An Inexact Trust-Region Feasible-Point Algorithm for Nonlinear Systems of Equalities and Inequalities

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AN INEXACT TRUST-REGION FEASIBLE-POINT ALGORITHM FOR NONLINEAR SYSTEMS OF EQUALITIES AND INEQUALITIES[3,4]

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Abstract. In this work we define a trust-region feasible-point algorithm for approximating solutions of the nonlinear system of equalities and inequalities $F(x,y) = 0, y \geq 0$, where $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$ is continuously differentiable. This formulation is quite general; the Karush-Kuhn-Tucker conditions of a general nonlinear programming problem are an obvious example, and a set of equalities and inequalities can be transformed, using slack variables, into such form. We will be concerned with the possibility that n, m, and p may be large and that the Jacobian matrix may be sparse and rank deficient. Exploiting the convex structure of the local model trust-region subproblem, we propose a globally convergent inexact trust-region feasible-point algorithm to minimize an arbitrary norm of the residual, say $\|F(x,y)\|_a$, subject to the nonnegativity constraints. This algorithm uses a trust-region globalization strategy to determine a descent direction as an inexact solution of the local model trust-region subproblem and then, it uses linesearch techniques to obtain an acceptable steplength. We demonstrate that, under rather weak hypotheses, any accumulation point of the iteration sequence is a constrained stationary point for $f = \|F\|_a$, and that the sequence of constrained residuals converges to zero.

Key Words: constrained nonlinear systems, hybrid method, trust-region, interior-point, feasible-point, linesearch, inexact Newton's method, global convergence, equalities, inequalities, singular Newton's method.

AMS subject classifications. 65K05, 49D37

1. Introduction.

In this paper we consider the problem of solving the nonlinear system of equations and inequalities

$$F(x,y) = 0\ ,\ y \geq 0 \tag{1.1}$$

where $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$ is continuously differentiable. We will be concerned with the possibility that n, m, and p may be large, and that the Jacobian of F at (x,y), say $F'(x,y)$, may be sparse and rank deficient.

Problem (1.1) is quite general; the Karush-Kuhn-Tucker conditions of a general nonlinear programming problem are an example of such a problem, and a general set of equalities and inequalities can be transformed into problem (1.1) using slack variables.

Recently in El Hallabi [6], the author proposed a globally and q-quadratically convergent hybrid algorithm to solve a nonlinear system of equations. The algorithm uses an arbitrary norm of the residual as merit function. It calculates a search direction as an approximate solution of a local

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model, then uses linesearch techniques to obtain an acceptable steplength. Moreover, for a specific choice of the norm, it can be implemented as a sequential linear programming method, and hence be solved by a simplex type method or by the more recent primal-dual feasible-point methods for linear programming. See Kojima, Mizuno, and Yoshise [10], Zhang and Tapia [19], Zhang, Tapia, and Dennis [20], and Lusting, Marsten, and Shanno [12]).

Also, recently there has been considerable activity in the area of feasible-point methods for general nonlinear programming problems. We cite for example Yamashita [18], Wright [16], Wright [17] Lasdon, Yu, and Plummer [11], McCormick [14], and Martinez, Tapia, and Parada [13]. El-Bakry, Tapia, Tsuchiya, and Zhang [3], propose a primal-dual Newton feasible-point method to solve the Karush-Kuhn-Tucker conditions of a general nonlinear programming problem using the squared l_2 norm of the residual as a merit function. Very much influenced by this latter approach, we extend the hybrid approach of El Hallabi [6] to problem (1.1).

We propose an inexact trust-region feasible-point (ITRFP) algorithm to solve problem (1.1) in the equivalent form

$$\begin{array}{llcl} \text{minimize}_{\,(x,y)\in \mathbb{R}^n\times\mathbb{R}^m} & f(x,y) & = & \|F(x,y)\|_a \\ \text{subject to} & y & \geq & 0, \end{array} \tag{1.2}$$

where $\| \, \|_a$ is an arbitrary (but fixed) norm on $\mathbb{R}^p$. The proposed algorithm combines trust-region, line-search and feasible-point strategies. At each iteration, the search direction (u_k, v_k) is obtained as an approximate solution of the local model trust-region subproblem

$$(LMTR) \equiv \left\{ \begin{array}{ll} \text{minimize} & m_k(u,v) = \|F(x_k,y_k) + F'(x_k,y_k)(u,v)\|_a \\ \text{subject to} & (1-\sigma_k)y_k + v \geq 0 \\ & \|(u,v)\|_b \leq \Delta_k. \end{array} \right. \tag{1.3}$$

where $\Delta_k > 0$ is the trust-region radius, $\| \, \|_a$ and $\| \, \|_b$ are two arbitrary (but fixed) norms on $\mathbb{R}^p$ and $\mathbb{R}^n \times \mathbb{R}^m$ respectively, and σ_k is an arbitrary scalar in $[0, \hat{\sigma}]$ for some $\hat{\sigma} \in [0, 1)$. To obtain an acceptable steplength, a linesearch in the direction (u_k, v_k) is performed.

In (1.3), the parameter σ_k is used to ensure the feasibility of the iterates $\{y_k\}$. We allow this parameter to be zero to provide a global convergence theory for the feasible-point variant of our approach. Indeed, with σ_k equal to zero, the iterates y_k are allowed to be on the boundary. On the other hand, we can force strict feasibility by choosing σ_k to be positive, and hence obtain a global convergence theory for the interior-point method variant of our approach. In any case, $1 - \sigma_k$ has to be bounded away from zero, which is obtained threw the constant $\hat{\sigma}$.

In (1.2) and (1.3) we use arbitrary norms for the convenience of the presentation and for the sake of mathematical generalization. Our goal is to use polyhedral norms, in which case the local model trust-region subproblem LMTR in (1.3) can be formulated as a linear programming problem.

In Section 2 we derive optimality conditions for problem (1.2). The inexact trust-region feasible-

point algorithm (ITRFP) is described in Section 3. In Section 4 we demonstrate that the ITRFP algorithm is globally convergent. In Section 5 we prove, under rather weak assumptions, that the sequence of constrained residuals $\{F(x_k, y_k)\}$ converges to zero. Finally, we give some concluding remarks in Section 6.

For convenience, we use z_k and (x_k, y_k) to denote the iterate and w_k and (u_k, v_k) to denote the search direction or the step. Also, we use z_k^i to denote the i^{tk} component of z_k.

2. Optimality Conditions. In this section, we define the optimality condition for problem (1.2), and we derive a practical condition for optimality in terms of minimizers of LMTR subproblem.

The locally Lipschitz composite function $f = \|F\|_a$ is regular, i.e. at any z and in any direction w in $\mathbb{R}^n \times \mathbb{R}^m$, its *generalized directional derivative*, denoted by $f^0(z; w)$, and its *one-sided directional derivative*, denoted by $f'(z; w)$, exist and are equal (see Clarke [2]). They are respectively defined by

$$f^0(z; w) = \limsup_{z' \to z,\ t \downarrow 0} \frac{f(z' + tw) - f(z')}{t} \tag{2.1}$$

and

$$f'(z; w) = \lim_{t \downarrow 0} \frac{f(z + tw) - f(z)}{t} \ . \tag{2.2}$$

For more details concerning properties of the various derivatives of $f = \|F\|_a$, we refer the reader to Clarke [2].

In this research, we use both derivatives althought they are equal. To study the optimality conditions, working with the one-sided directional derivative is sufficient. But to analyze the behavior of the algorithm near an iterate that is not a constrained stationary point of f, the generalized directional derivative is a powerful tool because its definition uses a hall neighborhood of z rather just the point z.

We now give the definition of stationarity that will be used in the present work.

DEFINITION 2.1. *A point $z_* = (x_*, y_*)$ is said to be a constrained stationary point of f if $y_* \geq 0$ and*

$$f'(z_*; w) \geq 0 \tag{2.3}$$

for all feasible directions $w = (u, v)$, i.e. for all $w = (u, v)$ such that $y_* + v \geq 0$.

The following lemma shows that the one-sided directional derivatives of both the function f and its local model

$$m_z(w) = \|F(z) + F'(z)w\|_a \tag{2.4}$$

are equal. This is important from an algorithmic point of view.

LEMMA 2.1. [El Hallabi and Tapia [8] *Assume that* $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$ *is a continuously differentiable function. Let* z *and* w *be respectively a feasible point and a feasible direction for problem (1.2). Then*

$$f'(z;w) = m_z'(0;w). \tag{2.5}$$

In the following proposition we give a practical criterion for optimality in terms of minimizers of the local model trust-region subproblem.

THEOREM 2.1. *Assume that* $z = (x, y) \in \mathbb{R}^n \times \mathbb{R}^m$ *is a feasible point of problem (1.1). Assume further that* $\alpha > 0$, *and* $\Delta > 0$. *Then* $(u_*, v_*) = (0, 0)$ *solves the local model trust-region subproblem*

$$\begin{array}{llcl} \text{minimize} & m_z(u,v) & = & \|F(x,y) + F'(x,y)(u,v)\|_a \\ \text{subject to} & \alpha y + v & \geq & 0 \\ & \|(u,v)\|_b & \leq & \Delta \end{array} \tag{2.6}$$

if and only if (x, y) *is a stationary point of* f.

Proof. Let $w = (u, v)$ be an arbitrary feasible direction of the minimization problem (1.2). It is obvious that if $y^i = 0$ then $\alpha y^i + tv^i \geq 0$ for all positive t. On the other hand, if $y^i > 0$, then for sufficiently small t, say $0 < t \leq t_* \leq 1$, we have $\alpha y^i + tv^i \geq 0$. In summary we obtain

$$\alpha y + tv \geq 0 \quad \text{and} \quad \|t(u,v)\|_b \leq \Delta \tag{2.7}$$

for sufficiently small t, say $t \in (0, t_*]$ for convenience, i.e. $t(u, v)$ is a feasible point of problem (2.6).

Assume that zero solves problem (2.6). This, together with (2.7), implies that

$$\frac{\|F(x,y) + tF'(x,y)(u,v)\|_a - \|F(x,y)\|_a}{t} \geq 0, \tag{2.8}$$

holds for all $t \in (0, t_*]$. Hence, by passing to the limit as t converges to zero and using Lemma 2.1, we obtain

$$f'(z;w) \geq 0. \tag{2.9}$$

Finally, since (2.9) holds for all feasible directions w, the feasible point $z = (x, y)$ is necessarily a stationary point of f.

Now, assume that $z = (x, y)$ is a stationary point of f. Let $w = (u, v)$ be a feasible point of problem (2.6). If $y^i = 0$ then

$$\alpha y^i + tv^i \geq 0, \quad \forall \quad t \geq 0. \tag{2.10a}$$

On the other hand, if $y^i > 0$ then there exists $\bar{t} \in (0,1]$ such that

$$\alpha y^i + \bar{t} v^i \geq 0. \tag{2.10b}$$

¿From (2.10a) and (2.10b), we obtain

$$y + \frac{\bar{t}}{\alpha} v \geq 0, \tag{2.11}$$

i.e. $\frac{\bar{t}}{\alpha}(u,v)$ is a feasible direction of problem (1.2). Because $z = (x,y)$ is a stationary point of f and $f'(z;\cdot)$ is positively homogeneous, we have

$$f'(z;w) \geq 0. \tag{2.12}$$

On the other hand, since $m_x(\cdot)$ is convex, we have

$$m_x'(0;w) \leq m_x(s) - m_x(0), \tag{2.13}$$

which, together with Lemma 2.1 and (2.12), implies

$$\|F(x,y)\|_a \leq \|F(x,y) + F'(x,y)(u,v)\|_a\ , \tag{2.13}$$

i.e. zero solves problem (2.6).□

3. The Inexact Trust-Region Feasible-Point Algorithm. In this section we define our algorithm for approximating a solution of the nondifferentiable optimization problem

$$\begin{array}{llcl} \min_{(x,y)\in \mathbb{R}^n \times \mathbb{R}^m} & f(x,y) & = & \|F(x,y)\|_a \\ \text{subject to} & y & \geq 0 & \end{array}$$

where $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$ is continuously differentiable and $\|\ \|_a$ is an arbitrary norm defined on $\mathbb{R}^p$. To obtain a descent direction, the local model subproblem LMTR is solved for an approximate solution in the sense given in the following definition.

DEFINITION 3.1. *Consider $z_k = (x_k, y_k) \in \mathbb{R}^n \times \mathbb{R}^m$ such that $y_k \geq 0$, $\varepsilon_k \geq 0$, $0 \leq \sigma_k < 1$, and $\Delta_k > 0$. Also let $\|\ \|_a$ and $\|\ \|_b$ be any two norms defined on $\mathbb{R}^p$ and $\mathbb{R}^n \times \mathbb{R}^m$ respectively . We say that $w_k = (u_k, v_k)$ is an ε_k-solution of the local model trust-region subproblem*

$$(LMTR) \equiv \left\{ \begin{array}{lccl} \textit{minimize} & m_k(u,v) & = & \|F(x_k,y_k) + F'(x_k,y_k)(u,v)\|_a \\ \textit{subject to} & (1-\sigma_k)y_k + v & \geq 0 & \\ & \|(u,v)\|_b & \leq \Delta_k & \end{array} \right.$$

if w_k satisfies

$$m_k(w_k) - m_k(0) < 0 \quad \textit{and} \quad m_k(w_k) \leq m_k(w) + \varepsilon_k$$

for all $w = (u, v)$ *satisfying* $\|w\|_b \leq \Delta_k$ *and* $(1 - \sigma_k)y_k + v \geq 0$.

Let (x_k, y_k) be a feasible point of problem (1.2), i.e $y_k \geq 0$, and consider any approximate solution of LMTR subproblem, say (u_k, v_k). We have

$$y_k + v_k \geq \sigma_k y_k, \tag{3.1}$$

which implies that $y_k + v_k$ is a feasible point of problem (1.2) for $\sigma_k \geq 0$. However if $\sigma_k < 0$, then $y_k + v_k$ might be negative. Therefore, in the following lemma, we describe the scheme to restore feasibility.

LEMMA 3.1. *Assume that* (x_k, y_k) *is a feasible point of problem (1.2). Assume further that* (u_k, v_k) *is a feasible point of problem (1.3). Let*

$$I_k = \{i \in [1...m] \quad | \quad v_k^i < 0\}. \tag{3.2}$$

Then for we have

$$y_k + \hat{t_k} v_k \geq 0 \tag{3.3}$$

and

$$\frac{1}{1 + \sigma_k} \leq \hat{t_k} \leq 1. \tag{3.4}$$

Moreover, if $\{\sigma_k > 0\}$ *converging to zero, then*

Proof. It is obvious that (3.2) holds for $\sigma_k \leq 0$. Therefore, we consider the case where $\sigma_k > 0$. Let us set

$$I_k = \{i \in [1...m] \quad | \quad v_k^i < 0\}.$$

It is obvious that

$$y_k^i + t v_k^i \geq 0 \quad \forall \quad t \geq 0 \tag{3.4}$$

holds for all $i \notin I_k$,i.e such that $v_k^i \leq 0$. For the indices $i \in I_k$, we set

$$\hat{t_k} = \min_{i \in I_k} \left(1, -\frac{-y_k^i}{v_k^i}\right), \tag{3.5}$$

which implies that

$$y_k^i + t v_k^i \geq 0 \quad \forall \quad i \in I_k. \tag{3.6}$$

¿From (3.4) and (3.6), we obtain (3.2).

Now, since (u_k, v_k) is a feasible point of subproblem LMTR, we have

$$(1+\sigma_k)y_k^i + v_k^i \geq 0, \quad \forall i \in [1...m],$$

which implies that,

$$-\frac{-y_k^i}{v_k^i} \geq \frac{1}{1+\sigma_k} \quad \forall i \in I_k. \tag{3.7}$$

¿From (3.5) and (3.7), we obtain (3.3).

Inexact Trust-Region Feasible-Point Algorithm (ITRFP)

Let $c_i, i = 0, \ldots, 5, \Delta_{\min}$, $\Delta_{\max}$ and β_0 be constants satisfying:

$$\begin{array}{ll} 0 < c_1 < c_2 < 1 \leq c_3 & 0 < c_4 < c_5 < 1 \\ 0 < \Delta_{\min} \ll 1 & 1 \ll \Delta_{\max} < +\infty \\ 0 < \beta_0 & 0 < \hat{\sigma} < 1 \end{array}$$

Let $z_0 = (x_0, y_0)$ be any point such that $y_0 \geq 0$. Assume that $\Delta_0 \geq \Delta_{\min}$, $\sigma_0 \in [0, \hat{\sigma}]$, and $\| \, \|_a$ and $\| \, \|_b$ be any two norms respectively on $\mathbb{R}^p$ and $\mathbb{R}^n \times \mathbb{R}^m$.

Suppose that $z_k = (u_k, v_k)$, Δ_k, σ_k, β_k has been determined by the algorithm at the k^{th} iteration. The algorithm determines z_{k+1}, Δ_{k+1}, σ_{k+1}, and β_{k+1} in the following manner:

STEP 1. Set $\varepsilon_k = \beta_k \|F(z_k)\|_a$ and obtain an ε_k-solution $w_k = (u_k, v_k)$ of subproblem LMTR

STEP 2. Set $t_k = 1$

Until

$$f(z_k + t_k w_k) \leq f(z_k) + c_1[m_k(t_k w_k) - f(z_k)]$$

Choose $\bar{t_k}$ such that

$$c_4 t_k \leq \bar{t_k} \leq c_5 t_k;$$

set $t_k = \bar{t_k}$.

End (Until)

Set $w_k := t_k w_k$ and $z_{k+1} = z_k + w_k$

STEP 4. If $f(z_{k+1}) \leq f(z_k) + c_2[m_k(w_k) - f(z_k)]$

choose Δ_{k+1} so that

$$\|w_k\|_b \leq \Delta_{k+1} \leq \max(\Delta_k, c_3\|w_k\|_b)$$

Else

Choose Δ_{k+1} such that

$$c_4\|w_k\|_b \leq \Delta_{k+1} \leq \|w_k\|_b$$

STEP 5. Set $\qquad \Delta_{k+1} = \min(\max(\Delta_{k+1}, \Delta_{\min}), \Delta_{\max})$

Choose $\quad \beta_{k+1} \in [0, \leq \beta_0]$, and $\quad \sigma_{k+1} \in [0, \hat{\sigma}]$.

DEFINITION 3.2 *The iterate* $(x_{k+1}, y_{k+1}, \Delta_{k+1}, \sigma_{k+1}, \beta_{k+1})$ *will be referred to as a successor of* $(x_k, y_k, \Delta_k, \sigma_k, \beta_k)$, *and* t_k *will be referred to as an acceptable steplength with respect to* $(x_k, y_k, \Delta_k, \sigma_k, \beta_k)$ *or just an acceptable steplength.*

4. Global Convergence for the ITRFP Algorithm. In this section we demonstrate that the inexact trust-region feasible-point algorithm (ITRFP) is globally convergent in the sense that any accumulation point of the iteration sequence is a constrained stationary point of f. This result will be established in Theorem 4.4 using a proof by contradiction. Therefore, a crucial role of our global convergence analysis is played by the derivation of some important properties of the ITRFP Algorithm near feasible points that are not constrained stationary points of f. First, in Theorem 4.1, we analyze the behavior of the ε_k-solution near such points; second, in Theorem 4.2, we analyze the behavior of the steplenght; and third we analyze the behavior of the successor iterate in Theorem 4.3.

Throughout this section, unless otherwise stated, $\varepsilon_k(\beta_k)$ is defined by

$$\varepsilon_k(\beta_k) = \beta_k \|F(z_k)\|_a \ .$$

We start by proving that any ε_k-solution of the local model trust-region subproblem (LMTR) is a descent direction for $f = \|F\|_a$ at the current iterate, and consequently, we can obtain an acceptable step by using a linesearch technique.

PROPOSITION 4.1. *Assume that* z_k *is a feasible point. If* z_k *is a not a constrained stationary point of* f, *then*

$$f'(z_k; w_k) < 0 \ . \tag{4.1}$$

for any ε_k*-solution* w_k *obtained in STEP 1 of the ITRFP Algorithm.*

Proof. First, we obtain from the convexity of $m_k(\cdot)$ that the inequality

$$m_k'(0; w) \leq m_k(w) - m_k(0) \tag{4.2}$$

holds for all w. Now, the proof follows from (4.2), Definition 1.1, Theorem 2.1, and Lemma 2.1. □

PROPOSITION 4.2. *Assume that a feasible* z_k *is not a constrained stationary point of* f. *Assume further that* w_k *is given by STEP 1 of the ITRFP Algorithm. Then there exists* $t_k \in (0, 1]$ *such that*

$$f(z_k + t_k w_k) \leq f(z_k) + c_1[m_k(t_k w_k) - f(z_k)] \ . \tag{4.3}$$

Proof. The proof is an obvious consequence of Lemma 2.1 and Proposition 4.1. □

At each iteration, we can consider the local model subproblem LMTR as a parameterized minimization problem. In the following theorem, we analyze the behavior of the ε_k-solutions, considered as functions of the parameters of the LMTR subproblem.

THEOREM 4.1. *Let $\{(x_k, y_k), \Delta_k, \sigma_k, \beta_k)\}$, where $y_k \geq 0$, $0 \leq \sigma_k \leq \hat{\sigma}$, and $\beta_k \geq 0$, be a sequence converging to $((x_*, y_*), \Delta_*, \sigma_*, 0)$, where (x_*, y_*) and (x_k, y_k) are not constrained stationary points of f, and where Δ_k and Δ_* are positive. Let (u_k, v_k) be an $\varepsilon_k(\beta_k)$-solution of the local subproblem*

$$
(4.4) \qquad \begin{array}{lll} \text{minimize} & m_k(u,v) & = \quad \|F(x_k, y_k) + F'(x_k, y_k)(u,v)\|_a \\ \text{subject to} & (1-\sigma_k)y_k + v & \geq 0 \\ & \|(u,v)\|_b & \leq \quad \Delta_k. \end{array}
$$

Then any accumulation point of $\{(u_k, v_k)\}$, say (x_*, y_*), is an exact solution of the local subproblem

$$
(4.5) \qquad \begin{array}{lll} \text{minimize} & m_*(u,v) & = \quad \|F(x_*, y_*) + F'(x_*, y_*)(u,v)\|_a \\ \text{subject to} & (1-\sigma_*)y_* + v & \geq 0 \\ & \|(u,v)\|_b & \leq \quad \Delta_*. \end{array}
$$

Proof. Since $\{\Delta_k\}$ converges to Δ_* and $\|(u_k, v_k)\|_b \leq \Delta_k$ for all k, the sequence $\{(u_k, v_k)\}$ is bounded. Consider any accumulation point (u_*, v_*) of this sequence. We prove that

$$
(4.6) \qquad \|F(x_*, y_*) + F'(x_*, y_*)(u_*, v_*)\|_a \leq \|F(x_*, y_*) + F'(x_*, y_*)(u,v)\|_a
$$

holds for all (u,v) such that $\|(u,v)\|_b \leq \Delta_*$ and $(1-\sigma_*)y_* + v \geq 0$, i.e., (u_*, v_*) is an exact solution of the minimization problem (4.5).

Let (u,v) satisfy $\|(u,v)\|_b \leq \Delta_*$ and $(1-\sigma_*)y_* + v \geq 0$.
First, we consider the indices i such that

$$
(1-\sigma_*)y_*^i + v^i > 0,
$$

i.e. a feasible point of problem (4.5). Since $\{(1-\sigma_k)y_k^i + \frac{\Delta_k}{\Delta_*}v^i\}$ converges to $(1-\sigma_*)y_*^i + v^i$, we have for sufficiently large k

$$
(1-\sigma_k)y_k^i + \frac{\Delta_k}{\Delta_*}v^i \geq 0.
$$

This implies , since $(1-\sigma_k)y_k^i \geq 0$, that

$$
(4.7) \qquad (1-\sigma_k)y_k^i + t\frac{\Delta_k}{\Delta_*}v^i \geq 0
$$

for all $t \in [0,1]$.
Now, we consider the indices i such that

$$
(1-\sigma_*)y_*^i + v^i = 0.
$$

Therefore $v^i \leq 0$ must hold. If $v^i = 0$ it is obvious that $(1-\sigma_k)y_k^i + tv^i \geq 0$ for all positive t. So we consider the case where $v^i < 0$. This implies that $y_*^i > 0$ and $(1-\sigma_k)y_k > 0$ for sufficiently large k, say $k \geq k_*$. Let us set, for $k \geq k_*$,

$$t_k^i = \min\Big(1, -\frac{(1-\sigma_k)y_k^i}{v^i}\Big),$$

which implies that

$$(1-\sigma_k)y_k^i + tv^i \geq 0 \tag{4.8}$$

for all $t \in [0, t_k^i]$. Let us define

$$I_-(v) = \{i \in [1,...,m] \mid v^i < 0\} \cap \{i \in [1,...,m] \mid (1-\sigma_*)y_*^i + v^i = 0\}$$

and

$$\hat{t_k} = \min_{i \in I_-(v)} \Big(t_k^i, \ \frac{\Delta_k}{\Delta_*}\Big). \tag{4.9}$$

We obtain from (4.7), (4.8), and (4.9) that

$$(1-\sigma_k)y_k + \hat{t_k}v \geq 0, \tag{4.10a}$$

and

$$\|\hat{t_k}\frac{\Delta_k}{\Delta_*}(u,v)\|_b = \Delta_k\frac{\|(u,v)\|_b}{\Delta_*} \leq \Delta_k. \tag{4.10b}$$

Observe that $\{\hat{t_k}\}$ converges to one and that $\{(1-\sigma_k)y_k + \hat{t_k}v\}$ converges to $(1-\sigma_*)y_* + v$. ¿From (4.10a,b) we obtain that $(u_k, v_k) = \hat{t_k}\frac{\Delta_k}{\Delta_*}(u,v)$ is a feasible point of the local model trust-region subproblem (4.4) whose $\varepsilon_k(\beta_k)$-solution is (u_k, v_k). This implies that

$$\begin{aligned}\|F(x_k,y_k) + F'(x_k,y_k)(u_k,v_k)\|_a \quad &\leq \quad \|F(x_k,y_k) + t_k\tfrac{\Delta_k}{\Delta_*}F'(x_k,y_k)(u,v)\|_a \\ &\quad\quad +\beta_k\|F(x_k,y_k)\|_a\ ,\end{aligned}$$

and by passing to the limit when $k \to +\infty$, we obtain (4.6).□

In the folowing theorem, we proove that the acceptable step is bounded away from zero near a feasible point that is not a constrained stationary point of f.

THEOREM 4.2. *Let $\{(x_k, y_k, \Delta_k, \sigma_k, \beta_k)\}$ where $y_k \geq 0$, $\Delta_k \geq \Delta_{\min}$, $0 \leq \sigma_k \leq \hat{\sigma}$, and $\beta_k \geq 0$, be a sequence that converges to some $(x_*, y_*, \Delta_*, \sigma_*, 0)$. Assume that (x_*, y_*) and (x_k, y_k) are not constrained stationary points of f. Then there exists a positive scalar $t(x_*, y_*, \Delta_*) > 0$ such that*

$$t_* \geq t(x_*, y_*, \Delta_*) \tag{4.10}$$

holds for any accumulation point t_* of $\{t_k\}$ where t_k is an acceptable steplength with respect to $(x_k, y_k, \Delta_k, \beta_k)$.

Proof. The proof is very similar to the proof of Theorem in El Hallabi and Tapia [8]. Observe that $\Delta_k \geq \Delta_{\min}$ implies that $\Delta_* > 0$. Assume that for any constant $\gamma > 0$, there exists an accumulation point of $\{t_k\}$, say $t_{*,\gamma}$, such that

$$0 \leq t_{*,\gamma} < \gamma \ .$$

Therefore there exists a subsequence $\{t_k, k \in N \subset \mathbb{N}\}$ converging to zero. Without loss of generality, we can assume that $\{t_k\}$ converges to zero. This implies that for sufficiently large k, we have $0 < t_k < 1$, and that a steplength of one is never accepted. Let $\bar{t_k}$ be the last non acceptable steplength in the direction (u_k, v_k), an $\varepsilon_k(\beta_k)$-solution of the local model trust-region subproblem (LMTR). We have that

$$0 < c_4\bar{t_k} \leq t_k \leq c_5\bar{t_k} \ . \tag{4.11}$$

Since $\{t_k\}$ converges to zero, we obtain from (4.11) that $\{\bar{t_k}\}$ converge to zero. On the other hand, the steplength $\bar{t_k}$ is not acceptable; therefore we have

$$f(z_k + \bar{t_k}w_k) - f(z_k) > c_1[\|F(z_k) + \bar{t_k}F'(z_k)w_k\|_a - \|F(z_k)\|_a \tag{4.12}$$

where $z_k = (x_k, y_k)$ and $w_k = (u_k, v_k)$. Also, since $\{\Delta_k\}$ converges to Δ_*, the sequence $\{u_k, v_k\}$ is bounded. Let $w_* = (u_*, v_*)$ be any accumulation point of this sequence. Without loss of generality we can assume that $\{u_k, v_k\}$ converges to (u_*, v_*). Let us rewrite (4.12) as follows

$$\begin{aligned}\frac{f(z_k + \bar{t_k}w_*) - f(z_k)}{\bar{t_k}} > {} & c_1\frac{\|F(z_k) + \bar{t_k}F'(z_k)w_*\|_a - \|F(z_k)\|_a}{\bar{t_k}} \\ & + \frac{f(z_k + \bar{t_k}w_*) - f(z_k + \bar{t_k}w_k)}{\bar{t_k}} \\ & + c_1 \ \frac{\|F(z_k) + \bar{t_k}F'(z_k)w_*\|_a - \|F(z_k) + \bar{t_k}F'(z_k)w_k\|_a}{\bar{t_k}}.\end{aligned} \tag{4.13}$$

But because F is continuously differentiable, we have

$$\|F(z_k) + \bar{t_k}F'(z_k)w_*\|_a - \|F(z_k)\|_a \geq f(z_k + \bar{t_k}w_*) - f(z_k) + o(\bar{t_k}), \tag{4.14}$$

where $\lim_{k\to+\infty} \frac{o(\bar{t_k})}{\bar{t_k}} = 0$, which, together with (4.13), implies

$$\begin{aligned}(1 - c_1)\,\frac{f(z_k + \bar{t_k}w_*) - f(z_k)}{\bar{t_k}} > {} & \frac{f(z_k + \bar{t_k}w_*) - f(z_k) + \bar{t_k}w_k)}{\bar{t_k}} \\ & c_1\frac{\|F(z_k) + \bar{t_k}F'(z_k)w_k\|_a - \|F(z_k) + \bar{t_k}F'(z_k)w_*\|_a}{\bar{t_k}} + \frac{o(\bar{t_k})}{\bar{t_k}}.\end{aligned} \tag{4.15}$$

Because the function f and the norm $\| \|_a$ are locally Lipschitz and because $c_1 \in (0,1)$, we obtain

$$\limsup_{k \to +\infty} \frac{f(z_k + \bar{t_k} w_*) - f(z_k)}{\bar{t_k}} \geq 0 \ ,$$

and hence

$$\limsup_{\substack{z \to z_* \\ t \downarrow 0}} \frac{f(z + t w_*) - f(z)}{t} \geq 0,$$

i.e. $f^0(z_*; w_*)) \geq 0$, and because $f = \|F\|_a$ is regular, we have

$$f'(z_*, w_*) \geq 0 \quad . \tag{4.16}$$

¿From the convexity of $m_*(\cdot)$ (see (4.2)), (4.16), and Lemma 2.1, we obtain

$$\|F(z_*)\|_a \leq \|F(z_*) + F'(z_*) w_*\|_a. \tag{4.17}$$

On the other hand, since $\Delta_* > 0$, we obtain from Theorem 4.1 that $w_* = (u_*, v_*)$ is an exact solution of the local model trust-region subproblem (4.5). This, together with (4.17), implies that zero is a solution of the local model trust-region subproblem (4.15). Therefore, by Theorem 2.1, we obtain that $w_* = (x_*, y_*)$ is a stationary point of f. This contradicts our hypothesis.

Consequently, there exists a positive scalar $t(x_*, y_*, \Delta_*)$ such that (4.10) holds for any accumulation point t_* of $\{t_k\}$. $\square$

COROLLARY 4.1. *Let* $\{(x_j, y_j, \Delta_j, \sigma_j, \beta_j)\}$*, where* $y_j \geq 0$*,* $\Delta_j \geq \Delta_{\min}$*,* $0 \leq \sigma_j \leq \hat{\sigma}$*, and* $\beta_j \geq 0$*, be a subsequence generated the the ITRFP algorithm that converges to some* $(x_*, y_*, \Delta_*, \sigma_*, 0)$*. If zero is an accumulation point of the sequence of the acceptable steplengths* $\{t_j\}$*, then* (x_*, y_*) *in necessarily a constrained stationary point of* f*.*

Now we establish that the ITRFP Algorithm satisfies a property that we call we *Local Uniform Decrease.* This property is a very powerful tool for obtaining global convergence results (see El Hallabi [4], El Hallabi [5], and El Hallabi [7]. It is the most important hypothesis used in Polak [15] and Huard [9] to obtain the global convergence of some conceptual algorithms.

THEOREM 4.3. *Consider* $(x_*, y_*, \Delta_*, \sigma_*)$ *where* $y_* \geq 0$*,* $\Delta_* > 0$*, and* $0 \leq \sigma_* \leq \hat{\sigma}$*. If* (x_*, y_*) *is not a stationary point of* f*, then there exists a neighborhood of* $(x_*, y_*, \Delta_*, \sigma_*, 0)$*, denoted* $N_* = N(x_*, y_*, \Delta_*, \sigma_*, 0)$*, and a positive scalar* $\rho_* = \rho(x_*, y_*, \Delta_*, \sigma_*)$ *such that for any* $(x, y, \Delta, \sigma, \beta) \in N_*$ *with* $\Delta \geq \Delta_{\min}$*,* $0 \leq \sigma \leq \hat{\sigma}$*, and* $\beta > 0$

$$f(x_+, y_+) < f(x_*, y_*) - \rho_* \tag{4.18}$$

holds for any successor $(x_+, y_+, \Delta_+, \sigma_+, \beta_+)$ *of* $(x, y, \Delta, \sigma, \beta)$*.*

Proof. We give a proof by contradiction. Assume that the theorem does not hold. Then there exists a sequence $\{x_k, y_k, \Delta_k, \sigma_k, \beta_k\}$, with $0 \le \sigma_k \le \hat{\sigma}$ and $\beta_k \ge 0$, converging to $(x_*, y_*, \Delta_*, \sigma_*, 0)$, a sequence $\{\rho_k\}$ converging to zero, and a sequence $\{(x_{k+}, y_{k+}, \Delta_{k+}, \sigma_{k+}, \beta_{k+})\}$ of successors of $(x_k, y_k, \Delta_k, \sigma_k, \beta_k)$, such that

$$f(x_{k+}, y_{k+}) \ge f(x_*, y_*) - \rho_k \tag{4.19}$$

holds for all k. Therefore, for all k, there exists and $t_k \in (0, 1]$ such that

$$(x_{k+}, y_{k+}) = (x_k, y_k) + t_k(u_k, v_k)$$

satisfies (4.19). Because (x_{k+}, y_{k+}) is a successor of (x_k, y_k), we have

$$\begin{aligned} f(x_{k+}, y_{k+}) \quad &\le \quad f(x_k, y_k) + c_1 \\ &\qquad [\|F(x_k, y_k) + t_k F'(x_k, y_k)u_k, v_k\|_a - \|F(x_k, y_k)\|_a] \ . \end{aligned} \tag{4.20}$$

¿From (4.19) and (4.20) we obtain

$$\begin{aligned} f(x_*, y_*) - \rho_k \quad &\le \quad f(x_k, y_k) + c_1 \\ &\qquad [\|F(x_k, y_k) + t_k F'(x_k, y_k)u_k, v_k\|_a - \|F(x_k, y_k)\|_a]. \end{aligned} \tag{4.21}$$

The sequence $\{(t_k, u_k, v_k)\}$ is bounded. Let (t_*, u_*, v_*) be any accumulation point of such a sequence where, by theorem 4.1, $t_* > 0$. We obtain from (4.21) that

$$\|F(x_*, y_*) + t_* F'(x_*, y_*)(u_*, v_*)\|_a - \|F(x_*, y_*)\|_a \ge 0 \ . \tag{4.22}$$

Let us set

$$\phi_*(t) = \|F(x_*, y_*) + tF'(x_*, y_*)(u_*, v_*)\|_a \ .$$

¿From (4.22) we obtain

$$\phi_*(0) \le \phi_*(t_*) \ ,$$

and since ϕ_* is convex and $0 < t_* \le 1$, we have necessarily

$$\phi_*(0) \le \phi_*(1) \ ,$$

or equivalently

$$\|F(x_*, y_*)\|_a \le \|F(x_*, y_*) + F'(x_*, y_*)(u_*, v_*)\|_a \ . \tag{4.23}$$

On the other hand we obtain from Theorem 4.1 that (u_*, v_*) is an exact minimizer of the local model trust-region subproblem (4.5), which, together with (4.23), implies that zero is a solution of

subproblem (4.5). Therefore, we obtain from Theorem 2.1 that (x_*, y_*) is a necessarily a stationary point of f, which contradicts our hypothesis. □

Finally, in the following theorem, we demonstrate that the inexact trust-region feasible-point Algorithm (ITRFP) described in Section 3 is globally convergent.

THEOREM 4.4 *Consider a continuously differentiable function* $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \times \mathbb{R}^m$. *Let* $\| \; \|_a$ *and* $\| \; \|_b$ *be arbitrary (but fixed) norms respectively on* $\mathbb{R}^p$ *and* $\mathbb{R}^n \times \mathbb{R}^m$, *let* $f(x, y) = \|F(x, y)\|_a$, *and finally let* $(x_0, y_0) \in \mathbb{R}^n \times \mathbb{R}^m$ *satisfy* $y_0 \geq 0$. *Assume that*
i) the sequence $\{\beta_k\}$ *converges to zero, and*
ii) the inequality $0 \leq \sigma_k \leq \hat{\sigma} < 1$ *holds for all* k.

Then any accumulation point of the sequence $\{(x_k, y_k)\}$ *generated by the ITRFP algorithm of Section 3 using* (x_0, y_0) *as initial iterate is a constrained stationary point of* f.

Proof. Let (x_*, y_*) be an accumulation point of the sequence $\{x_k, y_k\}$ generated by the algorithm. Without loss of generality (by considering a subsequence if necessary), we can assume that the sequence converges to (x_*, y_*), and hence the sequence $\{(x_k, y_k, \Delta_k, \sigma_k, \beta_k)\}$ can be assumed to be bounded. Let $\{(x_j, y_j, \Delta_j, \sigma_j, \beta_j)\}$ be a subsequence that converges to $(x_*, y_*, \Delta_*, \sigma_*, 0)$. Because the sequence $\{f(x_k, y_k)\}$ is decreasing, we have

$$f(x_j, y_j) < f(x_k, y_k) \quad \forall j \geq k \;, \quad \forall k \in \mathbb{N} \;,$$

which implies that

$$f(x_*, y_*) < f(x_k, y_k) \quad \forall k \in \mathbb{N} \;. \tag{4.24}$$

Suppose that (x_*, y_*) is not a stationary point of f. Since in $(x_*, y_*, \Delta_*, \sigma_*, 0)$, (x_*, y_*) is not a stationary point of f, $\Delta_* > 0$, and $0 \leq \sigma_* \leq \hat{\sigma}$, we obtain from Theorem 4.3, there exists a neighborhood $N_* = N_*(x_*, y_*, \Delta_*, \sigma_*, 0)$ and a positive scalar ρ_* such that for any $(x, y, \Delta, \sigma, \beta) \in N_*$, with $y \geq 0, \beta > 0$, and $0 \leq \sigma \leq \hat{\sigma}$, the inequality

$$f(x_+, y_+) < f(x_*, y_*) - \rho_*$$

holds for any successor (x_+, y_+) of (x, y). Now since the sequence $\{(x_j, y_j, \Delta_j, \sigma_j, \beta_j)\}$ converges to $(x_*, y_*, \Delta_*, \sigma_*, 0)$, there exists an integer j_* such that $(x_j, y_j, \Delta_j, \sigma_j, \beta_j) \in N_*$ for all $j \geq j_*$ and hence

$$f(x_{j+1}, y_{j+1}) < f(x_*, y_*) - \rho_* \quad \forall j \geq j_* \;. \tag{4.25}$$

Inequality (4.25) contradicts (4.24). Consequently, any accumulation point of the sequence $\{x_k, y_k\}$ generated by the algorithm in Section 3 is a stationary point of $f = \|F\|_a$. □

5. Convergence to a Solution of $F(x) = 0$. In this section we demonstrate that, under rather weak hypotheses, the sequence of constrained residuals $\{F(x_k, y_k)\}$ actually converges to zero.

THEOREM 5.1. *Assume the hypotheses of Theorem 4.4. Also assume that there exists a bounded subsequence* $\{(x_j, y_j), j \in J \subset \mathbb{N}\}$ *and a constant* $0 \leq \eta < 1$ *such that*

$$\|F(x_j, y_j) + F'(x_j, y_j)(u_j, v_j)\|_a \leq \eta \|F(x_j, y_j)\|_a \tag{5.1}$$

holds for all $j \in J$. *Then any accumulation point* (x_*, y_*) *of the iteration sequence* $\{(x_k, y_k)\}$ *is a solution of problem (1.1). Moreover, the sequence of constrained residuals* $\{F(x_k, y_k)\}$ *converges to zero.*

Proof. Let (x_*, y_*) be an accumulation point of the subsequence $\{(x_j, y_j), j \in J \subset \mathbb{N}\}$. Without loss of generality (by considering a sub-subsequence if necessary) we can assume that the subsequence converges to (x_*, y_*). From Theorem 4.4, we obtain that (x_*, y_*) is a stationary point of f, which implies that

$$\|F(x_*, y_*)\|_a \leq \|F(x_*, y_*) + F^{'}(x_*, y_*) w\|_a \tag{5.2}$$

for all feasible direction w of problem (2.1). On the other hand, since $\{(u_j, v_j)\}$ is bounded, we can assume without loss of generality that it converges to, say, (u_*, v_*). Therefore, inequality (5.1) implies that

$$\|F(x_*, y_*) + F'(x_*, y_*)(u_*, v_*)\|_a \leq \eta \|F(x_*, y_*)\|_a. \tag{5.3}$$

Finally, from (5.2), (5.3), and $0 \leq \eta < 1$, we obtain

$$F(x_*, y_*) = 0. \tag{5.4}$$

This implies that the sequence of residuals $\{\|F(x_k, y_k)\|_a\}$ has zero as accumulation point, and, since it is decreasing, it converges to zero. $\square$

REMARK. Condition (5.1) can be written as

$$m_k(w_k) \;\leq\; \eta \quad m_k(0) \tag{5.5}$$

where $w_k = (u_k, v_k)$. Because, first, at each iteration we minimize, within some tolerance (see Definition 3.1), the local model trust-region subproblem LMTR, second, zero is a feasible point for such minimization problem, and third, we are considering a constrained zero residual problem, the assumption that (5.1) holds for a subsequence does not seem to be restrictive.

6. Summary and Concluding Remarks. We have presented an inexact trust-region feasible-point algorithm to solve the nonlinear system of equations and inequalities

$$F(x,y)=0\ ,\quad y\geq 0 \tag{6.1}$$

where $F:\mathbb{R}^n\times\mathbb{R}^m\rightarrow\mathbb{R}^p$ is continuously differentiable. The algorithm solves the model problem

$$\begin{array}{llcl} \text{minimize}_{\ (x,y)\in\mathbb{R}^n\times\mathbb{R}^m} & f(x,y) & = & \|F(x,y)\|_a \\ \text{subject to} & y & \geq & 0, \end{array} \tag{6.2}$$

where $\|\ \|_a$ can be an arbitrary (but fixed) norm on $\mathbb{R}^p$. It combines trust-region, feasible-point, and linesearch strategies. At each iteration the search direction is obtained as an approximate solution of the local model trust-region subproblem

$$\begin{array}{llcl} \text{minimize} & m_k(u,v) & = & \|F(x_k,y_k)+F'(x_k,y_k)(u,v)\|_a \\ \text{subject to} & (1-\sigma_k)y_k+v & \geq 0 & \\ & \|(u,v)\|_b & \leq & \Delta_k, \end{array} \tag{6.3}$$

where $\Delta_k\geq\Delta_{min}>0$ is the trust-region radius, $0\leq\sigma_k\leq\hat{\sigma}<1$, and $\|\ \|_a$ and $\|\ \|_b$ are arbitrary (but fixed) norms on $\mathbb{R}^p$ and $\mathbb{R}^n\times\mathbb{R}^m$ respectively.

In our formulation, we use arbitrary norms for the convenience of the presentation and the seek of mathematical generalization. Motivated by the recent advances in the linear programming research area, our goal is to use a polyhedral norm, (especially $\|\ \|_a=\|\ \|_1$and $\|\ \|_b=\|\ \|_\infty$), so that the local model trust-region subproblem (6.3) can be formulated as a linear programming problem, which would be adequate for large systems, and hence take advantage of the recent advances in linear programming area. Also our formulation is adequate for cases where the Jacobian matrix F' may be singular.

We have established, under rather weak hypotheses, that any accumulation point of the iteration sequence is a stationarity point of $f=\|F\|_a$. We also showed that, under a weak forcing condition, the sequence of constrained residuals converges to zero. Moreover, since we are considering a constrained zero residual problem, in practice the forcing condition will be satisfied as a by-product of the local model trust-region subproblem minimization process.

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