

**Superconvergence and  
Postprocessing of Fluxes From  
Lowest Order Mixed Methods on  
Triangles and Tetrahedra**

*Todd DuPont*

*Philip Keenan*

**CRPC-TR94539**

**November 1994**

Center for Research on Parallel Computation  
Rice University  
6100 South Main Street  
CRPC - MS 41  
Houston, TX 77005

# SUPERCONVERGENCE AND POSTPROCESSING OF FLUXES FROM LOWEST ORDER MIXED METHODS ON TRIANGLES AND TETRAHEDRA

TODD F. DUPONT\* AND PHILIP T. KEENAN†

**Abstract.** Certain finite difference methods on rectangular grids for second order elliptic equations are known to yield superconvergent flux approximations. A class of related finite difference methods have recently been defined for triangular meshes by applying special quadrature rules to an extended version of a mixed finite element method [1, 2]; the usual hybrid mixed method can also be applied to meshes of triangular and tetrahedral elements. Unfortunately, the flux vectors from these methods are only first order accurate. Empirical evidence indicates that a local postprocessing technique described in [11] recovers second order accurate velocities at special points. In this paper, a class of local postprocessing techniques generalizing the one in [11] are presented and analyzed. These postprocessors are shown to recover second order accurate velocity fields on three lines meshes. Numerical experiments illustrate these results and investigate more general situations, including meshes of tetrahedral elements.

**Keywords:** Mixed finite element method, finite differences, elliptic partial differential equation, flux, velocity, triangles, tetrahedra, postprocessing, superconvergence.

**AMS(MOS) subject classification:** 65N30, 65N06, 65N22

**1. Introduction.** In this paper we discuss techniques for extracting additional information from numerical solutions to elliptic partial differential equations of the form

$$(1) \quad -\nabla \cdot (K(\mathbf{x})\nabla p(\mathbf{x})) = f(\mathbf{x}), \quad \mathbf{x} \in \Omega,$$

where  $\Omega$  is a polygonal domain in  $\mathbf{R}^2$  or  $\mathbf{R}^3$ , and  $K(\mathbf{x})$  is a positive definite matrix. Let the flux associated with (1) be

$$(2) \quad \mathbf{u} = -K\nabla p,$$

and assume boundary conditions of the form

$$(3) \quad \mathbf{u}(\mathbf{x}) \cdot \hat{\mathbf{n}}(\mathbf{x}) = g(\mathbf{x}), \quad \mathbf{x} \in \partial\Omega,$$

where  $\hat{\mathbf{n}}$  is the unit outward normal to  $\partial\Omega$ . In the context of flow in porous media the flux is called the Darcy velocity when  $p$  is the pressure and  $K$  is the permeability tensor; we will use the words flux and velocity interchangeably to describe  $\mathbf{u}$ .

Cell centered finite difference methods on rectangular grids are well known to yield superconvergent flux approximations [14, 15, 3, 4]. A class of finite difference methods have recently been defined for triangular meshes by applying special quadrature rules to an extended version of a mixed finite element method [1, 2]; the usual hybrid mixed method can also be applied to meshes of triangular and tetrahedral elements. Unfortunately, on triangles and tetrahedra, the resulting velocity vectors are only first order accurate. Empirical evidence indicates that a local postprocessing technique described

---

\* Department of Computer Science, University of Chicago, 1100 E. 58th Street, Chicago IL 60637. Supported in part by DOE grant DE-FG02-92ER25119.

† Department of Computational and Applied Mathematics, Rice University, P.O. Box 1892, Houston, TX 77251-1892. Supported in part by a National Science Foundation Postdoctoral Fellowship.

in [11] recovers second order accurate velocities at special points. In this paper, a class of local postprocessing techniques generalizing the one in [11] are presented and analyzed. These postprocessors are shown to recover second order accurate velocity fields on three lines meshes. Numerical experiments illustrate these results and investigate the performance of the postprocessors in more general situations, including on meshes of tetrahedral elements.

The rest of this paper is outlined as follows. In the next section, we give notation used throughout the paper. In section 3 we state the mixed finite element method. In section 4 we prove that the flux  $\mathbf{U}$  computed from the mixed method is super-close to the  $\pi$ -projection[13] of the true flux, on three lines triangular meshes. In section 5 we use that result to prove second order accuracy for all postprocessing methods satisfying certain conditions. In sections 6 and 7 we give examples of postprocessing schemes which satisfy those conditions. Finally in section 8 we give experimental results illustrating these schemes. In most of the paper we assume that (1) was solved using the mixed finite element method with lowest-order Raviart-Thomas approximating spaces ( $RT_0$ ) on triangular elements in two dimensions [13]. We remark on generalizations to the other methods described in [1, 2] in the final section of the paper.

**2. Notation.** For  $R$  a domain in  $\mathbf{R}^2$  or  $\mathbf{R}^3$  let  $L^2(R)$  denote the space of real-valued square-integrable functions. On this space use  $(\cdot, \cdot)_R$  and  $\|\cdot\|_R$  as the inner product and norm, respectively. Denote by  $\langle \cdot, \cdot \rangle_{\partial R}$  the  $L^2(\partial R)$  inner product. We also use the space

$$(4) \quad H(\text{div}; R) = \{\mathbf{u} = (u^1, u^2) : \mathbf{u} \in (L^2(R))^2 \text{ and } \nabla \cdot \mathbf{u} \in L^2(R)\},$$

with norm

$$(5) \quad \|\mathbf{u}\|_{H(\text{div}; R)}^2 = \int_R [|\mathbf{u}|^2 + |\nabla \cdot \mathbf{u}|^2] d\mathbf{x}.$$

Also define  $H_0(\text{div}; R) = \{\mathbf{v} \in H(\text{div}; R) : \mathbf{v} \cdot \hat{\mathbf{n}} = 0 \text{ on } \partial\Omega\}$ . The Sobolev space

$$(6) \quad H^2(R) = \{f \in L^2(R) \text{ and } |D^2 f| \in L^2(R)\},$$

with norm

$$(7) \quad \|f\|_{H^2(R)}^2 = \|f\|_R^2 + \|D^2 f\|_R^2.$$

is also used. When  $R = \Omega$ , we may omit it in the definitions above.

Let  $\mathbf{T}_h$  denote a triangulation of  $\Omega$  into triangles or tetrahedra with maximum diameter  $h$ . Associated with  $\mathbf{T}_h$ , the  $RT_0$  spaces  $V_h \subset H(\text{div}; \Omega)$  and  $W_h \subset L^2(\Omega)$  are characterized as follows. Let  $N_T$  denote the number of triangles (or tetrahedra, in  $\mathbf{R}^3$ ) in  $\mathbf{T}_h$  and  $N_e$  the number of edges (or faces, in  $\mathbf{R}^3$ ). Then

$$W_h = \text{span}\{w_i, i = 1, \dots, N_T : (w_i)|_{T_j} = \delta_{ij}, j = 1, \dots, N_T\}.$$

Thus  $W_h$  is a space of piecewise constant scalar functions. Letting  $\hat{\mathbf{n}}_\ell$  denote one of the unit vectors normal to edge  $\ell$ , denoted by  $e_\ell$ ,

$$\begin{aligned} V_h = \text{span}\{ & \mathbf{v}_k \in H(\text{div}; \Omega), \quad k = 1, \dots, N_e : \\ & \mathbf{v}_k|_T \in (\mathcal{P}^0(T))^2 \oplus \mathbf{x}\mathcal{P}^0(T) \quad \text{for all } T \in \mathbf{T}_h, \text{ and} \\ & \mathbf{v}_k \cdot \hat{\mathbf{n}}_\ell|_{e_\ell} = \delta_{k\ell}, \quad \ell = 1, \dots, N_e\}, \end{aligned}$$

where  $\mathcal{P}^0(T)$  denotes the set of constant functions defined on  $T \in \mathbf{T}_h$ . Specifically, in  $\mathbf{R}^2$ , the function  $\mathbf{v}_k = (v_k^1, v_k^2) \in V_h$  is given on  $T$  by

$$\begin{aligned} (v_k^1)|_T &= \alpha_T^1 + \beta x, \\ (v_k^2)|_T &= \alpha_T^2 + \beta y, \end{aligned}$$

with the three coefficients determined by the requirements

$$(8) \quad \mathbf{v}_k \cdot \hat{\mathbf{n}}_\ell|_{e_\ell} = \delta_{k\ell}.$$

Thus  $\mathbf{v}_k$  is nonzero only on the two elements which share edge  $k$ .  $V_h$  is thus a certain subset of the space of piecewise linear vector fields, which does not contain certain linear vector fields, such as  $(y, x)^T$ .

**3. A Mixed Method.** A mixed method is defined [13] by rewriting (1) as the pair

$$(9) \quad \mathbf{u}(\mathbf{x}) = -K(\mathbf{x})\nabla p(\mathbf{x}),$$

$$(10) \quad \nabla \cdot \mathbf{u} = f.$$

Multiplying (9)–(10) by appropriate test functions and integrating we obtain

$$(11) \quad (K^{-1}\mathbf{u}, \mathbf{v}) = (p, \nabla \cdot \mathbf{v}), \quad \mathbf{v} \in H_0(\text{div}; \Omega),$$

$$(12) \quad (\nabla \cdot \mathbf{u}, w) = (f, w), \quad w \in L^2(\Omega).$$

We define

$$\begin{aligned} V_h^g &= \{\mathbf{v} \in V_h : \mathbf{v} \cdot \hat{\mathbf{n}} = g \text{ on } \partial\Omega\}, \\ V_h^0 &= V_h \cap H_0(\text{div}; \Omega). \end{aligned}$$

In the lowest order case we then seek  $\mathbf{U} \in V_h^g$  and  $P \in W_h$  that satisfy

$$(13) \quad (K^{-1}\mathbf{U}, \mathbf{v}) = (P, \nabla \cdot \mathbf{v}), \quad \mathbf{v} \in V_h^0,$$

$$(14) \quad (\nabla \cdot \mathbf{U}, w) = (f, w), \quad w \in W_h.$$

Because we used Neumann boundary conditions the solution  $P$  is unique only up to an additive constant, but adding an additional constraint, such as  $\int_\Omega p(\mathbf{x})d\mathbf{x} = 0$ , makes it unique; the presence of a lower order term in (1) would have the same effect.

For rectangular meshes and  $K$  a scalar or diagonal matrix, superconvergence of order  $h^2$  of the velocity  $\mathbf{U}$  and potential  $P$  at Gauss points was proven in [12]. These results were extended in [8] to demonstrate superconvergence of velocities along lines connecting Gauss points. In [15], these results were shown to hold even for finite difference methods, which may be viewed as lowest order mixed methods where special numerical quadrature is used to approximate the integral  $(K^{-1}\mathbf{v}_j, \mathbf{v}_i)$ . On triangular meshes, superconvergence of the potential  $P$  of order  $h^2$  at the centroid of each triangle was proven in [9, 7]. No superconvergence results for velocity are known to us in this case. Our numerical results indicate that the normal fluxes may be pointwise superconvergent in certain situations, but that in others they are not. In the next

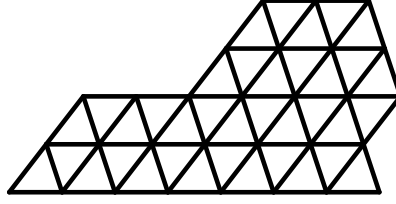


FIG. 1. A three lines mesh

section we prove an  $l^2$  super-closeness result for the normal fluxes on three-lines meshes of triangles.

Elements of  $V_h$  are defined by their normal components across edges. However, the space  $V_h$  is not rich enough to approximate all linear vector fields. The analysis of mixed methods depends on a projection  $\pi : H(\text{div}) \rightarrow V_h$  defined by letting  $\pi \mathbf{v}$  be the element in  $V_h$  satisfying

$$(15) \quad \langle \pi \mathbf{v} \cdot \hat{\mathbf{n}}_l, 1 \rangle_{e_l} = \langle \mathbf{v} \cdot \hat{\mathbf{n}}_l, 1 \rangle_{e_l},$$

on each edge  $e_l$  of the triangulation. It follows that

$$(16) \quad (\nabla \cdot (\mathbf{u} - \pi \mathbf{u}), w) = 0, \quad w \in W_h.$$

One can easily show by direct computation that in general, on triangular meshes,

$$\|\mathbf{u} - \pi \mathbf{u}\| \approx O(h),$$

even if  $\mathbf{u}$  is linear, and even in discrete norms. For this reason, even if the computed normal fluxes were exact (so that  $\mathbf{U} = \pi \mathbf{u}$ ), the error in the velocity *vector* field would still be limited to first order.

**4. A Super-closeness result.** In this section we restrict  $K$  to be the identity tensor and  $\mathbf{T}_h$  to be a three lines mesh. A three lines mesh is a mesh of similar triangles for which every edge is parallel to one of three designated lines. Figure 1 is an example of a three lines mesh.

The Theorem in the next section applies whenever a lemma such as this can be shown. It does not require these restrictions on  $K$  and  $\mathbf{T}_h$ , and numerical evidence suggests that these constraints can be weakened.

**LEMMA 4.1.** *If  $\mathbf{T}_h$  is a three-lines mesh and  $K$  is the identity tensor, then the velocity field  $\mathbf{U}$  computed by the mixed method is super-close to the  $\pi$ -projection of the true solution  $\mathbf{u}$ , in the sense that*

$$\|\mathbf{U} - \pi \mathbf{u}\| \leq Ch^2 \|\mathbf{u}\|_{H^2},$$

for  $\mathbf{u} \in (H^2(\Omega))^2$ .

**Proof:** Subtracting (12) and (14) and using (16) gives

$$(17) \quad (\nabla \cdot (\mathbf{U} - \pi \mathbf{u}), w) = 0, \quad w \in W_h.$$

Since  $\nabla \cdot \mathbf{v} \in W_h$  for  $\mathbf{v} \in V_h$ ,

$$(18) \quad (P_h p - p, \nabla \cdot \mathbf{v}) = 0, \quad \mathbf{v} \in V_h,$$

where  $P_h$  is the  $L^2$  projection into  $W_h$ . Subtracting (11) and (13) and using (18) implies

$$(19) \quad (K^{-1}(\mathbf{u} - \mathbf{U}), \mathbf{v}) = (P_h p - P, \nabla \cdot \mathbf{v}), \quad \mathbf{v} \in V_h^0.$$

When  $K$  is the identity tensor, (17) and (19) imply

$$(20) \quad (\mathbf{U} - \mathbf{u}, \mathbf{U} - \pi \mathbf{u}) = 0.$$

Hence,

$$(21) \quad \|\mathbf{U} - \pi \mathbf{u}\|^2 = (\mathbf{u} - \pi \mathbf{u}, \mathbf{U} - \pi \mathbf{u}).$$

If  $\mathbf{v}_i$  is the basis function associated with an interior edge of the triangulation, then direct calculation shows that

$$(\mathbf{w} - \pi \mathbf{w}, \mathbf{v}_i) = 0,$$

for any vector function  $\mathbf{w}$  linear on the two triangle support of  $\mathbf{v}_i$ . Since Neumann boundary conditions were used,

$$\mathbf{U} - \pi \mathbf{u} = \sum_i c_i \mathbf{v}_i,$$

for some constants  $c_i$ , where the sum is taken only over interior edges. Now let  $\mathbf{w}_i$  be a linear vector field that approximates  $\mathbf{u}$  well on the parallelogram  $P_i$  which is the support of  $\mathbf{v}_i$ . In particular, we require

$$(22) \quad \int_{P_i} |\mathbf{u} - \mathbf{w}_i|^2 + |\pi(\mathbf{u} - \mathbf{w}_i)|^2 dx \leq Ch^4 \int_{P_i} |D^2 \mathbf{u}|^2.$$

For instance,  $\mathbf{w}_i$  could be the Lagrange interpolant or the  $L^2$  projection. Then,

$$\begin{aligned} & \int (\mathbf{u} - \pi \mathbf{u}) \cdot \mathbf{v}_i dx \\ &= \int (\mathbf{u} - \mathbf{w}_i) \cdot \mathbf{v}_i + (\mathbf{w}_i - \pi \mathbf{w}_i) \cdot \mathbf{v}_i + \pi(\mathbf{w}_i - \mathbf{u}) \cdot \mathbf{v}_i dx, \\ &= \int (\mathbf{u} - \mathbf{w}_i) \cdot \mathbf{v}_i + \pi(\mathbf{w}_i - \mathbf{u}) \cdot \mathbf{v}_i dx. \end{aligned}$$

Also, there is a  $\beta > 0$  independent of  $h$ , such that if  $\mathbf{v} \in V_0^h$  and  $\mathbf{v} = \sum_i d_i \mathbf{v}_i$ , then

$$\beta \sum_i d_i^2 h^2 \leq \|\mathbf{v}\|^2 \leq \beta^{-1} \sum_i d_i^2 h^2.$$

Thus:

$$\begin{aligned} & \|\mathbf{U} - \pi \mathbf{u}\|^2 \\ & \leq \sum_i c_i h h^{-1} \int (\mathbf{u} - \mathbf{w}_i) \cdot \mathbf{v}_i + \pi(\mathbf{w}_i - \mathbf{u}) \cdot \mathbf{v}_i dx, \\ & \leq \left( \sum_i c_i^2 h^2 \right)^{1/2} \left( \sum_i h^{-2} \left[ \int (\mathbf{u} - \mathbf{w}_i) \cdot \mathbf{v}_i + \pi(\mathbf{w}_i - \mathbf{u}) \cdot \mathbf{v}_i dx \right]^2 \right)^{1/2}, \\ & \leq \beta^{-1/2} \|\mathbf{U} - \pi \mathbf{u}\| \left( \sum_i h^{-2} \left[ \int (\mathbf{u} - \mathbf{w}_i) \cdot \mathbf{v}_i + \pi(\mathbf{w}_i - \mathbf{u}) \cdot \mathbf{v}_i dx \right]^2 \right)^{1/2}. \end{aligned}$$

Thus,

$$\begin{aligned} & ||\mathbf{U} - \pi \mathbf{u}||^2 \\ & \leq Ch^{-2} \sum_i \left( \int_{P_i} |\mathbf{v}_i|^2 dx \right) \left( \int_{P_i} |\mathbf{u} - \mathbf{w}_i|^2 + |\pi(\mathbf{u} - \mathbf{w}_i)|^2 dx \right). \end{aligned}$$

But

$$||\mathbf{v}_i||_{L^\infty} \leq C,$$

so

$$||\mathbf{U} - \pi \mathbf{u}||^2 \leq C \sum_i \left( \int_{P_i} |\mathbf{u} - \mathbf{w}_i|^2 + |\pi(\mathbf{u} - \mathbf{w}_i)|^2 dx \right).$$

But by the approximation theory assumption (22) on  $\mathbf{w}_i$ , we have

$$\begin{aligned} ||\mathbf{U} - \pi \mathbf{u}||^2 & \leq Ch^4 \sum_i ||D^2 \mathbf{u}||_{L^2(P_i)}^2, \\ & \leq Ch^4 ||D^2 \mathbf{u}||_{L^2}^2. \end{aligned}$$

**5. On the Accuracy of Postprocessing Schemes.** We now give conditions which ensure that a given postprocessing method  $\mathbf{R}$  is extra order accurate.

**THEOREM 5.1.** *Let  $\mathbf{R}$  be a linear functional satisfying*

$$||\mathbf{u} - \mathbf{R}(\mathbf{u})|| \leq Ch^2 ||D^2 \mathbf{u}||,$$

*for all  $\mathbf{u} \in H^2(\Omega)$ , and*

$$||\mathbf{R}(\mathbf{v})|| \leq C ||\mathbf{v}||,$$

*for all  $\mathbf{v} \in V_h$ . Suppose moreover that  $\mathbf{R}(\mathbf{u})$  depends only on the normal components  $\mathbf{u} \cdot \hat{\mathbf{n}}$  at the midpoints of edges (or centers of faces, in  $\mathbf{R}^3$ ) of  $\mathbf{T}_h$ . Then in any context for which the velocity field  $\mathbf{U}$  computed by the mixed method is super-close to the  $\pi$ -projection of the true solution  $\mathbf{u}$ , in the sense that*

$$||\mathbf{U} - \pi \mathbf{u}|| \leq Ch^2 ||\mathbf{u}||_{H^2},$$

*for  $\mathbf{u} \in (H^2(\Omega))^2$ , we have that*

$$||\mathbf{R}(\mathbf{U}) - \mathbf{u}|| \leq Ch^2 ||\mathbf{u}||_2.$$

**Proof:** Directly from the linearity of  $\mathbf{R}$  and the assumptions, using the triangle inequality and (15):

$$||\mathbf{R}(\mathbf{U}) - \mathbf{u}|| \leq ||\mathbf{R}(\mathbf{U} - \pi \mathbf{u})|| + ||\mathbf{R}(\mathbf{u}) - \mathbf{u}||.$$

**COROLLARY 5.2.** *The same conditions on  $\mathbf{R}$  suffice for post-processors on quasi-uniform rectangular meshes.* This follows from known superconvergence results on rectangles. In section 7 a least square fitting method is used to construct a post-processed velocity field on triangles which is fully linear in each component. On rectangles or quadrilaterals one could extend this to make the resultant vector field  $\mathbf{R}(U)$  a full tensor linear on each element.

**6. Keenan's Postprocessing Scheme.** In this section we analyze the postprocessing scheme defined in [11].

**COROLLARY 6.1.** *The postprocessing scheme defined in [11] is second order accurate on interior triangles of a three lines mesh (when  $K$  is the identity), and is  $O(h^{1.5})$  accurate overall.*

**Proof:** For triangles in the interior of a three lines mesh, this scheme constructs  $\mathbf{R}(U)$  at the center of the triangle. Each component of  $\mathbf{R}(U)$  is a weighted average of normal fluxes  $U_i$  on edges of nearby triangles, with weights that depend only on the geometry. Moreover, the reconstruction process is exact on linear velocity fields given exact values for the normal fluxes. Thus,  $\mathbf{R}$  satisfies the hypotheses of Theorem 1 and hence is second order accurate on interior triangles. On boundary triangles the usual Raviart-Thomas interpolant is used, which is first order. Since there are only  $O(n)$  boundary triangles in a mesh of  $n^2$  triangular elements, the error in the postprocessed velocity field will go to zero like  $O(h^{1.5})$ , asymptotically.

**7. Least Squares Postprocessing Schemes.** In this section we construct a family of postprocessing schemes which satisfy the hypotheses of Theorem 1 and which are defined on the entire mesh, including on boundary triangles.

For triangles in the interior of the mesh, this class of schemes constructs  $\mathbf{R}$  as a vector field over the triangle. We select a functional form for  $\mathbf{R}(x, y)$  and determine the parameters by a least squares fit. For a triangle  $T$  with an edge on the boundary of  $\Omega$ , we retain symmetry by shifting attention to an adjacent triangle  $T'$  in the interior. The vector field  $\mathbf{R}$  computed for  $T'$  can then be evaluated at points in  $T$  as needed, without changing the order of the approximation.

For example, we can specify that on an interior triangle  $T$ ,

$$\mathbf{R}(x, y) = \begin{pmatrix} ax + by + c \\ dx + ey + f \end{pmatrix}.$$

We then compute the six coefficients  $a, \dots, f$  by minimizing the quadratic cost function

$$C(a, \dots, f) = \sum_i (\mathbf{R}(m_i) \cdot \hat{\mathbf{n}}_i - U_i)^2,$$

where the sum is taken over six or more edges  $i$  with midpoints  $m_i$  and preferred normals  $\hat{\mathbf{n}}_i$ . To preserve the symmetry of the situation we in fact use nine edges, namely the three edges of  $T$  and the two other edges of each of the three triangles bordering  $T$ .

Clearly if the computed normal fluxes were exact and the true velocity field was linear, the least squares fit would recover it. Moreover the components of  $\mathbf{R}(U)$  are weighted averages of the  $U_i$  with weights that depend only on the geometry, so  $\mathbf{R}$  is a bounded operator. Thus  $\mathbf{R}$  satisfies the hypotheses of Theorem 1.

**COROLLARY 7.1.** *Postprocessing schemes based on least squares fits to full linears (or richer spaces) are second order accurate on three lines meshes (when  $K$  is the identity).*

We note that other functional forms can be used. On rectangular or quadrilateral elements, for instance, one might naturally let  $\mathbf{R}$  be tensor linear in each component,



with the eight coefficients fit using the twelve symmetrically placed edges nearest the element.

In addition constraints can be incorporated. For instance, on triangles we might add the constraint that  $\mathbf{R}(U)$  have the same average divergence over the element as  $U$  had; this reduces the number of parameters to five.

**8. Numerical Results.** In this section we summarize the results of some numerical experiments using the postprocessing methods defined in the previous two sections.

Table 1 presents results in cases to which Theorem 1 directly applies, namely ones with three lines meshes, Neumann boundary conditions and  $K = I$ . In addition, a rectangular mesh example is included for comparison.

Mesh	$\ \mathbf{U} - \mathbf{u}\ _{l^2}$	$\ \mathbf{R}_k(\mathbf{U}) - \mathbf{u}\ _{l^2}$	$\ \mathbf{R}_l(\mathbf{U}) - \mathbf{u}\ _{l^\infty}$	$\ (\mathbf{U} - \mathbf{u}) \cdot \hat{\mathbf{n}}\ _{l^\infty}$
square	$h^2$	-n.a.-	-n.a.-	$h^2$
equilateral	$h$	$h^{1.75}$	$h^2$	$h^2$
right-triangle	$h$	$h^{1.75}$	$h^2$	$h^2$

TABLE 1  
*Measured Convergence Rates*

The first column in Table 1 describes the mesh. The second column presents the measured convergence rate for the error in the vector flux,  $\|\mathbf{U} - \mathbf{u}\|_{l^2}$ . The discrete  $l^2$  norm is computed using the midpoint rule, thus evaluating the error only at the centers of the elements. Convergence is only first order, as expected, except on rectangles, where the superconvergence phenomenon increases the accuracy.

The third column presents the convergence rate for the postprocessed fluxes on the triangular meshes using the postprocessor from [11]. The error measured here is  $\|\mathbf{R}_k(\mathbf{U}) - \mathbf{u}\|_{l^2}$ , where  $\mathbf{R}_k$  is the reconstructed vector defined at the element centers. Convergence is at least  $O(h^{1.5})$ , and in some cases is close to second order. As noted above, the accuracy of this procedure is reduced near the boundary of the domain. It is also primarily suited to three lines meshes, on which the weighted averages have a simple geometric interpretation.

The fourth column presents the convergence rate for the postprocessed fluxes on the triangular meshes using the full linear least squares postprocessor from Section 7. The error measured here is  $\|\mathbf{R}_l(\mathbf{U}) - \mathbf{u}\|_{l^\infty}$ , where  $\mathbf{R}_l$  is the reconstructed vector. The discrete  $l^\infty$  norm dominates the discrete  $l^2$  norm, so the observation of second order accuracy here is stronger than what was proved in Theorem 1. The same accuracy is obtained when the least squares procedure is constrained to respect the local divergence.

The fifth and final column presents the discrete max norm of the error in the normal fluxes,  $\|(\mathbf{U} - \mathbf{u}) \cdot \hat{\mathbf{n}}\|_{l^\infty}$ , taken at the midpoints of the edges in the mesh. This is equivalent to  $\|\mathbf{U} - \pi \mathbf{u}\|_{L^\infty}$  taken over all of  $\Omega$ . The observation of second order accuracy here is stronger than what was proved in Lemma 4.1.

The square mesh was a uniform refinement of the unit square into smaller square elements. The equilateral mesh consisted of a hexagon cut into equilateral triangles. The right-triangle mesh was created from the square mesh by cutting each square into two right triangles by adding a line parallel to the line  $y = x$ .

In all the cases in Table 1,  $K = I$  and Neumann boundary conditions were used. The Rice Unstructured Flow code[10], a C++ implementation of various mixed finite element methods, was used to perform the experiments. Meshes were refined until they had between  $2 \times 10^3$  and  $10^4$  elements. To improve the reliability of the estimates of convergence order, the results were compared to a known analytic solution involving a superposition of many trigonometric waves of various frequencies, rather than to a quadratic polynomial or to a computed solution on a finer grid. Although the constant factors in the convergence rates are not shown, they are such that post-processing does indeed significantly reduce the observed error, even after only moderate mesh refinement.

Table 2 presents results in cases which violate at least one hypothesis of Lemma 4.1. In many cases, second order accuracy is retained, but in some cases it is lost. We omit the  $\mathbf{U} - \mathbf{u}$  vector error column, since it is always first order, and the postprocessor from [11], since it was designed specifically for three lines meshes. In their place, we provide both  $l^2$  and  $l^\infty$  convergence rates for the least squares postprocessor and for the normal fluxes.

Mesh	$\ \mathbf{R}_l(\mathbf{U}) - \mathbf{u}\ _{l^2}$	$\ \mathbf{R}_l(\mathbf{U}) - \mathbf{u}\ _{l^\infty}$	$\ (\mathbf{U} - \mathbf{u}) \cdot \hat{\mathbf{n}}\ _{l^2}$	$\ (\mathbf{U} - \mathbf{u}) \cdot \hat{\mathbf{n}}\ _{l^\infty}$
curvilinear	$h^{1.5}$	$h$	$h^2$	$h^{1.75}$
tetrahedra	$h^{1.5}$	$h$	$h$	$h$
Dirichlet	$h^2$	$h^2$	$h^{1.9}$	$h$
tensor	$h^2$	$h^2$	$h^2$	$h^2$
(E) equilateral	$h^2$	$h^2$	$h^2$	$h^2$
(E) right-triangle	$h^{1.9}$	$h^{1.8}$	$h^{1.5}$	$h^{0.9}$
(E) Dirichlet	$h^2$	$h^2$	$h^{1.9}$	$h$
(E) tensor	$h^{1.9}$	$h$	$h^{1.6}$	$h$

TABLE 2  
*Some More General Situations*

We first consider the effect of removing the three lines mesh hypothesis. The curvilinear mesh was created by applying a smooth transformation to the each refinement of the right-triangle mesh. The curvilinear mesh is thus not a three lines mesh, though locally it is a smooth deformation of one. The tetrahedral mesh is discussed below. Both of these cases use Neumann boundary conditions and  $K = I$ . In both cases, the second order convergence rate is somewhat reduced.

Next we examine using Dirichlet boundary conditions, or using a more general tensor. Both of these cases use the equilateral mesh. The other factors are left

unchanged:  $K = I$  in the Dirichlet case, and Neumann boundary conditions are used in the tensor case, in which  $K$  is a globally constant full tensor, that is, a non-diagonal matrix. These do not seem to affect second order accuracy, at least in  $l^2$ ; Dirichlet boundary effects may degrade the  $l^\infty$  rate for the normal fluxes. In such cases the symmetry of the averaging process in evaluating the postprocessor at the element centroid probably contributes to the observed second order accuracy in the postprocessed vector.

These last two cases, as well as the cases in Table 1, are then repeated using the enhanced mixed method, which may be viewed as a generalization of cell centered finite differences to triangles, using special (lower order) quadrature rules to simplify the solution of an extended mixed method formulation as described in [1, 2]. These cases are labeled with an (E) in Table 2. The normal fluxes are less accurate, but the averaging effect of the post-processor continues to maintain close to second order accuracy, at least in  $l^2$ .

**8.1. Tetrahedra.** It is a geometric fact of life that regular tetrahedra do not fill space. Thus there is no easy way to generalize our three-lines mesh result to three dimensions. Computationally, however, the Least-Squares family of schemes are easily extended to meshes of tetrahedra, since no geometric regularity in the mesh is required by their construction. Numerically, we observe  $O(h^{1.5})$  convergence rates in  $l^2$ , as indicated in Table 2. This indicates that while some form of superconvergence is present for tetrahedral meshes, and similarly for non-three-lines meshes of triangles, a more subtle analysis will be required to explain it.

## REFERENCES

- [1] T. Arbogast, C. N. Dawson, and P. T. Keenan, *Mixed Finite Element Methods as Finite Difference Methods for Solving Elliptic Equations on Triangular Elements*, Dept. of Computational and Applied Mathematics Tech. Report #93-53, Rice University, 1993.
- [2] T. Arbogast, C. N. Dawson, and P. T. Keenan, *Efficient Mixed Methods for Groundwater Flow on Triangular or Tetrahedral Meshes*, Computational Methods in Water Resources X, (Peters et. al., editors), Kluwer (1994), pp. 3-10.
- [3] T. Arbogast, M. F. Wheeler, and I. Yotov, *Mixed methods for elliptic problems with a tensor coefficient*. Technical Report TR94-02, Department of Computational and Applied Mathematics, Rice University, 1994.
- [4] T. Arbogast, C. N. Dawson, P. T. Keenan, M. F. Wheeler, and I. Yotov, *Mixed finite element methods on general geometry*. (in preparation).
- [5] F. Brezzi and M. Fortin, *Mixed and hybrid finite elements*, Springer Series in Computational Mathematics, Vol. 15, Springer-Verlag, Berlin, 1991.
- [6] J. Douglas, Jr., R. E. Ewing, and M. F. Wheeler, *Approximation of the pressure by a mixed method in the simulation of miscible displacement*, RAIR Model. Math. Anal., Numer., 17, pp. 17-33, 1983.
- [7] J. Douglas, Jr., and J.E. Roberts, *Global estimates for mixed methods for second order elliptic equations*, Math. Comp., 44, pp. 39-52, 1985.
- [8] R. E. Ewing, R. D. Lazarov, and J. Wang, *Superconvergence of the velocity along the Gauss lines in mixed finite element methods*, SIAM J. Numer. Anal., **28**, pp. 1015-1029, 1991.
- [9] L. Gastaldi and R. Nochetto, *Optimal  $L^\infty$ -error estimates for nonconforming and mixed finite element methods of lowest order*, Numer. Math., **50**, pp. 587-611, 1987.
- [10] P. T. Keenan, *RUF 1.0 User Manual: The Rice Unstructured Flow Code*, Dept. of Computational and Applied Mathematics Tech. Report #94-30, Rice University, 1994.

- [11] P. T. Keenan, *An Efficient Postprocessor for Velocities from Mixed Methods on Triangular Elements*, Dept. of Computational and Applied Mathematics Tech. Report #94-22, Rice University, 1994.
- [12] M. Nakata, A. Weiser, and M. F. Wheeler, *Some superconvergence results for mixed finite element methods for elliptic problems on rectangular domains*, in *The Mathematics of Finite Elements and Applications V*, J. R. Whiteman, ed., Academic Press, London, 1985.
- [13] P. A. Raviart and J. M. Thomas, *A mixed finite element method for 2nd order elliptic problems*, in *Mathematical Aspects of the Finite Element Method*, Lecture Notes in Math., Springer-Verlag, Berlin, 1977.
- [14] T. F. Russell and M. F. Wheeler, *Finite element and finite difference methods for continuous flows in porous media*, in *The Mathematics of Reservoir Simulation*, R. E. Ewing, ed., SIAM, 1983.
- [15] A. Weiser and M. F. Wheeler, *On convergence of block-centered finite differences for elliptic problems*, *SIAM J. Numer. Anal.*, **25**, pp. 351-375, 1988.