking advantage of superconvergent node points, we are able to show superconvergence of the flux and the pressure in certain discrete H^1 and L^2 norms.

This paper is organized into five sections. We establish notation and some basic approximation results in Section 2. The coarse grid scheme is presented in Section 3 along with optimal order error estimates. The fine grid scheme with error estimates is presented in Section 4. In Section 5 we give conclusions, remarks and extensions of this work.

2. Notation and Approximation Results.

In this section we define some notation.

Let $0 = t^0 < t^1 < \dots < t^N = T$ be a given sequence, $\Delta t^n = t^n - t^{n-1}$, $\Delta t = \max_n \Delta t^n$, and for $\phi = \phi(t, \cdot)$, let $\phi^n = \phi(t^n, \cdot)$ and

$$d_t \phi^n = \frac{\phi^n - \phi^{n-1}}{\Delta t^n}$$

Let (\cdot, \cdot) denote the $L^2(\Omega)$ inner product, scalar and vector. Let $V = H(\Omega, \text{div}) = \{\mathbf{v} \in (L^2(\Omega))^d : \nabla \cdot \mathbf{v} \in L^2(\Omega)\}$ and $W = L^2(\Omega)$. We denote the subspaces of V containing functions with normal traces weakly equal to 0 and g^n as V^0 and V^n , respectively.

We will consider two quasi-uniform triangulations of Ω , a coarse triangulation with mesh size H denoted by \mathcal{T}_H , and a refinement of this triangulation with mesh size h denoted by \mathcal{T}_h . Both of these triangulations will consist of rectangles in two dimension or bricks in three dimensions. We consider the lowest order Raviart-Thomas-Nedelec space on rectangles, [8, 6]. Thus, on an element $E \in \mathcal{T}_k, k = h$ or H , we have

$$V_k(E) = \{(\alpha_1 x_1 + \beta_1, \alpha_2 x_2 + \beta_2, \alpha_3 x_3 + \beta_3)^T : \alpha_i, \beta_i \in \mathbb{R}\},$$

$$W_k(E) = \{\alpha : \alpha \in \mathbb{R}\},$$

where the last component in V_k should be deleted in two dimensions. Define $V_k^0 = V^0 \cap V_k$ and $V_k^n = V^n \cap V_k$.

We use the standard nodal basis, where for V_k the nodes are at the midpoints of edges or faces of the elements, and for W_k the nodes are at the centers of the elements. We denote the grid points of the fine grid by

$$(x_{i+1/2}, y_{j+1/2}), i = 0, \dots, N_x, j = 0, \dots, N_y,$$

and define

$$x_i = \frac{1}{2}(x_{i+1/2} + x_{i-1/2}), i = 1, \dots, N_x,$$

$$y_j = \frac{1}{2}(y_{j+1/2} + y_{j-1/2}), j = 1, \dots, N_y,$$

$$h_{i+1/2}^x = x_{i+1} - x_i, i = 1, \dots, N_x - 1,$$

$$h_{j+1/2}^y = y_{j+1} - y_j, j = 1, \dots, N_y - 1,$$

$$h_i^x = x_{i+1/2} - x_{i-1/2}, i = 1, \dots, N_x,$$

$$h_j^y = y_{j+1/2} - y_{j-1/2}, j = 1, \dots, N_y,$$

$$h = \max_{i,j} (h_i^x, h_j^y),$$

with corresponding notation for a third dimension. For the coarse grid, similar quantities are defined, but the number of points in each direction are notated as \hat{N}_x and \hat{N}_y .

We define discrete inner products corresponding to applications of the midpoint (M), trapezoidal (T) and midpoint by trapezoidal (TM) quadrature rules by

$$\begin{aligned}
(r, s)_M &= \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} h_i^x h_j^y r_{ij} s_{ij}, \\
(\mathbf{v}, \mathbf{q})_{TM} &= \sum_{i=0}^{N_x} \sum_{j=1}^{N_y} h_{i+1/2}^x h_j^y v_{i+1/2}^x q_{i+1/2}^x + \sum_{i=1}^{N_x} \sum_{j=0}^{N_y} h_i^x h_{j+1/2}^y v_{i+1/2}^y q_{i+1/2}^y, \\
(\mathbf{v}, \mathbf{q})_T &= \sum_{i=0}^{N_x} \sum_{j=1}^{N_y} h_{i+1/2}^x h_j^y \frac{1}{2} (v_{i+1/2}^x q_{i+1/2}^x + v_{i+1/2}^x q_{i+1/2}^x) \\
&\quad + \sum_{i=1}^{N_x} \sum_{j=0}^{N_y} h_i^x h_{j+1/2}^y \frac{1}{2} (v_{i+1/2}^y q_{i+1/2}^y + v_{i+1/2}^y q_{i+1/2}^y),
\end{aligned}$$

where we add a third sum in each for the case of three dimensions. We denote the associated norms by $\|\cdot\|_R$, where $R = M, T$ or TM and by $E_R(\mathbf{q}, \mathbf{r})$, the error in approximating an integral by the given rule, i.e. $E_T(\mathbf{q}, \mathbf{r}) = (\mathbf{q}, \mathbf{r}) - (\mathbf{q}, \mathbf{r})_T$. The error in approximating an integral by either the trapezoidal or the trapezoidal by midpoint rule is [3],

$$(5) \quad |E_Q(\mathbf{q}, \mathbf{v})| \leq C \sum_{E \in \mathcal{T}_h} \sum_{|\alpha|=2} \left\| \frac{\partial^\alpha}{\partial \mathbf{x}^\alpha} (\mathbf{q} \cdot \mathbf{v}) \right\|_{L^1, E} h^2.$$

For any $\phi \in L^2(\Omega)$ we let $\hat{\phi}_k$ denote the L^2 projection of ϕ onto W_k , i.e.

$$(6) \quad (\phi, w) = (\hat{\phi}_k, w), \quad \forall w \in W_k.$$

This L^2 projection operator has the following approximation property for $\phi \in H^j(\Omega)$,

$$(7) \quad \|\hat{\phi}_k - \phi\| \leq C \|\phi\|_j k^j, \quad 0 \leq j \leq 1,$$

for $k = h$ or H .

Associated with the RTN mixed finite element spaces is the projection operator $\Pi : (H^1(\Omega))^d \rightarrow V_k$, such that

$$(8) \quad (\nabla \cdot \Pi \mathbf{q}, w) = (\nabla \cdot \mathbf{q}, w), \quad \forall w \in W_k,$$

$$(9) \quad \|\mathbf{q} - \Pi \mathbf{q}\| \leq C \|\mathbf{q}\|_1 k,$$

$$(10) \quad \|\nabla \cdot (\mathbf{q} - \Pi \mathbf{q})\| \leq C \|\nabla \cdot \mathbf{q}\|_1 k.$$

We will use the estimate [5]

$$(11) \quad \|\Pi \mathbf{u}^n - \mathbf{u}^n\|_{TM} + \|\Pi \tilde{\mathbf{u}}^n - \tilde{\mathbf{u}}^n\|_{TM} \leq C k^2,$$

where, again, we let $k = h$ or H .

We will also make use of the following lemma proven in Arbogast, Wheeler and Yotov [1].

LEMMA 2.1. *For the lowest order RTN spaces on rectangles, for any $\mathbf{q} = (q^x, q^y) \in H^1(\Omega)$ and $E \in \mathcal{T}_k$,*

$$(12) \quad \left\| \frac{\partial}{\partial x} (\Pi \mathbf{q})^x \right\|_{0, E} \leq \left\| \frac{\partial q^x}{\partial x} \right\|_{0, E},$$

$$(13) \quad \left\| \frac{\partial}{\partial y} (\Pi \mathbf{q})^y \right\|_{0, E} \leq \left\| \frac{\partial q^y}{\partial y} \right\|_{0, E}.$$

In the following arguments, C will represent a generic constant independent of H, h and Δt . We will use the standard inequality,

$$(14) \quad ab \leq \frac{\delta}{2}a^2 + \frac{1}{2\delta}b^2 \quad a, b, \delta \in \mathbb{R}, \quad \delta > 0.$$

3. A Coarse Grid Nonlinear Finite Difference Scheme. In this section we develop and give convergence estimates for a nonlinear cell-centered finite difference scheme on the coarse grid. For simplicity we consider two dimensions and note that extensions to three dimensions are straightforward.

3.1. Definition of the Scheme. The variational formulation for (1) at time t^n is to find $(p^n, \tilde{\mathbf{u}}^n, \mathbf{u}^n) \in (W \times V \times V^n)$ satisfying

$$(15) \quad (p_t^n, w) + (\nabla \cdot \mathbf{u}^n, w) = (f^n, w), \quad \forall w \in W,$$

$$(16) \quad (\tilde{\mathbf{u}}^n, \mathbf{v}) = (p^n, \nabla \cdot \mathbf{v}), \quad \forall \mathbf{v} \in V^0,$$

$$(17) \quad (\mathbf{u}^n, \mathbf{v}) = (K(p^n)\tilde{\mathbf{u}}^n, \mathbf{v}), \quad \forall \mathbf{v} \in V,$$

We choose cell-centered finite difference approximations $P_H^n \in W_H, \tilde{\mathbf{U}}_H^n \in V_H$ and $\mathbf{U}_H^n \in V_H^n$ to the functions $p(t^n, \cdot), \tilde{\mathbf{u}}(t^n, \cdot)$ and $\mathbf{u}(t^n, \cdot)$, respectively, for each $n = 1, \dots, N$, satisfying

$$(18) \quad (d_t P_H^n, w) + (\nabla \cdot \mathbf{U}_H^n, w) = (f^n, w), \quad \forall w \in W_H,$$

$$(19) \quad (\tilde{\mathbf{U}}_H^n, \mathbf{v})_{\text{TM}} = (P_H^n, \nabla \cdot \mathbf{v}), \quad \forall \mathbf{v} \in V_H^0,$$

$$(20) \quad (\mathbf{U}_H^n, \mathbf{v})_{\text{TM}} = (K(\mathcal{P}_H(P_H^n))\tilde{\mathbf{U}}_H^n, \mathbf{v})_{\text{T}}, \quad \forall \mathbf{v} \in V_H,$$

and we take $P_H^0 = \hat{p}_H(t^0, \cdot)$. This scheme is based on an expansion of the standard mixed finite element method that was formulated for linear elliptic problems in [1].

We define $\mathcal{P}_H(p)$ from the values of p_{ij} for $i = 1, \dots, \hat{N}_x$ and $j = 1, \dots, \hat{N}_y$ as follows. For points (x, y) such that $x_i \leq x \leq x_{i+1}, i \in \{1, \dots, \hat{N}_x\}$ and $y_j \leq y \leq y_{j+1}, j \in \{1, \dots, \hat{N}_y\}$, we take $\mathcal{P}_H(p)(x, y)$ to be the bilinear interpolant,

$$\begin{aligned} \mathcal{P}_H(p)(x, y) &= (p_{ij}(\frac{x_{i+1}-x}{x_{i+1}-x_i}) + p_{i+1,j}(\frac{x-x_i}{x_{i+1}-x_i}))(\frac{y_{j+1}-y}{y_{j+1}-y_j}) \\ &\quad + (p_{ij+1}(\frac{x_{i+1}-x}{x_{i+1}-x_i}) + p_{i+1,j+1}(\frac{x-x_i}{x_{i+1}-x_i}))(\frac{y-y_j}{y_{j+1}-y_j}). \end{aligned}$$

For $i = 1, \dots, \hat{N}_x - 1$, we set

$$\mathcal{P}_H(p)(x_i, y_{1/2}) = \frac{(2H_1^y + H_2^y)p_{i1} - H_1^y p_{i2}}{H_1^y + H_2^y}.$$

This is a two point extrapolation, and by Taylor's theorem we have $|(\mathcal{P}_H(p) - p)(x_i, y_{1/2})| = CH^2$. For points (x, y) such that $x_i \leq x \leq x_{i+1}$ and $y_{1/2} \leq y \leq y_1$, we define $\mathcal{P}_H(p)$ as the bilinear interpolant between $p_{i,1}, p_{i+1,1}, \mathcal{P}_H(p)(x_i, y_{1/2})$ and $\mathcal{P}_H(p)(x_{i+1}, y_{1/2})$. By interpolation theory $|\mathcal{P}_H(p) - p| = CH^2$ for these points. In a similar way we can define $\mathcal{P}_H(p)$ for (x, y) such that $x_i \leq x \leq x_{i+1}$ and $y_{\hat{N}_y} \leq y \leq y_{\hat{N}_y+1/2}$ as well as for points (x, y) where $x_{1/2} \leq x \leq x_1$ or $x_{\hat{N}_x} \leq x \leq x_{\hat{N}_x+1/2}$ and $y_j \leq y \leq y_{j+1}$ for j such that

$1 \leq j \leq \hat{N}_y$. Lastly, we define $\mathcal{P}_H(p)$ at the corners of the domain. Here, we use three point extrapolation,

$$\begin{aligned}\mathcal{P}_H(p)(x_{1/2}, y_{1/2}) &= \mathcal{P}_H(p)_{1,1/2} + \mathcal{P}_H(p)_{1/2,1} - p_{1,1} \\ &= p_{1,1/2} + p_{1/2,1} - p_{1,1} + O(H^2).\end{aligned}$$

By Taylor's theorem, $|(\mathcal{P}_H(p) - p)(x_{1/2}, y_{1/2})| \leq CH^2$. For points (x, y) such that $x_{1/2} \leq x \leq x_1$ and $y_{1/2} \leq y \leq y_1$, we define $\mathcal{P}_H(p)(x, y)$ as the bilinear interpolant of $\mathcal{P}_H(p)(x_{1/2}, y_{1/2})$, $\mathcal{P}_H(p)(x_{1/2}, y_1)$, $\mathcal{P}_H(p)(x_1, y_{1/2})$ and $p_{1,1}$ which is an $O(H^2)$ approximation to $p(x, y)$ within this "corner region". Similarly, we can define $\mathcal{P}_H(p)$ as an $O(H^2)$ approximation to p in the other three "corner" regions.

We have just proven the following lemma,

LEMMA 3.1. *If p is twice differentiable, then for $\mathcal{P}_H(p)$ defined above,*

$$\|\mathcal{P}_H(p) - p\|_\infty \leq CH^2.$$

If a uniform mesh is used and K is a diagonal tensor, equations (18)-(20) reduce to a standard nonlinear finite difference procedure. Denoting P_H^n by P^n , we have in the interior of Ω :

$$\begin{aligned}f_{ij}^n H^2 + P_{ij}^{n-1} \frac{H^2}{\Delta t} &= \frac{1}{2}[(K_{11}(\mathcal{P}_H(P^n))_{i+1/2,j+1/2} + K_{11}(\mathcal{P}_H(P^n))_{i+1/2,j-1/2})(P_{ij}^n - P_{i+1,j}^n) \\ &\quad + (K_{11}(\mathcal{P}_H(P^n))_{i-1/2,j+1/2} + K_{11}(\mathcal{P}_H(P^n))_{i-1/2,j-1/2})(P_{ij}^n - P_{i-1,j}^n) \\ &\quad + (K_{22}(\mathcal{P}_H(P^n))_{i+1/2,j+1/2} + K_{22}(\mathcal{P}_H(P^n))_{i-1/2,j+1/2})(P_{ij}^n - P_{ij+1}^n) \\ &\quad + (K_{22}(\mathcal{P}_H(P^n))_{i+1/2,j-1/2} + K_{22}(\mathcal{P}_H(P^n))_{i-1/2,j-1/2})(P_{ij}^n - P_{ij-1}^n)] \\ &\quad + \frac{H^2}{\Delta t} P_{ij}^n.\end{aligned}$$

Existence and uniqueness of a solution to this discrete nonlinear problem is given in the following theorem.

THEOREM 3.2. *For time t^n and Δt sufficiently small, there exists a unique solution to equations (18)-(20).*

Proof. We are seeking a unique solution to the nonlinear equation $F(P^n) = 0$, where $F(P^n) = b^n + P^n + \frac{\Delta t}{H^2} A(P^n) P^n$. Here, b^n is a vector whose entry corresponding to grid cell (x_i, y_j) is $-\frac{\Delta t}{H^2} \int_{\Omega_{ij}} f_{ij}^n - P_{ij}^{n-1}$, P is a vector whose ij th entry corresponds to the value of the scalar variable P_{ij}^n and A is a matrix function of P^n given by the stencil above. By Theorem 5.4.5 of Ortega and Reinbolt [7], if F is continuously differentiable and uniformly monotone on \mathbb{R}^n , then a unique solution to $F(P^n) = 0$ exists. It is easily verified that the F defined above is continuously differentiable. In order to prove that F is uniformly monotonic we note that uniform monotonicity is equivalent to positive definiteness of the Jacobian, $J = F'$, and that a real matrix J is positive definite if and only if its symmetric part, $(J + J^T)/2$, is positive definite [2, Lemma 3.1]. Furthermore, we know that if a matrix is strictly diagonal dominant with positive diagonal entries, then the eigenvalues of the matrix have positive real parts [2, Theorem 4.9]. Now, $J = I + \frac{\Delta t}{H^2} A(P^n) + \frac{\Delta t}{H^2} A'(P^n) P^n$. Thus, with $\frac{\Delta t}{H^2}$ sufficiently small, we have that the symmetric part of J has positive real eigenvalues and, hence, is positive definite, making J positive definite and F uniformly monotonic. \square

3.2. Preliminary Estimates. Before we show convergence estimates for this finite difference scheme, we show convergence for a related linear scheme. The arguments given below closely follow those of Arbogast, Wheeler and Yotov [1] except that we extend their work to time differenced time dependent problems.

THEOREM 3.3. *For each $n = 1, \dots, N$, let $(\underline{P}_H^n, \tilde{\underline{U}}_H^n, \underline{U}_H^n) \in (W_H \times V_H \times V_H^n)$ satisfy*

$$(21) \quad (\nabla \cdot \underline{U}_H^n, w) = (b^n, w), \quad \forall w \in W_H,$$

$$(22) \quad (\tilde{\underline{U}}_H^n, \mathbf{v})_{\text{TM}} = (\underline{P}_H^n, \nabla \cdot \mathbf{v}), \quad \forall \mathbf{v} \in V_H^0,$$

$$(23) \quad (\underline{U}_H^n, \mathbf{v})_{\text{TM}} = (K(\mathcal{P}_H(p^n))\tilde{\underline{U}}_H^n, \mathbf{v})_{\text{T}}, \quad \forall \mathbf{v} \in V_H,$$

with $b^n = f^n - p_t^n$ and $\underline{P}_H^0 = \hat{p}_H^0$. Then,

$$(24) \quad \|\underline{U}_H^n - \mathbf{u}^n\|_{\text{TM}} + \|\tilde{\underline{U}}_H^n - \tilde{\mathbf{u}}^n\|_{\text{TM}} \leq CH^2,$$

$$(25) \quad \|\underline{P}_H^n - p^n\|_{\text{M}} \leq CH^2,$$

$$(26) \quad \|d_t \underline{P}_H^n - d_t p^n\|_{\text{M}} \leq C(H^2 + \Delta t).$$

In order to prove this theorem, we will first prove two preliminary lemmas.

LEMMA 3.4. *Assume for each $n = 1, \dots, N$, that $p^n, p_t^n \in W_3^4(\Omega)$. Then, there exist $\tilde{\mathbf{U}}^{*,n} \in V_H, P^{*,n} \in W_H, \mathbf{Z}^{*,n} \in V_H^n, \tilde{\mathbf{Z}}^{*,n} \in V_H$ and $W^{*,n} \in W_H$ such that*

$$(27) \quad (\tilde{\mathbf{U}}^{*,n}, \mathbf{v})_{\text{TM}} = (P^{*,n}, \nabla \cdot \mathbf{v}), \quad \forall \mathbf{v} \in V_H^0,$$

$$(28) \quad (\tilde{\mathbf{Z}}^{*,n}, \mathbf{v})_{\text{TM}} = (W^{*,n}, \nabla \cdot \mathbf{v}), \quad \forall \mathbf{v} \in V_H^0,$$

$$(29) \quad (\mathbf{Z}^{*,n}, \mathbf{v})_{\text{TM}} = (K(p^n)\tilde{\mathbf{Z}}^{*,n}, \mathbf{v})_{\text{T}} + (K(p^n)_t \tilde{\mathbf{U}}^{*,n}, \mathbf{v})_{\text{T}}, \quad \forall \mathbf{v} \in V_H,$$

and there exists a constant C independent of H such that for all i, j ,

$$(30) \quad |P_{i,j}^{*,n} - p_{i,j}^n| \leq CH^2,$$

$$(31) \quad |W_{i,j}^{*,n} - p_{t,i,j}^n| \leq CH^2,$$

$$(32) \quad |\tilde{U}_{x,i+1/2j}^{*,n} - \tilde{u}_{x,i+1/2j}^n| + |\tilde{U}_{y,ij+1/2}^{*,n} - \tilde{u}_{y,ij+1/2}^n| \leq CH^2,$$

$$(33) \quad |\tilde{Z}_{x,i+1/2j}^{*,n} - \tilde{u}_{t,x,i+1/2j}^n| + |\tilde{Z}_{y,ij+1/2}^{*,n} - \tilde{u}_{t,y,ij+1/2}^n| \leq CH^2,$$

$$(34) \quad |Z_{x,i+1/2j}^{*,n} - u_{t,x,i+1/2j}^n| + |Z_{y,ij+1/2}^{*,n} - u_{t,y,ij+1/2}^n| \leq CH^2.$$

Proof. Arbogast, Wheeler and Yotov [1] present a lemma which gives the desired $P^{*,n}$ and $\tilde{\mathbf{U}}^{*,n}$ above. In order to derive (31) and (33), we apply a lemma due to Weiser and Wheeler [9] to the solution pair $(\tilde{\mathbf{u}}_t^n, p_t^n)$ satisfying the elliptic problem

$$\begin{aligned} \nabla \cdot \tilde{\mathbf{u}}_t^n &= F^n, \text{ in } \Omega, \\ \tilde{\mathbf{u}}_t^n &= -\nabla p_t^n, \text{ in } \Omega, \\ p_t^n &= G^n \equiv \frac{\partial p^n}{\partial t} \Big|_{\partial\Omega}, \text{ on } \partial\Omega, \end{aligned}$$

where $F^n = f_t^n + p_{tt}^n$. This result gives a $W^{*,n}$ satisfying (31) and through (28), $\tilde{\mathbf{Z}}^{*,n}$ satisfies (33) in the interior of Ω . Define $\tilde{\mathbf{Z}}$ on Γ by,

$$\begin{aligned} \tilde{Z}_{x,i+1/2j}^{*,n} &= \tilde{u}_{t,x,i+1/2j}^n, \\ \tilde{Z}_{y,ij+1/2}^{*,n} &= \tilde{u}_{t,y,ij+1/2}^n. \end{aligned}$$

Then, (33) clearly holds on Γ .

Choosing \mathbf{v} in (29) to be the basis function associated with node $(x_{i+1/2}, y_j)$, we have for $i = 1, \dots, \hat{N}_x - 1$,

$$\begin{aligned} \mathbf{Z}_{x,i+1/2j}^{*,n} &= \frac{1}{2}[K_{11}(p^n)_{i+1/2j+1/2} + K_{11}(p^n)_{i+1/2j-1/2}]\tilde{\mathbf{Z}}_{x,i+1/2j}^{*,n} \\ &\quad + \frac{1}{2}[K_{11}(p^n)_{t,i+1/2j+1/2} + K_{11}(p^n)_{t,i+1/2j-1/2}]\tilde{\mathbf{U}}_{x,i+1/2j}^{*,n}. \end{aligned}$$

Since $\mathbf{u}_t^n = K(p^n)\tilde{\mathbf{u}}_t^n + K(p^n)_t\tilde{\mathbf{u}}^n$, Taylor's theorem gives for $i = 1, \dots, \hat{N}_x - 1$,

$$\begin{aligned} \mathbf{u}_{t,x,i+1/2j}^n &= \frac{1}{2}[K_{11}(p^n)_{i+1/2j+1/2} + K_{11}(p^n)_{i+1/2j-1/2}]\tilde{\mathbf{u}}_{t,x,i+1/2j}^n \\ &\quad + \frac{1}{2}[K_{11}(p^n)_{t,i+1/2j+1/2} + K_{11}(p^n)_{t,i+1/2j-1/2}]\tilde{\mathbf{u}}_{x,i+1/2j}^n + O(H^2). \end{aligned}$$

Therefore,

$$|\mathbf{Z}_{x,i+1/2j}^{*,n} - \mathbf{u}_{t,x,i+1/2j}^n| \leq C|\tilde{\mathbf{Z}}_{x,i+1/2j}^{*,n} - \tilde{\mathbf{u}}_{t,x,i+1/2j}^n| + O(H^2).$$

In a similar manner we can bound $|\mathbf{Z}_{y,ij+1/2}^{*,n} - \mathbf{u}_{t,y,ij+1/2}^n|$, and (34) follows. \square

We can now extend a corollary from Arbogast, Wheeler and Yotov [1] to give: For the $\tilde{\mathbf{U}}^{*,n}, P^{*,n}, \mathbf{Z}^{*,n}, \tilde{\mathbf{Z}}^{*,n}$ and $W^{*,n}$ in Lemma 3.4, there exists a constant C , independent of H , such that

$$\begin{aligned} \|\tilde{\mathbf{U}}^{*,n} - \tilde{\mathbf{u}}^n\|_{\text{TM}} &\leq CH^2, \\ \|\tilde{\mathbf{Z}}^{*,n} - \tilde{\mathbf{u}}_t^n\|_{\text{TM}} &\leq CH^2, \\ \|\mathbf{Z}^{*,n} - \mathbf{u}_t^n\|_{\text{TM}} &\leq CH^2. \end{aligned}$$

LEMMA 3.5. *There exists a constant C independent of H and Δt such that*

$$(35) \quad \|\nabla \cdot (d_t \mathbf{u}^n - d_t \underline{\mathbf{U}}_H^n)\| \leq CH,$$

$$(36) \quad \|d_t \tilde{\mathbf{u}}^n - d_t \tilde{\underline{\mathbf{U}}}_H^n\|_{\text{TM}} + \|d_t \mathbf{u}^n - d_t \underline{\mathbf{U}}_H^n\|_{\text{TM}} \leq C(H^2 + \Delta t).$$

Proof. To prove this lemma, we consider the time difference of (15)-(17),

$$(37) \quad (\nabla \cdot d_t \mathbf{u}^n, w) = (d_t b^n, w), \quad \forall w \in W,$$

$$(38) \quad (d_t \tilde{\mathbf{u}}^n, \mathbf{v}) = (d_t p^n, \nabla \cdot \mathbf{v}), \quad \forall \mathbf{v} \in V^0,$$

$$(39) \quad (d_t \mathbf{u}^n, \mathbf{v}) = (d_t(K(p^n))\tilde{\mathbf{u}}^n, \mathbf{v}) + (K(p^{n-1})d_t \tilde{\mathbf{u}}^n, \mathbf{v}), \quad \forall \mathbf{v} \in V,$$

and the time difference of (21)-(23),

$$(40) \quad (\nabla \cdot d_t \underline{\mathbf{U}}_H^n, w) = (d_t b^n, w), \quad \forall w \in W_H,$$

$$(41) \quad (d_t \tilde{\underline{\mathbf{U}}}_H^n, \mathbf{v})_{\text{TM}} = (d_t \underline{P}_H^n, \nabla \cdot \mathbf{v}), \quad \forall \mathbf{v} \in V_H^0,$$

$$(42) \quad (d_t \underline{\mathbf{U}}_H^n, \mathbf{v})_{\text{TM}} = (d_t(K(\mathcal{P}_H(p^n)))\tilde{\underline{\mathbf{U}}}_H^n, \mathbf{v})_{\text{T}} + (K(\mathcal{P}_H(p^{n-1}))d_t \tilde{\underline{\mathbf{U}}}_H^n, \mathbf{v})_{\text{T}}, \quad \forall \mathbf{v} \in V_H.$$

We subtract (40) from (37), and we subtract (41) and (42) from (28) and (29) to give

$$(43) \quad (\nabla \cdot (d_t \mathbf{u}^n - d_t \underline{\mathbf{U}}_H^n), w) = 0, \quad \forall w \in W_H,$$

$$(44) \quad (\tilde{\mathbf{Z}}^{*,n} - d_t \tilde{\underline{\mathbf{U}}}_H^n, \mathbf{v})_{\text{TM}} = (W^{*,n} - d_t \underline{P}_H^n, \nabla \cdot \mathbf{v}), \quad \forall \mathbf{v} \in V_H^0,$$

$$(45) \quad \begin{aligned} (\mathbf{Z}^{*,n} - d_t \underline{\mathbf{U}}_H^n, \mathbf{v})_{\text{TM}} &= (K(p^n)\tilde{\mathbf{Z}}^{*,n} - K(\mathcal{P}_H(p^{n-1}))d_t \tilde{\underline{\mathbf{U}}}_H^n, \mathbf{v})_{\text{T}} \\ &\quad + (K(p^n)_t \tilde{\mathbf{U}}^{*,n} - d_t(K(\mathcal{P}_H(p^n)))\tilde{\underline{\mathbf{U}}}_H^n, \mathbf{v})_{\text{T}}, \quad \forall \mathbf{v} \in V_H. \end{aligned}$$

Using (43) and applying the Cauchy-Schwarz inequality we have,

$$\begin{aligned} \|\nabla \cdot (d_t \mathbf{u}^n - d_t \underline{\mathbf{U}}_H^n)\|^2 &= (\nabla \cdot (d_t \mathbf{u}^n - d_t \underline{\mathbf{U}}_H^n), \nabla \cdot (d_t \mathbf{u}^n - \Pi d_t \mathbf{u}^n)) \\ &\leq \|\nabla \cdot (d_t \mathbf{u}^n - d_t \underline{\mathbf{U}}_H^n)\| \|\nabla \cdot (d_t \mathbf{u}^n - \Pi d_t \mathbf{u}^n)\|. \end{aligned}$$

Thus, by (10) the first part of the lemma is obtained.

Now, let $\mathbf{v} = \Pi d_t \mathbf{u}^n - d_t \underline{\mathbf{U}}_H^n$ in (44) and $\mathbf{v} = \tilde{\mathbf{Z}}^{*,n} - d_t \tilde{\underline{\mathbf{U}}}_H^n$ in (45), use (43) and combine to get

$$\begin{aligned} (K(p^n) \tilde{\mathbf{Z}}^{*,n} - K(\mathcal{P}_H(p^{n-1})) d_t \tilde{\underline{\mathbf{U}}}_H^n, \tilde{\mathbf{Z}}^{*,n} - d_t \tilde{\underline{\mathbf{U}}}_H^n)_T &= \\ -(\tilde{\mathbf{Z}}^{*,n} - d_t \tilde{\underline{\mathbf{U}}}_H^n, \Pi d_t \mathbf{u}^n - d_t \underline{\mathbf{U}}_H^n)_{\text{TM}} + (\mathbf{Z}^{*,n} - d_t \underline{\mathbf{U}}_H^n, \tilde{\mathbf{Z}}^{*,n} - d_t \tilde{\underline{\mathbf{U}}}_H^n)_{\text{TM}} \\ (46) \quad &- (K(p^n)_t \tilde{\underline{\mathbf{U}}}_H^n - d_t K(\mathcal{P}_H(p^n)) \tilde{\underline{\mathbf{U}}}_H^n, \tilde{\mathbf{Z}}^{*,n} - d_t \tilde{\underline{\mathbf{U}}}_H^n)_T \end{aligned}$$

Adding $(K(\mathcal{P}_H(p^{n-1})) \tilde{\mathbf{Z}}^{*,n}, \tilde{\mathbf{Z}}^{*,n} - d_t \tilde{\underline{\mathbf{U}}}_H^n)_T$ to both sides of (46), using the boundedness assumption on K , Taylor's Theorem, the Cauchy-Schwarz inequality and (14) we have

$$\begin{aligned} \|\tilde{\mathbf{Z}}^{*,n} - d_t \tilde{\underline{\mathbf{U}}}_H^n\|_{\text{TM}} &\leq C(\|\Pi d_t \mathbf{u}^n - \mathbf{Z}^{*,n}\|_{\text{TM}} + \Delta t \|\tilde{\underline{\mathbf{U}}}_H^n\|_{\text{TM}} \\ &\quad + \|d_t(K(p^n) - K(\mathcal{P}_H(p^n))) \tilde{\underline{\mathbf{U}}}_H^n\|_T \\ &\quad + \|d_t K(\mathcal{P}_H(p^n))(\tilde{\underline{\mathbf{U}}}_H^n - \underline{\tilde{\mathbf{U}}}_H^n)\|_T + (h^2 + \Delta t) \|\tilde{\mathbf{Z}}^{*,n}\|_T). \end{aligned}$$

Taylor's theorem, the estimate (11) and Lemma 3.4 imply that $\|\Pi d_t \mathbf{u}^n - \mathbf{Z}^{*,n}\|_{\text{TM}} \leq C(H^2 + \Delta t)$. By Lemma 3.4 $\|\tilde{\underline{\mathbf{U}}}_H^n\|_{\text{TM}}$ and $\|\tilde{\mathbf{Z}}^{*,n}\|_T$ are bounded. Thus, by Taylor's theorem, the Lipschitz condition on K_t , the boundedness of K_t and the approximation properties of \mathcal{P}_H ,

$$(47) \quad \|\tilde{\mathbf{Z}}^{*,n} - d_t \tilde{\underline{\mathbf{U}}}_H^n\|_{\text{TM}} \leq C(H^2 + \Delta t + \|\tilde{\underline{\mathbf{U}}}_H^n - \underline{\tilde{\mathbf{U}}}_H^n\|_T).$$

By results from Arbogast, Wheeler and Yotov [1], $\|\tilde{\underline{\mathbf{U}}}_H^n - \underline{\tilde{\mathbf{U}}}_H^n\|_T \leq CH^2$. Hence, by the triangle inequality and Lemma 3.4,

$$\|d_t \tilde{\mathbf{u}}^n - d_t \tilde{\underline{\mathbf{U}}}_H^n\|_{\text{TM}} \leq C(H^2 + \Delta t).$$

Now, let $\mathbf{v} = \mathbf{Z}^{*,n} - d_t \underline{\mathbf{U}}_H^n$ in (45) and use the Cauchy-Schwarz inequality to get

$$\begin{aligned} \|\mathbf{Z}^{*,n} - d_t \underline{\mathbf{U}}_H^n\|_{\text{TM}} &\leq \|K(p^n) \tilde{\mathbf{Z}}^{*,n} - K(\mathcal{P}_H(p^{n-1})) d_t \tilde{\underline{\mathbf{U}}}_H^n\|_T \\ &\quad + \|K(p^n)_t \tilde{\underline{\mathbf{U}}}_H^n - d_t(K(\mathcal{P}_H(p^n))) \tilde{\underline{\mathbf{U}}}_H^n\|_T. \end{aligned}$$

By the Lipschitz assumptions on K and K_t , Taylor's theorem, the approximation properties of \mathcal{P}_H and the boundedness of K_t ,

$$\begin{aligned} \|\mathbf{Z}^{*,n} - d_t \underline{\mathbf{U}}_H^n\|_{\text{TM}} &\leq \|(K(p^n) - K(\mathcal{P}_H(p^{n-1}))) \tilde{\mathbf{Z}}^{*,n}\|_T + \|K(\mathcal{P}_H(p^{n-1}))(\tilde{\mathbf{Z}}^{*,n} - d_t \tilde{\underline{\mathbf{U}}}_H^n)\|_T \\ &\quad + \|(K(p^n)_t - d_t K(\mathcal{P}_H(p^n))) \tilde{\underline{\mathbf{U}}}_H^n\|_T + \|d_t K(\mathcal{P}_H(p^n))(\tilde{\underline{\mathbf{U}}}_H^n - \underline{\tilde{\mathbf{U}}}_H^n)\|_T \\ &\leq C(H^2 + \Delta t). \end{aligned}$$

The triangle inequality and Lemma 3.4 result in,

$$\|d_t \mathbf{u}^n - d_t \underline{\mathbf{U}}_H^n\|_{\text{TM}} \leq C(H^2 + \Delta t).$$

□

REMARK 3.1. By the inverse assumption, the definition of Π and (11) we have

$$\begin{aligned} \|\tilde{\underline{\mathbf{U}}}_H^n\|_\infty &\leq \|\tilde{\underline{\mathbf{U}}}_H^n - \Pi \tilde{\mathbf{u}}^n\|_\infty + \|\Pi \tilde{\mathbf{u}}^n - \tilde{\mathbf{u}}^n\|_\infty + \|\tilde{\mathbf{u}}^n\|_\infty \\ &\leq CH^{-d/2} \|\tilde{\underline{\mathbf{U}}}_H^n - \Pi \tilde{\mathbf{u}}^n\|_{\text{TM}} + \|\Pi \tilde{\mathbf{u}}^n - \tilde{\mathbf{u}}^n\|_\infty + \|\tilde{\mathbf{u}}^n\|_\infty \\ &\leq C(H^{-d/2}H^2 + H + 1), \end{aligned}$$

where d is the space dimension. Thus, we know that $\|\tilde{\underline{\mathbf{U}}}_H^n\|_\infty$ is bounded.

Proof. (Of Theorem 3.3) Results (24) and (25) have been proven by Arbogast, Wheeler and Yotov [1].

In order to derive (26), we subtract (41)-(42) from (38)-(39) and use the definition of the L^2 projection to give

$$\begin{aligned} (48) \quad (d_t \tilde{\mathbf{u}}^n - d_t \tilde{\underline{\mathbf{U}}}_H^n, \mathbf{v}) + E_{\text{TM}}(d_t \tilde{\underline{\mathbf{U}}}_H^n, \mathbf{v}) &= (d_t \hat{p}_H^n - d_t \underline{P}_H^n, \nabla \cdot \mathbf{v}), \mathbf{v} \in V_H^0, \\ (d_t \mathbf{u}^n - d_t \underline{\mathbf{U}}_H^n, \mathbf{v}) + E_{\text{TM}}(d_t \underline{\mathbf{U}}_H^n, \mathbf{v}) &= (d_t(K(p^n))\tilde{\mathbf{u}}^n - d_t(K(\mathcal{P}_H(p^n)))\tilde{\underline{\mathbf{U}}}_H^n, \mathbf{v}) \\ &\quad + (K(p^{n-1})d_t \tilde{\mathbf{u}}^n - K(\mathcal{P}_H(p^{n-1}))d_t \tilde{\underline{\mathbf{U}}}_H^n, \mathbf{v}) \\ &\quad + E_{\text{T}}(d_t(K(\mathcal{P}_H(p^n)))\tilde{\underline{\mathbf{U}}}_H^n, \mathbf{v}) \\ (49) \quad &\quad + E_{\text{T}}(K(\mathcal{P}_H(p^{n-1}))d_t \tilde{\underline{\mathbf{U}}}_H^n, \mathbf{v}), \mathbf{v} \in V_H, \\ (50) \end{aligned}$$

Let ϕ satisfy the auxilliary problem with $\rho^n \in L^2(\Omega)$

$$(51) \quad -\nabla \cdot K(\mathcal{P}_H(p^{n-1}))\nabla \phi^n = \rho^n, \Omega,$$

$$(52) \quad -K(\mathcal{P}_H(p^{n-1}))\nabla \phi^n \cdot \nu = 0, \Gamma.$$

Elliptic regularity implies that

$$(53) \quad \|\phi^n\|_2 \leq C\|\rho^n\|.$$

By equations (51) and (48) and the definition of Π ,

$$\begin{aligned} &(d_t \hat{p}_H^n - d_t \underline{P}_H^n, \rho^n) \\ &= -(d_t \hat{p}_H^n - d_t \underline{P}_H^n, \nabla \cdot \Pi K(\mathcal{P}_H(p^{n-1}))\nabla \phi^n) \\ &= -(d_t \tilde{\mathbf{u}}^n - d_t \tilde{\underline{\mathbf{U}}}_H^n, \Pi K(\mathcal{P}_H(p^{n-1}))\nabla \phi^n) \\ &\quad - E_{\text{TM}}(d_t \tilde{\underline{\mathbf{U}}}_H^n, \Pi K(\mathcal{P}_H(p^{n-1}))\nabla \phi^n) \\ &= -(d_t \tilde{\mathbf{u}}^n - d_t \tilde{\underline{\mathbf{U}}}_H^n, \Pi K(\mathcal{P}_H(p^{n-1}))\nabla \phi^n - K(\mathcal{P}_H(p^{n-1}))\nabla \phi^n) \\ &\quad - (K(\mathcal{P}_H(p^{n-1}))(d_t \tilde{\mathbf{u}}^n - d_t \tilde{\underline{\mathbf{U}}}_H^n), \nabla \phi^n - \Pi \nabla \phi^n) \\ &\quad - (K(\mathcal{P}_H(p^{n-1}))(d_t \tilde{\mathbf{u}}^n - d_t \tilde{\underline{\mathbf{U}}}_H^n), \Pi \nabla \phi^n) \\ (54) \quad &\quad - E_{\text{TM}}(d_t \tilde{\underline{\mathbf{U}}}_H^n, \Pi K(\mathcal{P}_H(p^{n-1}))\nabla \phi^n). \end{aligned}$$

By (49)

$$\begin{aligned} -(K(\mathcal{P}_H(p^{n-1}))(d_t \tilde{\mathbf{u}}^n - d_t \tilde{\underline{\mathbf{U}}}_H^n), \Pi \nabla \phi^n) &= ((K(p^{n-1}) - K(\mathcal{P}_H(p^{n-1})))d_t \tilde{\mathbf{u}}^n, \Pi \nabla \phi^n) \\ &\quad - (d_t \mathbf{u}^n - d_t \underline{\mathbf{U}}_H^n, \Pi \nabla \phi^n) \\ &\quad + (d_t(K(p^n))\tilde{\mathbf{u}}^n - d_t(K(\mathcal{P}_H(p^n)))\tilde{\underline{\mathbf{U}}}_H^n, \Pi \nabla \phi^n) \\ &\quad - E_{\text{TM}}(d_t \underline{\mathbf{U}}_H^n, \Pi \nabla \phi^n) \\ &\quad + E_{\text{T}}(d_t(K(\mathcal{P}_H(p^n)))\tilde{\underline{\mathbf{U}}}_H^n, \Pi \nabla \phi^n) \\ (55) \quad &\quad + E_{\text{T}}(K(\mathcal{P}_H(p^{n-1}))d_t \tilde{\underline{\mathbf{U}}}_H^n, \Pi \nabla \phi^n). \end{aligned}$$

We also have by integration by parts, (43) and (45)

$$\begin{aligned}
& -(d_t \mathbf{u}^n - d_t \underline{\mathbf{U}}_H^n, \Pi \nabla \phi^n) \\
& = -(d_t \mathbf{u}^n - d_t \underline{\mathbf{U}}_H^n, \Pi \nabla \phi^n - \nabla \phi^n) - (d_t \mathbf{u}^n - d_t \underline{\mathbf{U}}_H^n, \nabla \phi^n) \\
(56) \quad & = -(d_t \mathbf{u}^n - d_t \underline{\mathbf{U}}_H^n, \Pi \nabla \phi^n - \nabla \phi^n) + (\nabla \cdot (d_t \mathbf{u}^n - d_t \underline{\mathbf{U}}_H^n), \phi^n - \hat{\phi}_H^n).
\end{aligned}$$

Furthermore, we can write

$$\begin{aligned}
(d_t K(p^n) \tilde{\mathbf{u}}^n - d_t K(\mathcal{P}_H(p^n)) \tilde{\underline{\mathbf{U}}}_H^n, \Pi \nabla \phi^n) & = ((d_t K(p^n) - d_t K(\mathcal{P}_H(p^n))) \tilde{\mathbf{u}}^n, \Pi \nabla \phi^n) \\
(57) \quad & + (d_t K(\mathcal{P}_H(p^n)) (\tilde{\mathbf{u}}^n - \tilde{\underline{\mathbf{U}}}_H^n), \Pi \nabla \phi^n).
\end{aligned}$$

Using (5) gives

$$\begin{aligned}
|E_{\text{TM}}(d_t \tilde{\underline{\mathbf{U}}}_H^n, \Pi K(p^n) \nabla \phi^n)| & \leq C \sum_E \sum_{|\alpha|=2} \left\| \frac{\partial^\alpha}{\partial \mathbf{x}^\alpha} (d_t \tilde{\underline{\mathbf{U}}}_H^n \cdot \Pi K(p^n) \nabla \phi^n) \right\|_{L^1, E} H^2 \\
& \leq C \sum_E \left(\left\| \frac{\partial d_t \tilde{\underline{\mathbf{U}}}_{H,x}^n}{\partial x} \right\|_{0,E} \left\| \frac{\partial}{\partial x} (\Pi K(p^n) \nabla \phi^n)^x \right\|_{0,E} \right. \\
(58) \quad & \left. + \left\| \frac{\partial d_t \tilde{\underline{\mathbf{U}}}_{H,y}^n}{\partial y} \right\|_{0,E} \left\| \frac{\partial}{\partial y} (\Pi K(p^n) \nabla \phi^n)^y \right\|_{0,E} \right) H^2.
\end{aligned}$$

By Lemma 2.1 and the inverse inequality we have

$$\begin{aligned}
\left\| \frac{\partial d_t \tilde{\underline{\mathbf{U}}}_{H,x}^n}{\partial x} \right\|_{0,E} & \leq \left\| \frac{\partial}{\partial x} (d_t \tilde{\underline{\mathbf{U}}}_{H,x}^n - \Pi d_t \tilde{\mathbf{u}}_x^n) \right\|_{0,E} + \left\| \frac{\partial}{\partial x} \Pi d_t \tilde{\mathbf{u}}_x^n \right\|_{0,E} \\
& \leq C \|d_t \tilde{\underline{\mathbf{U}}}_{H,x}^n - \Pi d_t \tilde{\mathbf{u}}_x^n\|_{0,E} H^{-1} + \left\| \frac{\partial}{\partial x} d_t \tilde{\mathbf{u}}_x^n \right\|_{0,E} \\
(59) \quad & \leq C \|d_t \tilde{\underline{\mathbf{U}}}_H^n - \Pi d_t \tilde{\mathbf{u}}^n\|_{0,E} H^{-1} + \|d_t \tilde{\mathbf{u}}^n\|_{1,E}.
\end{aligned}$$

Since $d_t \tilde{\underline{\mathbf{U}}}_H^n$ and $\Pi K(p^n) \nabla \phi^n$ are in V_H , by (58) and the Cauchy-Schwarz inequality,

$$\begin{aligned}
|E_{\text{TM}}(d_t \tilde{\underline{\mathbf{U}}}_H^n, \Pi K(p^n) \nabla \phi^n)| & \leq C \sum_E (\|d_t \tilde{\underline{\mathbf{U}}}_H^n - \Pi d_t \tilde{\mathbf{u}}^n\|_{0,E} + \|d_t \tilde{\mathbf{u}}^n\|_{1,E} H) \|\phi^n\|_{2,E} H \\
(60) \quad & \leq C (\|d_t \tilde{\underline{\mathbf{U}}}_H^n - \Pi d_t \tilde{\mathbf{u}}^n\|_0 + \|d_t \tilde{\mathbf{u}}^n\|_1 H) \|\phi^n\|_2 H.
\end{aligned}$$

As done above and noting that K has bounded second derivatives,

$$(61) \quad |E_{\text{TM}}(d_t \underline{\mathbf{U}}_H^n, \Pi \nabla \phi^n)| \leq C (\|d_t \underline{\mathbf{U}}_H^n - \Pi d_t \mathbf{u}^n\|_0 + \|d_t \mathbf{u}^n\|_1 H) \|\phi^n\|_2 H,$$

$$(62) \quad |E_{\text{T}}(d_t(K(\mathcal{P}_H(p^n))) \tilde{\underline{\mathbf{U}}}_H^n, \Pi \nabla \phi^n)| \leq (C + \Delta t) (\|\tilde{\underline{\mathbf{U}}}_H^n - \Pi \tilde{\mathbf{u}}^n\|_0 + \|\tilde{\mathbf{u}}^n\|_1 H) \|\phi^n\|_2 H,$$

$$(63) \quad |E_{\text{T}}(K(\mathcal{P}_H(p^{n-1})) d_t \tilde{\underline{\mathbf{U}}}_H^n, \Pi \nabla \phi^n)| \leq C (\|d_t \tilde{\underline{\mathbf{U}}}_H^n - \Pi d_t \tilde{\mathbf{u}}^n\|_0 + \|d_t \tilde{\mathbf{u}}^n\|_1 H) \|\phi^n\|_2 H,$$

where C in the second and third inequalities depends on K^* .

Combining (54) with (55)-(63), applying approximation properties of the L^2 and Π projections, using Lemma 3.5, and equations (24) and (11) gives

$$\begin{aligned}
& (d_t \hat{p}_H^n - d_t \underline{p}_H^n, \rho^n) \\
& \leq \|d_t \tilde{\mathbf{u}}^n - d_t \tilde{\underline{\mathbf{U}}}_H^n\| \|\Pi K(p^n) \nabla \phi^n - K(p^n) \nabla \phi^n\| \\
& \quad + \|K(\mathcal{P}_H(p^{n-1})) (d_t \tilde{\mathbf{u}}^n - d_t \tilde{\underline{\mathbf{U}}}_H^n)\| \|\nabla \phi^n - \Pi \nabla \phi^n\|
\end{aligned}$$

$$\begin{aligned}
& + \|(d_t K(p^n) - d_t K(\mathcal{P}_H(p^n)))\tilde{\mathbf{u}}^n\| \|\Pi \nabla \phi^n\| \\
& + \|d_t K(\mathcal{P}_H(p^n))(\tilde{\mathbf{u}}^n - \tilde{\mathbf{U}}^n)\| \|\Pi \nabla \phi^n\| \\
& + \|(K(p^{n-1}) - K(\mathcal{P}_H(p^{n-1})))d_t \tilde{\mathbf{u}}^n\| \|\Pi \nabla \phi^n\| \\
& + \|d_t \mathbf{u}^n - d_t \underline{\mathbf{U}}_H^n\| \|\Pi \nabla \phi^n - \nabla \phi^n\| \\
& + \|\nabla \cdot (d_t \mathbf{u}^n - d_t \underline{\mathbf{U}}_H^n)\| \|\phi^n - \hat{\phi}_H^n\| \\
& + \|d_t(K(p^n))\tilde{\mathbf{u}}^n - d_t(K(\mathcal{P}_H(p^n)))\tilde{\mathbf{U}}_H^n\| \|\Pi \nabla \phi^n\| \\
& + |E_{\text{TM}}(d_t \tilde{\mathbf{U}}_H^n, \Pi K(p^n) \nabla \phi^n)| + |E_{\text{TM}}(d_t \underline{\mathbf{U}}_H^n, \Pi \nabla \phi^n)| \\
& + |E_{\text{T}}(d_t(K(\mathcal{P}_H(p^n)))\tilde{\mathbf{U}}_H^n, \Pi \nabla \phi^n)| + |E_{\text{T}}(K(\mathcal{P}_H(p^{n-1}))d_t \tilde{\mathbf{U}}_H^n, \Pi \nabla \phi^n)| \\
& \leq C(H^2 + \Delta t)\|\phi^n\|_2.
\end{aligned}$$

Applying equation (53) and taking $\rho^n = d_t \hat{p}^n - d_t \underline{P}_H^n$ we have

$$(64) \quad \|d_t \hat{p}^n - d_t \underline{P}_H^n\| \leq C(H^2 + \Delta t).$$

□

3.3. Convergence Estimate of the Nonlinear Scheme. We now prove the following theorem about the convergence of the above finite difference scheme:

THEOREM 3.6. *Let P_H^n , $\tilde{\mathbf{U}}_H^n$ and \mathbf{U}_H^n , $n = 1, \dots, N$ be defined as in (18) - (20) with initial values $P_H^0 = \hat{p}_H(t^0, \cdot)$. Then, there exists a positive constant C , independent of H and Δt such that*

$$(65) \quad \|P_H^N - p^N\|_{\text{M}} + \left\{ \Delta t \sum_{n=1}^N K_* \|\tilde{\mathbf{U}}_H^n - \tilde{u}^n\|_{\text{T}}^2 \right\}^{1/2} \leq C(H^2 + \Delta t).$$

Proof. Let $\gamma^n = P_H^n - \underline{P}_H^n$, $\boldsymbol{\eta}^n = \tilde{\mathbf{U}}_H^n - \tilde{\mathbf{U}}_H^n$, $\boldsymbol{\xi}^n = \mathbf{U}_H^n - \underline{\mathbf{U}}_H^n$ and $\alpha^n = \underline{P}_H^n - p^n$. Subtracting $(d_t \underline{P}_H^n, w) + (\nabla \cdot \underline{\mathbf{U}}_H^n, w)$ from both sides of (18) and using equations (21) and (15) we have

$$\begin{aligned}
(d_t \gamma^n, w) + (\nabla \cdot \boldsymbol{\xi}^n, w) &= (f^n, w) - (\nabla \cdot \underline{\mathbf{U}}_H^n, w) - (d_t \underline{P}_H^n, w) \\
&= (p_t^n, w) - (d_t p^n, w) - (d_t \alpha^n, w) \\
(66) \quad &= (\epsilon^n, w) - (d_t \alpha^n, w), \quad w \in W_H,
\end{aligned}$$

where ϵ is a time truncation term. Subtracting (22) from (19) results in

$$(67) \quad (\boldsymbol{\eta}^n, \mathbf{v})_{\text{TM}} = (\gamma^n, \nabla \cdot \mathbf{v}), \quad \mathbf{v} \in V_H^0,$$

and subtracting (23) from (20) gives

$$\begin{aligned}
(\boldsymbol{\xi}^n, \mathbf{v})_{\text{TM}} &= (K(\mathcal{P}_H(P^n))\tilde{\mathbf{U}}_H^n, \mathbf{v})_{\text{T}} - (K(\mathcal{P}_H(p^n))\tilde{\mathbf{U}}_H^n, \mathbf{v})_{\text{T}} \\
&= (K(\mathcal{P}_H(P_H^n))\boldsymbol{\eta}^n, \mathbf{v})_{\text{T}} + ((K(\mathcal{P}_H(P_H^n)) - K(\mathcal{P}_H(\underline{P}_H^n)))\tilde{\mathbf{U}}_H^n, \mathbf{v})_{\text{T}} \\
(68) \quad &- ((K(\mathcal{P}_H(p^n)) - K(\mathcal{P}_H(\underline{P}_H^n)))\tilde{\mathbf{U}}_H^n, \mathbf{v})_{\text{T}}, \quad \mathbf{v} \in V_H.
\end{aligned}$$

Letting $w = \gamma^n$ in (66), $\mathbf{v} = \boldsymbol{\xi}^n$ in (67) and $\mathbf{v} = \boldsymbol{\eta}^n$ in (68) gives

$$(69) \quad (d_t \gamma^n, \gamma^n) = -(\nabla \cdot \boldsymbol{\xi}^n, \gamma^n) + (\epsilon^n, \gamma^n)_{\text{M}} - (d_t \alpha^n, \gamma^n),$$

$$(70) \quad (\boldsymbol{\eta}^n, \boldsymbol{\xi}^n)_{\text{TM}} = (\gamma^n, \nabla \cdot \boldsymbol{\xi}^n),$$

$$\begin{aligned}
(\boldsymbol{\xi}^n, \boldsymbol{\eta}^n)_{\text{TM}} &= (K(\mathcal{P}_H(P_H^n))\boldsymbol{\eta}^n, \boldsymbol{\eta}^n)_{\text{T}} + ((K(\mathcal{P}_H(P_H^n)) - K(\mathcal{P}_H(\underline{P}_H^n)))\tilde{\mathbf{U}}_H^n, \boldsymbol{\eta}^n)_{\text{T}} \\
(71) \quad &- ((K(\mathcal{P}_H(p^n)) - K(\mathcal{P}_H(\underline{P}_H^n)))\tilde{\mathbf{U}}_H^n, \boldsymbol{\eta}^n)_{\text{T}}.
\end{aligned}$$

Combining equations (69)-(71), applying the Cauchy-Schwarz inequality and (14) we have

$$\begin{aligned}
& \frac{1}{2\Delta t} [\|\gamma^n\|_M^2 - \|\gamma^{n-1}\|_M^2] + \|K(\mathcal{P}_H(P_H^n))^{1/2} \boldsymbol{\eta}^n\|_T^2 \\
& \leq (d_t \gamma^n, \gamma^n)_M + \|K(\mathcal{P}_H(P_H^n))^{1/2} \boldsymbol{\eta}^n\|_T^2 \\
& \leq (\epsilon^n, \gamma^n)_M - (d_t \alpha^n, \gamma^n)_M + ((K(\mathcal{P}_H(p^n)) - K(\mathcal{P}_H(\underline{P}_H^n))) \tilde{\mathbf{U}}_H^n, \boldsymbol{\eta}^n)_T \\
& \quad - ((K(\mathcal{P}_H(P_H^n)) - K(\mathcal{P}_H(\underline{P}_H^n))) \tilde{\mathbf{U}}_H^n, \boldsymbol{\eta}^n)_T \\
& \leq \frac{1}{2} \|\epsilon^n\|_M^2 + \|\gamma^n\|_M^2 + \frac{1}{2} \|d_t \alpha^n\|_M^2 + C \| (K(\mathcal{P}_H(p^n)) - K(\mathcal{P}_H(\underline{P}_H^n))) \tilde{\mathbf{U}}_H^n \|_T^2 \\
& \quad + C \| (K(\mathcal{P}_H(P_H^n)) - K(\mathcal{P}_H(\underline{P}_H^n))) \tilde{\mathbf{U}}_H^n \|_T^2 + \delta \|\boldsymbol{\eta}^n\|_{TM}^2,
\end{aligned}$$

where $\delta \leq K_*/2$.

Now,

$$|\epsilon_{ij}^n| = \frac{1}{\Delta t} \left| \int_{t^{n-1}}^{t^n} p_{tt}(x_i, y_j, t) (t - t^n) dt \right| \leq \|p_{tt}(x_i, y_j, \cdot)\|_{L^2(t^{n-1}, t^n)} (\Delta t)^{\frac{1}{2}}.$$

So,

$$(72) \quad \|\epsilon^n\|_M^2 \leq \Delta t \sum_{ij} H_i^x H_j^y \|p_{tt}(x_i, y_j, \cdot)\|_{L^2(t^{n-1}, t^n)}^2.$$

By the triangle inequality and Theorem 3.3,

$$\|d_t \alpha^n\|_M^2 \leq C(H^4 + \Delta t^2).$$

By the Lipschitz assumption on K , the definition of \mathcal{P}_H and Theorem 3.3

$$(73) \quad \|(K(\mathcal{P}_H(p^n)) - K(\mathcal{P}_H(\underline{P}_H^n))) \tilde{\mathbf{U}}_H^n\|_T^2 \leq C \|p^n - \underline{P}_H^n\|_M^2 \leq C H^4,$$

$$(74) \quad \|(K(\mathcal{P}_H(P_H^n)) - K(\mathcal{P}_H(\underline{P}_H^n))) \tilde{\mathbf{U}}_H^n\|_T^2 \leq C \|\gamma^n\|_M^2,$$

where we have used the boundedness of $\|\tilde{\mathbf{U}}_H^n\|_\infty$ as per Remark 3.1.

Multiplying by $2\Delta t$, bringing the $\delta \|\boldsymbol{\eta}^n\|_{TM}^2$ term to the left-hand side, summing on $n, n = 1, \dots, N$, using (72)-(74) and applying Gronwall's Lemma gives,

$$\begin{aligned}
& \|\gamma^N\|_M^2 - \|\gamma^0\|_M^2 + \Delta t \sum_{n=1}^N \|K(\mathcal{P}_H(P_H^n))^{1/2} \boldsymbol{\eta}^n\|_T^2 \\
& \leq C \Delta t \sum_{n=1}^N (\|\epsilon^n\|_M^2 + \|d_t \alpha^n\|_M^2 + \|\alpha^n\|_M^2) + C H^4 + C \Delta t \sum_{n=1}^N \|\gamma^n\|_M^2 \\
& \leq C(\Delta t^2 + H^4).
\end{aligned}$$

The proof is completed by applying the initial conditions on P_H^0 and \underline{P}_H^0 , Theorem 3.3 and the triangle inequality. \square

4. Fine Grid Linear Scheme. We now consider a linear cell-centered finite difference scheme on the fine grid where we make use of the nonlinear solution on the coarse grid.

We solve the following problem for $P_h^n \in W_h$, $\tilde{\mathbf{U}}_h^n \in V_h$ and $\mathbf{U}_h^n \in V_H^n$ at each $n = 1, \dots, N$,

$$(75) \quad (d_t P_h^n, w) = -(\nabla \cdot \mathbf{U}_h^n, w) + (f^n, w), w \in W_h,$$

$$(76) \quad (\tilde{\mathbf{U}}_h^n, \mathbf{v})_{\text{TM}} = (P_h^n, \nabla \cdot \mathbf{v}), \mathbf{v} \in V_h^0,$$

$$(77) \quad \begin{aligned} (\mathbf{U}_h^n, \mathbf{v})_{\text{TM}} &= (K(\mathcal{P}_H(P_H^n))\tilde{\mathbf{U}}_h^n, \mathbf{v})_{\text{T}} \\ &\quad + (K'(\mathcal{P}_H(P_H^n))\mathcal{Q}_H(\tilde{\mathbf{U}}_H^n)(\mathcal{P}_h(P_h^n) - \mathcal{P}_H(P_H^n)), \mathbf{v})_{\text{T}}, \mathbf{v} \in V_h. \end{aligned}$$

We define $\mathcal{Q}_H(\tilde{\mathbf{u}})$ as a vector quantity with entries $\mathcal{Q}_H^x(\tilde{u}^x)$ and $\mathcal{Q}_H^y(\tilde{u}^y)$. The entry $\mathcal{Q}_H^x(\tilde{u}^x)$ is defined from the values of $\tilde{u}_{i+1/2,j}^x$ for $i = 0, \dots, \hat{N}_x$ and $j = 1, \dots, \hat{N}_y$ as follows. For points (x, y) such that $x_{i-1/2} \leq x \leq x_{i+1/2}$, $i \in \{1, \dots, \hat{N}_x\}$ and $y_j \leq y \leq y_{j+1}$, $j \in \{1, \dots, \hat{N}_y\}$, we take $\mathcal{Q}_H^x(\tilde{u}^x)$ to be the bilinear interpolant of $\tilde{u}_{i-1/2,j}^x$, $\tilde{u}_{i+1/2,j}^x$, $\tilde{u}_{i-1/2,j+1}^x$ and $\tilde{u}_{i+1/2,j+1}^x$. This leaves a strip half a cell in height along the top and bottom of the domain. We will consider the bottom strip. For $i = 0, \dots, \hat{N}_x$, we set

$$\mathcal{Q}_H^x(\tilde{u}^x)(x_{i+1/2}, j_{1/2}) = \frac{(2H_1^y + H_2^y)\tilde{u}_{i+1/2,1}^x - H_1^y\tilde{u}_{i+1/2,2}^x}{H_1^y + H_2^y}.$$

Now, for points (x, y) such that $x_{i-1/2} \leq x \leq x_{i+1/2}$, $i \in \{1, \dots, \hat{N}_x\}$ and $y_{1/2} \leq y \leq y_1$, we let $\mathcal{Q}_H^x(\tilde{u}^x)(x, y)$ be the bilinear interpolant of $\mathcal{Q}_H^x(\tilde{u}^x)(x_{i-1/2}, y_{1/2})$, $\mathcal{Q}_H^x(\tilde{u}^x)(x_{i+1/2}, y_{1/2})$, $\tilde{u}_{i-1/2,1}^x$ and $\tilde{u}_{i+1/2,1}^x$. An analogous definition is made along the top strip of the domain. The definition of $\mathcal{Q}_H^y(\tilde{u}^y)$ is similar to the above, except that the strips are along the left and right sides of the domain.

We have the following lemma for the approximation error of \mathcal{Q}_H ,

LEMMA 4.1. *If each component of $\tilde{\mathbf{u}}$ is twice differentiable, then for $\mathcal{Q}_H(\tilde{\mathbf{u}})$ defined above,*

$$\|\mathcal{Q}_H(\tilde{\mathbf{u}}) - \tilde{\mathbf{u}}\|_\infty \leq CH^2.$$

Proof. By Taylor's theorem we have that the two point extrapolation for the boundary points described above is $O(H^2)$ accurate. Thus, since bilinear interpolation is also $O(H^2)$ accurate, the lemma is proven. \square

We now prove the following theorem about the convergence of the above linear finite difference scheme,

THEOREM 4.2. *Let P_h^n , $\tilde{\mathbf{U}}_h^n$ and \mathbf{U}_h^n , $n = 1, \dots, N$ be defined as in (75) - (77) with initial values $P_h^0 = \hat{p}_h(t^0, \cdot)$. Then, there exists a positive constant C , independent of h , H and Δt such that*

$$(78) \quad \|P_h^N - p^N\|_{\text{M}} + \left\{ \Delta t \sum_{n=1}^N K_* \|\tilde{\mathbf{U}}_h^n - \tilde{\mathbf{u}}^n\|_{\text{T}}^2 \right\}^{1/2} \leq C(H^{4-d/2} + h^2 + \Delta t).$$

Proof. By Section 3 we can define $\underline{P}_h^n \in W_h$, $\tilde{\underline{\mathbf{U}}}_h^n \in V_h$ and $\underline{\mathbf{U}}_h^n \in V_H^n$ at each $n = 1, \dots, N$ satisfying equations (21)-(23) and Theorem 3.3 on the fine grid.

Let $\gamma^n = P_h^n - \underline{P}_h^n$, $\boldsymbol{\eta}^n = \tilde{\mathbf{U}}_h^n - \tilde{\underline{\mathbf{U}}}_h^n$, $\boldsymbol{\xi}^n = \mathbf{U}_h^n - \underline{\mathbf{U}}_h^n$ and $\alpha^n = \underline{P}_h^n - p^n$. As done in Theorem 3.6, we subtract $(d_t \underline{P}_h^n, w) + (\nabla \cdot \underline{\mathbf{U}}_h^n, w)$ from both sides of equation (75) and

combine with equation (21) applied to the fine grid. We also subtract (22) and (23) from (76) and (77) to give the error equations,

$$(79) \quad (d_t \gamma^n, w) = -(\nabla \cdot \xi^n, w) + (\epsilon^n, w) - (d_t \alpha^n, w),$$

$$(80) \quad (\eta^n, \mathbf{v})_{\text{TM}} = (\gamma^n, \nabla \cdot \mathbf{v}),$$

$$(81) \quad (\xi^n, \mathbf{v})_{\text{TM}} = (K(\mathcal{P}_H(P_H^n))\tilde{\mathbf{U}}_h^n, \mathbf{v})_{\text{T}} - (K(\mathcal{P}_h(p^n))\tilde{\mathbf{U}}_h^n, \mathbf{v})_{\text{T}} \\ + (K'(\mathcal{P}_H(P_H^n))\mathcal{Q}_H(\tilde{\mathbf{U}}_H^n)(\mathcal{P}_h(P_h^n) - \mathcal{P}_H(P_H^n)), \mathbf{v})_{\text{T}}.$$

Using Taylor's Theorem, $K(\mathcal{P}_h(p^n))$ can be written as

$$K(\mathcal{P}_h(p^n)) = K(\mathcal{P}_h(P_H^n)) + K'(\mathcal{P}_H(P_H^n))(\mathcal{P}_h(p^n) - \mathcal{P}_H(P_H^n)) \\ + \frac{K''(\theta^n)}{2}(\mathcal{P}_h(p^n) - \mathcal{P}_H(P_H^n))^2,$$

where θ^n is between $\mathcal{P}_h(p^n)$ and $\mathcal{P}_H(P_H^n)$.

Using this expression in (81), adding and subtracting $(K'(\mathcal{P}_H(P_H^n))\mathcal{Q}_H(\tilde{\mathbf{U}}_H^n)\mathcal{P}_h(p^n), \mathbf{v})_{\text{T}}$ we have

$$(\xi^n, \mathbf{v})_{\text{TM}} = (K(\mathcal{P}_H(P_H^n))\eta^n, \mathbf{v})_{\text{T}} \\ + (K'(\mathcal{P}_H(P_H^n))(\mathcal{Q}_H(\tilde{\mathbf{U}}_H^n) - \tilde{\mathbf{U}}_h^n)(\mathcal{P}_h(p^n) - \mathcal{P}_H(P_H^n)), \mathbf{v})_{\text{T}} \\ + (K'(\mathcal{P}_H(P_H^n))\mathcal{Q}_H(\tilde{\mathbf{U}}_H^n)(\mathcal{P}_h(P_h^n) - \mathcal{P}_h(p^n)), \mathbf{v})_{\text{T}} \\ + (\frac{K''(\theta^n)}{2}(\mathcal{P}_h(p^n) - \mathcal{P}_H(P_H^n))^2\tilde{\mathbf{U}}_h^n, \mathbf{v})_{\text{T}}.$$

Let $w = \gamma^n$, $\mathbf{v} = \xi^n$ and $\mathbf{v} = \eta^n$ in (79), (80) and (82), respectively, and combine to give

$$(82) \quad \frac{1}{2\Delta t}[\|\gamma^n\|^2 - \|\gamma^{n-1}\|^2] + K_*\|\eta^n\|_{\text{T}}^2 \\ \leq (d_t \gamma^n, \gamma^n) + \|K(\mathcal{P}_H(P_H^n))^{1/2}\eta^n\|_{\text{T}}^2 \\ \leq \frac{1}{2}\|\epsilon^n\|^2 + \|\gamma^n\|^2 + \frac{1}{2}\|d_t \alpha^n\|^2 + \delta\|\eta^n\|^2 \\ + C\|K'(\mathcal{P}_H(P_H^n))(\mathcal{Q}_H(\tilde{\mathbf{U}}_H^n) - \tilde{\mathbf{U}}_h^n)(\mathcal{P}_h(p^n) - \mathcal{P}_H(P_H^n))\|_{\text{T}}^2 \\ + C\|K'(\mathcal{P}_H(P_H^n))\mathcal{Q}_H(\tilde{\mathbf{U}}_H^n)(\mathcal{P}_h(P_h^n) - \mathcal{P}_h(p^n))\|_{\text{T}}^2 \\ + C\|\frac{K''(\theta^n)}{2}(\mathcal{P}_h(p^n) - \mathcal{P}_H(P_H^n))^2\tilde{\mathbf{U}}_h^n\|_{\text{T}}^2,$$

where $\delta \leq K_*/2$.

We will consider the last three terms of (82). The first of these can be bounded as follows.

$$(83) \quad \|K'(\mathcal{P}_H(P_H^n))(\mathcal{Q}_H(\tilde{\mathbf{U}}_H^n) - \tilde{\mathbf{U}}_h^n)(\mathcal{P}_h(p^n) - \mathcal{P}_H(P_H^n))\|_{\text{T}}^2 \\ \leq C\|\mathcal{Q}_H(\tilde{\mathbf{U}}_H^n) - \tilde{\mathbf{U}}_h^n\|_{\text{TM}}^2\|\mathcal{P}_h(p^n) - \mathcal{P}_H(P_H^n)\|_{\infty}^2,$$

where we can write,

$$\|\mathcal{Q}_H(\tilde{\mathbf{U}}_H^n) - \tilde{\mathbf{U}}_h^n\|_{\text{TM}}^2 \leq \|\mathcal{Q}_H(\tilde{\mathbf{U}}_H^n) - \mathcal{Q}_H(\tilde{\mathbf{u}}^n)\|_{\text{TM}}^2 + \|\mathcal{Q}_H(\tilde{\mathbf{u}}^n) - \tilde{\mathbf{u}}^n\|_{\text{TM}}^2 \\ + \|\tilde{\mathbf{u}}^n - \tilde{\mathbf{U}}_h^n\|_{\text{TM}}^2.$$

Since $\mathcal{Q}_H(\tilde{\mathbf{U}}_H^n)$ is a bilinear interpolant of terms that can be expressed in terms of nodal values of $\tilde{\mathbf{U}}_H^n$ on the coarse grid, it can be shown that,

$$\|\mathcal{Q}_H(\tilde{\mathbf{U}}_H^n) - \mathcal{Q}_H(\tilde{\mathbf{u}}^n)\|_{\text{TM}}^2 \leq C\|\tilde{\mathbf{U}}_H^n - \tilde{\mathbf{u}}^n\|_{\text{TM},H}^2,$$

where $\|\cdot\|_{\text{TM},H}$ denotes the midpoint by trapezoidal norm on the coarse grid. We also have $\|\mathcal{Q}_H(\tilde{\mathbf{u}}^n) - \tilde{\mathbf{u}}^n\|_{\text{TM}}^2 \leq CH^4$ by Lemma 4.1. In order to bound the second term in (83) we write it as,

$$\begin{aligned} \|\mathcal{P}_h(p^n) - \mathcal{P}_H(P_H^n)\|_\infty^2 &\leq \|\mathcal{P}_H(P_H^n) - \mathcal{P}_H(p^n)\|_\infty^2 + \|\mathcal{P}_H(p^n) - p^n\|_\infty^2 \\ &\quad + \|p^n - \mathcal{P}_h(p^n)\|_\infty^2. \end{aligned}$$

By the definition of \mathcal{P}_H , the quasi-uniformity assumption on the coarse grid and Theorem 3.6, we have

$$\begin{aligned} \|\mathcal{P}_H(P_H^n) - \mathcal{P}_H(p^n)\|_\infty^2 &\leq \frac{C}{H^d} \|P_H^n - p^n\|_{M,H}^2 \\ &\leq CH^{-d}(H^4 + \Delta t^2), \end{aligned}$$

where d is the space dimension. By Lemma 3.1, we have $\|\mathcal{P}_H(p^n) - p^n\|_\infty^2 \leq CH^4$ and $\|\mathcal{P}_h(p^n) - p^n\|_\infty^2 \leq Ch^4$. Thus,

$$\begin{aligned} &\|K'(\mathcal{P}_H(P_H^n))(\mathcal{Q}_H(\tilde{\mathbf{U}}_H^n) - \tilde{\mathbf{u}}^n)(\mathcal{P}_h(p^n) - \mathcal{P}_H(P_H^n))\|_{\text{T}}^2 \\ (84) \quad &\leq C(\|\tilde{\mathbf{U}}_H^n - \tilde{\mathbf{u}}^n\|_{\text{TM},H}^2 + \|\tilde{\mathbf{u}}^n - \tilde{\mathbf{u}}^n\|_{\text{TM}}^2 + H^4)(H^{4-d} + H^{-d}\Delta t^2 + h^4). \end{aligned}$$

The second to last term in (82) can be bounded by,

$$\begin{aligned} &\|K'(\mathcal{P}_H(P_H^n))\mathcal{Q}_H(\tilde{\mathbf{U}}_H^n)(\mathcal{P}_h(P_h^n) - \mathcal{P}_h(p^n))\|_{\text{T}}^2 \\ &\leq C\|\tilde{\mathbf{U}}_H^n\|_\infty^2 \|\mathcal{P}_h(P_h^n) - \mathcal{P}_h(p^n)\|_{\text{T}}^2 \\ &\leq C(H^{-d}\|\tilde{\mathbf{U}}_H^n - \tilde{\mathbf{u}}^n\|_{\text{TM}}^2 + \|\tilde{\mathbf{u}}^n\|_\infty^2)(\|P_h^n - \underline{P}_h^n\|_{\text{M}}^2 + \|\underline{P}_h^n - p^n\|_{\text{M}}^2) \\ (85) \quad &\leq C(H^{-d}\|\tilde{\mathbf{U}}_H^n - \tilde{\mathbf{u}}^n\|_{\text{TM}}^2 + C)(\|\gamma^n\|^2 + h^4). \end{aligned}$$

The last term in (82) can be bounded by,

$$\begin{aligned} &\|\frac{K''(\theta^n)}{2}(\mathcal{P}_h(p^n) - \mathcal{P}_H(P_H^n))^2 \tilde{\mathbf{u}}^n\|_{\text{T}}^2 \\ &\leq C\|\mathcal{P}_h(p^n) - \mathcal{P}_H(P_H^n)\|_\infty^2 \|\mathcal{P}_h(p^n) - \mathcal{P}_H(P_H^n)\|_{\text{T}}^2 \\ &\leq C(H^{4-d} + H^{-d}\Delta t^2 + h^4)(h^4 + H^4 + \|p^n - P_H^n\|_{\text{M},H}^2) \\ &\leq C(H^{4-d} + H^{-d}\Delta t^2 + h^4)(h^4 + H^4 + \Delta t^2) \\ (86) \quad &\leq C(H^{8-d} + h^4 + \Delta t^2 + H^{-d}h^4\Delta t^2 + H^{-d}\Delta t^4). \end{aligned}$$

Combining equation (82) with equations (84)-(86), taking the $\delta\|\boldsymbol{\eta}^n\|^2$ term to the left side, multiplying by $2\Delta t$ and summing over $n, n = 1, \dots, N^*$ where N^* is the time step at which $\|\gamma^n\|$ achieves its maximum value gives,

$$\|\gamma^{N^*}\|^2 - \|\gamma^0\| + \Delta t \sum_{n=1}^{N^*} K_* \|\boldsymbol{\eta}^n\|_{\text{TM}}^2$$

$$\begin{aligned}
&\leq \Delta t \sum_{n=1}^{N^*} (\|\epsilon^n\|^2 + \|d_t \alpha^n\|^2) + C \Delta t \sum_{n=1}^{N^*} \|\gamma^n\|^2 \\
&\quad + C(H^{4-d} + H^{-d} \Delta t^2 + h^4) \Delta t \sum_{n=1}^{N^*} (\|\tilde{\mathbf{U}}_H^n - \tilde{\mathbf{u}}^n\|_{\text{TM},H}^2 + \|\tilde{\mathbf{u}}^n - \tilde{\mathbf{U}}_h^n\|_{\text{TM}}^2 + H^4) \\
&\quad + C(h^4 + \|\gamma^{N^*}\|^2) \Delta t \sum_{n=1}^{N^*} H^{-d} \|\tilde{\mathbf{U}}_H^n - \tilde{\mathbf{u}}^n\|^2 \\
&\quad + C(H^{8-d} + h^4 + \Delta t^2 + H^{-d} h^4 \Delta t^2 + H^{-d} \Delta t^4).
\end{aligned}$$

Recalling the bound on ϵ^n , using Theorem 3.3 and recalling the initial conditions on P_h^0 and P_h^0 gives,

$$\begin{aligned}
\|\gamma^{N^*}\|^2 + \Delta t \sum_{n=1}^{N^*} K_* \|\boldsymbol{\eta}^n\|_{\text{TM}}^2 &\leq C(H^{8-d} + h^4 + \Delta t^2 + H^{-d} h^4 \Delta t^2 + H^{-d} \Delta t^4) \\
&\quad + C \Delta t \sum_{n=1}^{N^*} \|\gamma^n\|^2 + \tilde{C} \|\gamma^{N^*}\|^2 (H^{4-d} + H^{-d} \Delta t^2).
\end{aligned}$$

We can choose H and Δt such that $H^{4-d} + H^{-d} \Delta t^2 \leq \frac{1}{2\tilde{C}}$, and the last term can be moved to the left-hand side. Applying Gronwall's Lemma gives

$$\|\gamma^{N^*}\|^2 + \Delta t \sum_{n=1}^{N^*} K_* \|\boldsymbol{\eta}^n\|_{\text{TM}}^2 \leq C(H^{8-d} + h^4 + \Delta t^2).$$

Applying Theorem 3.3 and the triangle inequality give the desired result. \square

5. Conclusions. We have presented and derived error estimates for a two level finite difference scheme for nonlinear parabolic equations. Through the use of the $\mathcal{P}_h, \mathcal{P}_H$ and \mathcal{Q}_H operators, we have taken advantage of superconvergent node points and have shown optimal order convergence in both H^1 and L^2 for the coarse and fine grids.

We remark that we have only considered the case of Neumann boundary conditons and a diagonal tensor K . The expanded mixed method employed here was developed in order to handle a full symmetric tensor for K . However, in the case of a full tensor, convergence may be lost on the boundary. In this case we can show a coarse grid estimate of $H^r + \Delta t$ and a fine grid estimate of $h^r + H^{2r-d/2} + \Delta t$, where $r = 2$ if K is diagonal and no Dirichlet conditions are enforced, $r = 3/2$ if K is diagonal or the grids are generated by a C^2 map, and $r = 1$ otherwise for 2 dimensions and no convergence otherwise for 3 dimensions. These estimates for a full tensor use the inverse estimate and may be improved through the derivation of explicit L^∞ estimates. The loss of convergence due to the boundary is shown for the expanded mixed method in [1].

The two level scheme described above could be extended by adding more levels and expanding about the next coarser solution in the nonlinear term at each new level. This corresponds to adding more Newton-like iterations with each iteration taking place on the next finer grid. We are currently investigating this possibility. Computational results for the scheme described here are also in progress.

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