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The following two theorems provide a priori error estimates for the implicit-in-time scheme.

THEOREM 6.1. *If*

$$\mathbf{z} \in L^\infty(0,T;H^r(\Omega)), \qquad \frac{\partial^m u}{\partial t^m} \in L^1(0,T;L^2(\Omega)),\ m \le 4, \qquad \mathbf{z}_{tt}\cdot\nu \in L^1(0,T;L^2(\partial\Omega)),$$

then for $\{U^n, \mathbf{Z}^n\}$ *defined by (84)–(88), there exists a constant* C *such that*

$$\|U - P_h u\|_{L^\infty_{\Delta t}(0,T;L^2)} \le C(\Delta t^2 + h^r). \tag{89}$$

If additionally

$$\mathbf{z}_t \in L^\infty(0,T;H^r(\Omega)), \qquad \mathbf{z}_{ttt} \in L^1(0,T;L^2(\Omega)),$$

then there exists a constant C *such that*

$$\|\partial_t U - P_h(u_t)\|_{\widetilde{L}^\infty_{\Delta t}(0,T;L^2)} + \|\mathbf{Z} - \Pi_h \mathbf{z}\|_{\widetilde{L}^\infty_{\Delta t}(0,T;L^2)} \le C(\Delta t^2 + h^r). \tag{90}$$

THEOREM 6.2. *Assume that (89) holds and* $u \in L^\infty(0,T;H^r(\Omega))$*, then* $\{U^n, \mathbf{Z}^n\}$ *defined by (84)–(88), there exists a constant* C *such that*

$$\|U - u\|_{L^\infty_{\Delta t}(0,T;L^2)} \le C(\Delta t^2 + h^r). \tag{91}$$

Moreover, if (90) holds and $u_t \in L^\infty(0,T;H^r(\Omega))$, $\mathbf{z} \in L^\infty(0,T;H^r(\Omega))$*, then there exists a constant* C *such that*

$$\|\partial_t U - u_t\|_{\widetilde{L}^\infty_{\Delta t}(0,T;L^2)} + \|\mathbf{Z} - \mathbf{z}\|_{\widetilde{L}^\infty_{\Delta t}(0,T;L^2)} \le C(\Delta t^2 + h^r). \tag{92}$$

Proof. The proof is very similar to Theorem 5.1; hence, we comment only on the key differences. The error τ and σ satisfy (57),(58),(61) and

$$(\frac{2}{\Delta t}\partial_t \tau^{\frac{1}{2}}, w) + (\nabla\cdot\sigma^{\frac{1}{2}}, w) = (2r^0, w), \qquad w \in W_h, \tag{93}$$

$$(\partial_t^2 \tau^n, w) + (\nabla\cdot\sigma^{n;\frac{1}{4}}, w) = (r^n, w), \qquad n \ge 1,\ w \in W_h, \tag{94}$$

with r^0 defined as in Section 5 and

$$r^n \equiv (u_{tt})^{n;\frac{1}{4}} - \partial_t^2 u^n = \frac{1}{12}\int_{-\Delta t}^{\Delta t} (|t| - \Delta t)(3 - 2(1 - |t|/\Delta t)^2)\frac{\partial^4 u}{\partial t^4}(t^n + t)dt.$$

Define

$$\phi^0 = 0, \qquad \phi^n = \Delta t\sum_{i=0}^{n-1}\sigma^{i+\frac{1}{2}}, \quad n \ge 1.$$

Summing (94) and using (93), we find that

$$(\partial_t \tau^{n+\frac{1}{2}}, w) + (\nabla\cdot\phi^{n+\frac{1}{2}}, w) = (\Delta t\sum_{i=0}^{n} r^i, w) \equiv (R^n, w), \quad n \ge 0. \tag{95}$$

Averaging (70) at time level $n+1$ and n and using (58), we have

$$\begin{aligned}(A^{-1}\partial_t\phi^{n+\frac{1}{2}}, \mathbf{v}) - (\tau^{n+\frac{1}{2}}, \nabla\cdot\mathbf{v}) + \ll \alpha\phi^{n+\frac{1}{2}}, \mathbf{v}\gg \\ = (A^{-1}\eta^{n+\frac{1}{2}}, \mathbf{v}) + \ll \alpha E^{n+\frac{1}{2}}, \mathbf{v}\gg, \quad \mathbf{v} \in \mathbf{V}_h,\ n \ge 0.\end{aligned} \tag{96}$$

The estimate on $\|U - P_h u\|$ follows by adding (95) and (96) with $w = \tau^{n+\frac{1}{2}}$ and $\mathbf{v} = \phi^{n+\frac{1}{2}}$ and using the argument in the proof of Theorem 5.1. Likewise, (90) is derived similarly by using the test functions $w = \partial_t\tau^{n+\frac{1}{2}} + \partial_t\tau^{n-\frac{1}{2}}$ and $\mathbf{v} = \sigma^{n+\frac{1}{2}}$ in (94) and (61) directly. Theorem 6.2 follows exactly the same argument as Theorem 5.2. ☐

If Δt is chosen such that $\frac{C_4 \Delta t}{2h} < 1$, then we may bound the term in (77) involving the divergence by using Cauchy-Schwarz and the inverse hypothesis (19) since

$$
\begin{aligned}
(79) \qquad \Delta t |(\nabla\cdot \phi^{n+1}, \tau^{n+1})| &\leq \Delta t \|\nabla\cdot \phi^{n+1}\| \|\tau^{n+1}\| \\
&\leq \Delta t C_4 h^{-1} \|\phi^{n+1}\|_A \|\tau^{n+1}\| \\
&< \|\phi^{n+1}\|_A^2 + \|\tau^{n+1}\|^2.
\end{aligned}
$$

Taking the maximum of (77) over $0 \leq n \leq N-1$, and using $\|v\|_{\tilde{L}^\infty_{\Delta t}(0,T;L^2)} \leq \|v\|_{L^\infty_{\Delta t}(0,T;L^2)}$ with (78) and (79), we conclude that

$$
\begin{aligned}
(80) \quad \|\tau\|^2_{L^\infty_{\Delta t}(0,T;L^2)} + \|\phi\|^2_{L^\infty_{\Delta t}(0,T;L^2)} + \sum_{i=0}^{N} \Delta t \ll \alpha \phi^{i+\frac{1}{2}}, \phi^{i+\frac{1}{2}} \gg \\
\leq C \Bigg[h \|\eta\cdot\nu\|^2_{\tilde{L}^\infty_{\Delta t}(0,T;L^2(\partial\Omega))} + (\Delta t)^4 \|\mathbf{z}_{tt}\cdot\nu\|^2_{L^1(0,T;L^2(\partial\Omega))} \\
+ \left(\sum_{i=0}^{N-1} \Delta t \|R^i\| \right)^2 + \left(\sum_{i=0}^{N} \Delta t \|\eta^i\| \right)^2 \Bigg].
\end{aligned}
$$

Since

$$
(81) \qquad \|R^i\| \leq \sum_{i=0}^{N} \Delta t \|r^i\| \leq C(\Delta t)^2 \left(\|\frac{\partial^3 u}{\partial t^3}\|_{L^1(0,T;L^2(\Omega))} + \|\frac{\partial^4 u}{\partial t^4}\|_{L^1(0,T;L^2(\Omega))} \right)
$$

by (65) and (66), we have that

$$
(82) \qquad \left(\sum_{i=0}^{N-1} \Delta t \|R^i\| \right)^2 \leq C(\Delta t)^4 \left(\|\frac{\partial^3 u}{\partial t^3}\|^2_{L^1(0,T;L^2(\Omega))} + \|\frac{\partial^4 u}{\partial t^4}\|^2_{L^1(0,T;L^2(\Omega))} \right).
$$

The a priori bound (53) follows from (80) by using the approximation properties of the projections and the estimates (78),(82) and (79).

The bound (55) follows from (53) and the triangle inequality.

The estimates for $\|(\mathbf{Z}-\mathbf{z})^n\|$ and $\|\partial_t U^{n+\frac{1}{2}} - (u_t)^{n+\frac{1}{2}}\|$ are derived similarly by choosing the test functions $w = \partial_t \tau^{n+\frac{1}{2}} + \partial_t \tau^{n-\frac{1}{2}}$ and $\mathbf{v} = \sigma^{n+\frac{1}{2}}$ in (60) and (61) directly and using (78), (81) and the bound

$$
(83) \qquad \sum_{i=0}^{N-1} \Delta t \|\partial_t \eta^{i+\frac{1}{2}}\| \leq C \left(h^r \|\mathbf{z}\|_{L^\infty(0,T;H^r(\Omega))} + (\Delta t)^2 \|\mathbf{z}_{ttt}\|_{L^1(0,T;L^2(\Omega))} \right)
$$

for the resulting time truncation terms. □

6. Implicit Method. The *implicit-in-time mixed finite element method approximation* to (24)–(28) is given by a sequence of pairs $\{U^n, Z^n\} \in W_h \times \mathbf{V}_h$, $0 \leq n \leq N$ satisfying

$$
\begin{aligned}
(84) \qquad & (U^0, w) = (u_0, w), \qquad w \in W_h, \\
(85) \qquad & (A^{-1} Z^0, \mathbf{v}) - (U^0, \nabla\cdot\mathbf{v}) = - < u_0, \mathbf{v}\cdot\nu >, \qquad \mathbf{v} \in \mathbf{V}_h, \\
(86) \qquad & (\frac{2}{\Delta t} \partial_t U^{\frac{1}{2}}, w) + (\nabla\cdot \mathbf{Z}^{\frac{1}{2}}, w) = (f^{\frac{1}{2}} + \frac{2}{\Delta t} u_1, w) \qquad w \in W_h, \\
(87) \qquad & (\partial_t^2 U^n, w) + (\nabla\cdot \mathbf{Z}^{n;\frac{1}{4}}, w) = (f^{n;\frac{1}{4}}, w), \qquad w \in W_h,\ n \geq 1, \\
(88) \qquad & (A^{-1} \partial_t \mathbf{Z}^{n+\frac{1}{2}}, \mathbf{v}) - (\partial_t U^{n+\frac{1}{2}}, \nabla\cdot\mathbf{v}) + \ll \alpha \mathbf{Z}^{n+\frac{1}{2}}, \mathbf{v} \gg \\
& \qquad = - < \int_{t^n}^{t^{n+1}} g(t) dt, \mathbf{v}\cdot\nu >, \qquad \mathbf{v} \in \mathbf{V}_h,\ n \geq 0.
\end{aligned}
$$

The existence and uniqueness of the solution to the resulting linear system follows from the unisolvancy of the mixed formulation of the following elliptic problem with Robin boundary conditions:

$$
\begin{aligned}
\frac{4}{(\Delta t)^2} \phi - \nabla\cdot A \nabla \phi = 0 \qquad & \text{in } \Omega, \\
\phi + \alpha \Delta t (A \nabla \phi)\cdot\nu = 0 \qquad & \text{on } \partial\Omega.
\end{aligned}
$$

where $R^n = \Delta t \sum_{i=0}^n r^i$.

Multiplying (61) by Δt and summing over time levels, we see that

$$(70)\qquad (A^{-1}(\sigma^{n+1}-\sigma^0),\mathbf{v}) - (\tau^{n+1}-\tau^0,\nabla\cdot\mathbf{v}) + \ll\alpha\Delta t\sum_{i=0}^{n}\sigma^{i+\frac{1}{2}},\mathbf{v}\gg$$
$$= (A^{-1}(\eta^{n+1}-\eta^0),\mathbf{v}) + \ll\alpha E^{n+1},\mathbf{v}\gg,\quad \mathbf{v}\in\mathbf{V}_h,\ n\geq 0,$$

where $E^{n+1} = \Delta t\sum_{i=0}^n e^{i+\frac{1}{2}} \equiv \widehat{E}^{n+1} + \Delta t\sum_{i=0}^n \eta^{i+\frac{1}{2}}$. Noting that

$$(71)\qquad \phi^{n+\frac{1}{2}} = \Delta t\sum_{i=1}^{n}\sigma^{i+\frac{1}{2}},\quad n\geq 0,$$

$$(72)\qquad \partial_t\phi^{n+\frac{1}{2}} = \sigma^{n+1},\quad n\geq 0.$$

and using (57), (58), (71) and (72), we have

$$(73)\qquad (A^{-1}\partial_t\phi^{n+\frac{1}{2}},\mathbf{v}) - (\tau^{n+1},\nabla\cdot\mathbf{v}) + \ll\alpha\phi^{n+\frac{1}{2}},\mathbf{v}\gg$$
$$= (A^{-1}\eta^{n+1},\mathbf{v}) + \ll\alpha E^{n+1},\mathbf{v}\gg,\quad \mathbf{v}\in\mathbf{V}_h,\ n\geq 0.$$

Adding (69) and (73) with $w=\tau^{n+\frac{1}{2}}$ and $\mathbf{v}=\phi^{n+\frac{1}{2}}$, we find that

$$(74)\qquad \frac{1}{2\Delta t}\left(\|\tau^{n+1}\|^2 - \|\tau^n\|^2 + \|\phi^{n+1}\|_A^2 - \|\phi^n\|_A^2\right)$$
$$+ \ll\alpha\phi^{n+\frac{1}{2}},\phi^{n+\frac{1}{2}}\gg + (\nabla\cdot\phi^n,\tau^{n+\frac{1}{2}}) - (\nabla\cdot\phi^{n+\frac{1}{2}},\tau^{n+1})$$
$$= (R^n,\tau^{n+\frac{1}{2}}) + (A^{-1}\eta^{n+1},\phi^{n+\frac{1}{2}}) + \ll\alpha E^{n+1},\phi^{n+\frac{1}{2}}\gg,\quad n\geq 0.$$

Multiplying (74) by $2\Delta t$ and using Cauchy-Schwarz along with

$$(75)\qquad (\nabla\cdot\phi^n,\tau^{n+\frac{1}{2}}) - (\nabla\cdot\phi^{n+\frac{1}{2}},\tau^{n+1}) = \frac{1}{2}\left[(\nabla\cdot\phi^n,\tau^n) - (\nabla\cdot\phi^{n+1},\tau^{n+1})\right],$$

we have

$$(76)\qquad \left(\|\tau^{n+1}\|^2 - \|\tau^n\|^2 + \|\phi^{n+1}\|_A^2 - \|\phi^n\|_A^2\right) + 2\Delta t\ll\alpha\phi^{n+\frac{1}{2}},\phi^{n+\frac{1}{2}}\gg$$
$$+ \Delta t\left[(\nabla\cdot\phi^n,\tau^n) - (\nabla\cdot\phi^{n+1},\tau^{n+1})\right]$$
$$\leq 2\Delta t(\|R^n\|\|\tau\|_{\widetilde{L}^\infty_{\Delta t}(0,T;L^2)} + C\|\eta^{n+1}\|\|\phi\|_{\widetilde{L}^\infty_{\Delta t}(0,T;L^2)}$$
$$+\ll\alpha E^{n+1},\phi^{n+\frac{1}{2}}\gg),\quad n\geq 0.$$

Summing over time levels and using (57) and (58), we see that

$$(77)\ \|\tau^{n+1}\|^2 + \|\phi^{n+1}\|_A^2 + 2\sum_{i=0}^{n}\Delta t\ll\alpha\phi^{i+\frac{1}{2}},\phi^{i+\frac{1}{2}}\gg - \Delta t(\nabla\cdot\phi^{n+1},\tau^{n+1})$$
$$\leq \frac{(\Delta t)^2}{4}\|\eta^0\|_A^2 + \sum_{i=0}^{n}\Delta t(\|R^i\|\|\tau\|_{\widetilde{L}^\infty_{\Delta t}(0,T;L^2)} + C\|\eta^{i+1}\|\|\phi\|_{\widetilde{L}^\infty_{\Delta t}(0,T;L^2)})$$
$$+\sum_{i=0}^{n}\Delta t\ll\alpha E^{i+1},\phi^{i+\frac{1}{2}}\gg,\quad n\geq 0.$$

Since $\phi\in\mathbf{V}_h$ and using (67), we see that

$$(78)\ll\alpha E^{i+1},\phi^{i+\frac{1}{2}}\gg = \sum_{j=0}^{i}\Delta t\ll\alpha\mathbf{e}^{j+\frac{1}{2}},\phi^{i+\frac{1}{2}}\gg$$
$$\leq C\|\alpha^{\frac{1}{2}}\phi^{i+\frac{1}{2}}\cdot\nu\|_{L^2(\partial\Omega)}(\sum_{j=0}^{i}\Delta t\ h^{\frac{1}{2}}\|\eta^{j+\frac{1}{2}}\cdot\nu\|_{L^2(\partial\Omega)}$$
$$+\sum_{j=0}^{i}(\Delta t)^2\int_{t^j}^{t^{j+1}}\|\mathbf{z}_{tt}\cdot\nu(t)\|_{L^2(\partial\Omega)}dt)$$
$$\leq C\|\alpha^{\frac{1}{2}}\phi^{i+\frac{1}{2}}\cdot\nu\|_{L^2(\partial\Omega)}(h^{\frac{1}{2}}\|\eta\cdot\nu\|_{\widetilde{L}^\infty_{\Delta t}(0,T;L^2(\partial\Omega))} + (\Delta t)^2\|\mathbf{z}_{tt}\cdot\nu\|_{L^1(0,T;L^2(\partial\Omega))}).$$

Subtracting (24)–(28) from (48)–(52) and using properties (12) and (13) of the projections, we find that

$$(57)\qquad (\tau^0, w) = 0, \quad w \in W_h,$$

$$(58)\qquad (A^{-1}\sigma^0, \mathbf{v}) = (A^{-1}\eta^0, \mathbf{v}), \quad \mathbf{v} \in \mathbf{V}_h,$$

$$(59)\qquad (\frac{2}{\Delta t}\partial_t \tau^{\frac{1}{2}}, w) + (\nabla\cdot\sigma^0, w) = (2r^0, w), \quad w \in W_h,$$

$$(60)\qquad (\partial_t^2 \tau^n, w) + (\nabla\cdot\sigma^n, w) = (r^n, w), \quad w \in W_h,\ n \geq 1,$$

$$(61)\qquad (A^{-1}\partial_t\sigma^{n+\frac{1}{2}}, \mathbf{v}) - (\partial_t \tau^{n+\frac{1}{2}}, \nabla\cdot\mathbf{v}) + \ll \alpha\sigma^{n+\frac{1}{2}}, \mathbf{v} \gg$$
$$= (A^{-1}\partial_t\eta^{n+\frac{1}{2}}, \mathbf{v}) + \ll \alpha \mathbf{e}^{n+\frac{1}{2}}, \mathbf{v} \gg, \quad \mathbf{v} \in \mathbf{V}_h,\ n \geq 0,$$

where

$$(62)\qquad r^0 \equiv \frac{1}{2}(u_{tt})^0 + \frac{1}{\Delta t}\left((u_t)^0 - \partial_t u^{\frac{1}{2}}\right) = -\frac{1}{2(\Delta t)^2}\int_0^{\Delta t}(t-\Delta t)^2 \frac{\partial^3 u}{\partial t^3}(t)dt,$$

$$(63)\qquad r^n \equiv (u_{tt})^n - \partial_t^2 u^n = \frac{1}{6(\Delta t)^2}\int_{-\Delta t}^{\Delta t}(|t| - \Delta t)^3 \frac{\partial^4 u}{\partial t^4}(t^n + t)dt,$$

$$(64)\qquad \mathbf{e}^{n+\frac{1}{2}} \equiv \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}}(\mathbf{z}(t) - \mathbf{z}^{n+\frac{1}{2}})dt + \eta^{n+\frac{1}{2}}$$
$$= \frac{1}{2\Delta t}\int_{-\frac{\Delta t}{2}}^{\frac{\Delta t}{2}}(t^2 - (\Delta t)^2/4)\frac{\partial^2 \mathbf{z}}{\partial t^2}(t^{n+\frac{1}{2}} + t)dt + \eta^{n+\frac{1}{2}}.$$

Therefore, we have

$$(65)\qquad \|r^0\| \leq C\Delta t \|\frac{\partial^3 u}{\partial t^3}\|_{L^\infty(0,T;L^2(\Omega))}$$
$$\leq C\Delta t\left(\|\frac{\partial^3 u}{\partial t^3}\|_{L^1(0,T;L^2(\Omega))} + \|\frac{\partial^4 u}{\partial t^4}\|_{L^1(0,T;L^2(\Omega))}\right),$$

$$(66)\qquad \|r^n\| \leq C\Delta t\int_{t^{n-1}}^{t^{n+1}}\|\frac{\partial^4 u}{\partial t^4}\|(t)dt, \quad n \geq 1,.$$

And letting

$$\widehat{\mathbf{e}}^{n+\frac{1}{2}} \equiv \frac{1}{2\Delta t}\int_{-\frac{\Delta t}{2}}^{\frac{\Delta t}{2}}(t^2 - (\Delta t)^2/4)\frac{\partial^2 \mathbf{z}}{\partial t^2}(t^{n+\frac{1}{2}} + t)dt,$$

we have from (17) that for every $\mathbf{v} \in \mathbf{V}_h$,

$$(67)\qquad \ll \alpha\mathbf{e}^{n+\frac{1}{2}}, \mathbf{v} \gg = \ll \alpha\eta^{n+\frac{1}{2}}, \mathbf{v} \gg + \ll \alpha\widehat{\mathbf{e}}^{n+\frac{1}{2}}, \mathbf{v} \gg$$
$$= \ll (\alpha - \alpha_h^0)\eta^{n+\frac{1}{2}}, \mathbf{v} \gg + \ll \alpha\widehat{\mathbf{e}}^{n+\frac{1}{2}}, \mathbf{v} \gg$$
$$\leq C\|\alpha^{\frac{1}{2}}\mathbf{v}\cdot\nu\|_{L^2(\partial\Omega)}\left(h^{\frac{1}{2}}\|\eta^{n+\frac{1}{2}}\cdot\nu\|_{L^2(\partial\Omega)} + \Delta t\int_{t^n}^{t^{n+1}}\|\mathbf{z}_{tt}\cdot\nu\|_{L^2(\partial\Omega)}dt\right), \quad n \geq 0.$$

Analogous to the techniques used in the previous section, we consider the "discrete integral" of σ defined by

$$\phi^0 = \frac{\Delta t}{2}\sigma^0, \qquad \phi^n = \frac{\Delta t}{2}\sigma^0 + \Delta t\sum_{i=1}^n \sigma^i, \quad n \geq 1.$$

Recalling that $\partial_t^2\tau^n = (\partial_t\tau^{n+\frac{1}{2}} - \partial_t\tau^{n-\frac{1}{2}})/\Delta t$ and summing (60) over time levels, we have

$$(68)\qquad (\partial_t\tau^{n+\frac{1}{2}} - \partial_t\tau^{\frac{1}{2}}, w) + (\nabla\cdot(\phi^n - \phi^0), w) = (\Delta t\sum_{i=1}^n r^i, w), \quad n \geq 1.$$

Using (59), we see that

$$(69)\qquad (\partial_t\tau^{n+\frac{1}{2}}, w) + (\nabla\cdot\phi^n, w) = (R^n, w), \quad n \geq 0,$$

5. Explicit Method. The *explicit-in-time mixed finite element approximation* to (24)–(28) is given by a sequence of pairs $\{U^n, Z^n\} \in W_h \times \mathbf{V}_h$, $0 \le n \le N$ satisfying

$$(U^0, w) = (u_0, w), \quad w \in W_h, \tag{48}$$

$$(A^{-1}Z^0, \mathbf{v}) - (U^0, \nabla\cdot\mathbf{v}) = -< u_0, \mathbf{v}\cdot\nu>, \quad \mathbf{v} \in \mathbf{V}_h, \tag{49}$$

$$(\frac{2}{\Delta t}\partial_t U^{\frac{1}{2}}, w) + (\nabla\cdot\mathbf{Z}^0, w) = (f^0 + \frac{2}{\Delta t}u_1, w), \quad w \in W_h, \tag{50}$$

$$(\partial_t^2 U^n, w) + (\nabla\cdot\mathbf{Z}^n, w) = (f^n, w), \quad w \in W_h,\ n \ge 1, \tag{51}$$

$$(A^{-1}\partial_t\mathbf{Z}^{n+\frac{1}{2}}, \mathbf{v}) - (\partial_t U^{n+\frac{1}{2}}, \nabla\cdot\mathbf{v}) + \ll \alpha\mathbf{Z}^{n+\frac{1}{2}}, \mathbf{v}\gg = -< \int_{t^n}^{t^{n+1}} g(t)dt, \mathbf{v}\cdot\nu>, \quad \mathbf{v} \in \mathbf{V}_h,\ n \ge 0. \tag{52}$$

Equation (50) arises naturally by defining a fictitious value U^{-1} satisfying the condition

$$P_h u_1 = \frac{U^1 - U^{-1}}{2\Delta t},$$

and considering (51) with $n = 0$.

The method is explicit in time in the sense that the calculation of $\{U^n, Z^n\}$, $0 \le n \le N$ involves only the inversion of mass-type matrices associated with the spaces $\mathbf{V}_h$ and W_h. In particular, U^0, $\mathbf{Z}^0$, and U^1 are determined sequentially by solving (48), (49) and (50), respectively. The explicit calculations proceeds by alternately solving (51) for U^{n+1} having already calculated U^n, U^{n-1} and $\mathbf{Z}^n$ and solving (52) for $\mathbf{Z}^{n+1}$ having calculated $\mathbf{Z}^n$, U^{n+1} and U^n.

As expected for such an explicit scheme, this method is only conditionally stable. A sufficient condition for stability for the Dirichlet problem is derived in Theorem 5.1 of [4]. The condition, $\Delta t < 2h/C_4$, arises naturally in the proof of the following theorem, and so we do not repeat the stability argument here.

THEOREM 5.1. *If $\Delta t < 2h/C_4$ and*

$$\mathbf{z} \in L^\infty(0, T; H^r(\Omega)), \qquad \frac{\partial^m u}{\partial t^m} \in L^1(0, T; L^2(\Omega)),\ m \le 4, \qquad \mathbf{z}_{tt}\cdot\nu \in L^1(0, T; L^2(\partial\Omega)),$$

then for $\{U^n, \mathbf{Z}^n\}$ defined by (48)–(52), there exists a constant C such that

$$\|U - P_h u\|_{L^\infty_{\Delta t}(0,T;L^2)} \le C(\Delta t^2 + h^r). \tag{53}$$

If additionally

$$\mathbf{z}_t \in L^\infty(0, T; H^r(\Omega)), \quad \mathbf{z}_{ttt} \in L^1(0, T; L^2(\Omega)),$$

then there exists a constant C such that

$$\|\partial_t U - P_h(u_t)\|_{\widetilde{L}^\infty_{\Delta t}(0,T;L^2)} + \|\mathbf{Z} - \Pi_h \mathbf{z}\|_{L^\infty_{\Delta t}(0,T;L^2)} \le C(\Delta t^2 + h^r). \tag{54}$$

THEOREM 5.2. *Assume that (53) holds and $u \in L^\infty(0, T; H^r(\Omega))$, then for $\{U^n, \mathbf{Z}^n\}$ defined by (48)–(52), there exists a constant C such that*

$$\|U - u\|_{L^\infty_{\Delta t}(0,T;L^2)} \le C(\Delta t^2 + h^r). \tag{55}$$

Moreover, if (54) holds and $u_t \in L^\infty(0, T; H^r(\Omega))$, $\mathbf{z} \in L^\infty(0, T; H^r(\Omega))$, then there exists a constant C such that

$$\|\partial_t U - u_t\|_{\widetilde{L}^\infty_{\Delta t}(0,T;L^2)} + \|\mathbf{Z} - \mathbf{z}\|_{L^\infty_{\Delta t}(0,T;L^2)} \le C(\Delta t^2 + h^r). \tag{56}$$

Proof. Let

$$\tau^n = U^n - P_h u^n, \quad \sigma^n = \mathbf{Z}^n - \Pi_h \mathbf{z}^n, \quad \eta^n = (\mathbf{z} - \Pi_h \mathbf{z})^n.$$

Motivated by the work of Baker [2], we consider

$$\phi(t) = \int_0^t \sigma(s)ds, \quad \xi(t) = \int_0^t \eta(s)ds.$$

Integrating (37) and (38) in time and using (34)–(36), we have

$$(\tau_t, w) + (\nabla\cdot\phi, w) = 0, \qquad w \in W_h,\ t > 0, \tag{39}$$

$$(A^{-1}\phi_t, \mathbf{v}) - (\tau, \nabla\cdot\mathbf{v}) + \ll \alpha\phi, \mathbf{v}\gg = (A^{-1}\eta, \mathbf{v}) + \ll \alpha\xi, \mathbf{v}\gg, \qquad \mathbf{v} \in \mathbf{V}_h,\ t > 0. \tag{40}$$

Adding (39) and (40) with $w = \tau$ and $\mathbf{v} = \phi$ and using (17), we see that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}(\|\tau\|^2 + \|\phi\|_A^2) + \|\alpha^{\frac{1}{2}}\phi\cdot\nu\|^2_{L^2(\partial\Omega)} &= (A^{-1}\eta, \phi) + \ll \alpha\xi, \phi\gg \\ &= (A^{-1}\eta, \phi) + \ll (\alpha - \alpha_h^0)\xi, \phi\gg \\ &\le \frac{1}{2}\|\alpha^{\frac{1}{2}}\phi\cdot\nu\|^2_{L^2(\partial\Omega)} + (A^{-1}\eta, \phi) + Ch\|\xi\cdot\nu\|^2_{L^2(\partial\Omega)}. \end{aligned} \tag{41}$$

Integrating in time, using (34), $\phi(0) = 0$, $\tau(0) = 0$, and Cauchy-Schwarz, we have for $t > 0$ that

$$\begin{aligned} \|\tau\|^2(t) + \|\phi\|_A^2(t) + \int_0^t \|\alpha^{\frac{1}{2}}\phi(s)\cdot\nu\|^2_{L^2(\partial\Omega)}ds \\ \le 2\int_0^t (A^{-1}\eta(s), \phi(s))ds + Ch\int_0^t \|\xi(s)\cdot\nu\|^2_{L^2(\partial\Omega)}ds \\ \le C\left(\|\phi\|_{L^\infty(0,T;L^2)}\int_0^t \|\eta(s)\|ds + h\int_0^t \|\xi(s)\cdot\nu\|^2_{L^2(\partial\Omega)}ds\right). \end{aligned} \tag{42}$$

Hence,

$$\begin{aligned} \|\tau\|_{L^\infty(0,T;L^2)} + \|\phi\|_{L^\infty(0,T;L^2)} + \|\alpha^{\frac{1}{2}}\phi\cdot\nu\|_{L^2(0,T;L^2(\partial\Omega))} \\ \le C\left(\|\eta\|_{L^1(0,T;L^2)} + h^{\frac{1}{2}}\|\xi\cdot\nu\|_{L^2(0,T;L^2(\partial\Omega))}\right). \end{aligned} \tag{43}$$

Using the approximation properties of the Π_h-projection, we have proven the first part of the following theorem. The estimates for $(\mathbf{Z} - \Pi_h\mathbf{z})$ and $(U_t - P_h u_t)$ are derived via similar arguments by using the test functions $w = \tau_t$ and $\mathbf{v} = \sigma$ in (37) and (38).

THEOREM 4.1. *If $\mathbf{z} \in L^1(0,T;H^r)$ then there exists a constant C independent of h such that*

$$\|U - P_h u\|_{L^\infty(0,T;L^2)} \le Ch^r. \tag{44}$$

Moreover, if in addition $\mathbf{z}_t \in L^1(0,T;H^r)$ then

$$\|U_t - P_h u_t\|_{L^\infty(0,T;L^2)} + \|\mathbf{Z} - \Pi_h\mathbf{z}\|_{L^\infty(0,T;L^2)} \le Ch^r. \tag{45}$$

By an application of the triangle inequality and an appeal to the approximation properties (14)-(15) of the projections, the following estimates are easily deduced from (44) and (45).

THEOREM 4.2. *Assume (44) holds. If $u \in L^\infty(0,T;H^r)$, then there exists a constant C independent of h such that*

$$\|U - u\|_{L^\infty(0,T;L^2)} \le Ch^r. \tag{46}$$

Moreover, if $u_t \in L^\infty(0,T;H^r)$, $\mathbf{z} \in L^\infty(0,T;H^r)$ and (45) holds, then there exists a constant C independent of h such that

$$\|U_t - u_t\|_{L^\infty(0,T;L^2)} + \|\mathbf{Z} - \mathbf{z}\|_{L^\infty(0,T;L^2)} \le Ch^r. \tag{47}$$

4. Weak formulation and Continuous-Time Estimates. In this section we choose a weak form of (1)–(4) conducive to approximation by mixed finite elements. We then introduce the continuous-time mixed finite element approximation of this weak form and derive a priori estimates of its error. In particular, we reduce the question of the convergence of the transient problem to the approximation properties of the projections introduced in the previous section using a non-standard energy argument similar to that used by Baker [2].

We now consider a weak formulation of (1)–(4). We write the differential equation (1) as the first order system with respect to the spatial derivatives:

$$u_{tt} + \nabla\cdot\mathbf{z} = f \quad \text{in } \Omega \times (0,T), \tag{20}$$

$$A^{-1}\mathbf{z} + \nabla u = 0 \quad \text{in } \Omega \times (0,T). \tag{21}$$

For smooth test functions w and $\mathbf{v}$, we have

$$(u_{tt}, w) + (\nabla\cdot\mathbf{z}, w) = (f, w), \tag{22}$$

$$(A^{-1}\mathbf{z}, \mathbf{v}) - (u, \nabla\cdot\mathbf{v}) = - < u, \mathbf{v}\cdot\nu > . \tag{23}$$

Differentiating (23) with respect to time and using the boundary condition (2), we arrive at the weak form we shall consider:

$$u \in C^2(0,T; L^2(\Omega)), \qquad \mathbf{z} \in C^1(0,T; \widehat{\mathbf{H}}(\Omega; \mathrm{div})),$$

satisfying

$$(u(0), w) = (u_0, w), \qquad w \in L^2(\Omega), \tag{24}$$

$$(A^{-1}\mathbf{z}(0), \mathbf{v}) - (u_0, \nabla\cdot\mathbf{v}) = - < u_0, \mathbf{v}\cdot\nu >, \qquad \mathbf{v} \in \widehat{\mathbf{H}}(\Omega; \mathrm{div}), \tag{25}$$

$$(u_t(0), w) = (u_1, w), \qquad w \in L^2(\Omega), \tag{26}$$

$$(u_{tt}, w) + (\nabla\cdot\mathbf{z}, w) = (f, w), \qquad w \in L^2(\Omega),\ t > 0, \tag{27}$$

$$(A^{-1}\mathbf{z}_t, \mathbf{v}) - (u_t, \nabla\cdot\mathbf{v}) + \ll \alpha\mathbf{z}, \mathbf{v} \gg = - < g, \mathbf{v}\cdot\nu >, \qquad \mathbf{v} \in \widehat{\mathbf{H}}(\Omega; \mathrm{div}),\ t > 0. \tag{28}$$

As mentioned in the introduction, when $\alpha \equiv 0$, this is a non-standard formulation for the Dirichlet problem and $\widehat{\mathbf{H}}(\Omega; \mathrm{div}) \equiv \mathbf{H}(\Omega; \mathrm{div})$. The usual weak formulation follows by integrating (28) in time, cf. [4]. The error estimates presented below and in subsequent sections hold for this case.

By the *continuous-time mixed finite element approximation* to (24)–(28), we mean

$$U \in C^2(0,T; W_h), \qquad \mathbf{z} \in C^1(0,T; \mathbf{V}_h),$$

satisfying

$$(U(0), w) = (u_0, w), \qquad w \in W_h, \tag{29}$$

$$(A^{-1}\mathbf{Z}(0), \mathbf{v}) - (U(0), \nabla\cdot\mathbf{v}) = - < u_0, \mathbf{v}\cdot\nu >, \qquad \mathbf{v} \in \mathbf{V}_h, \tag{30}$$

$$(U_t(0), w) = (u_1, w), \qquad w \in W_h, \tag{31}$$

$$(U_{tt}, w) + (\nabla\cdot\mathbf{Z}, w) = (f, w), \qquad w \in W_h,\ t > 0, \tag{32}$$

$$(A^{-1}\mathbf{Z}_t, \mathbf{v}) - (U_t, \nabla\cdot\mathbf{v}) + \ll \alpha\mathbf{Z}, \mathbf{v} \gg = - < g, \mathbf{v}\cdot\nu >, \qquad \mathbf{v} \in \mathbf{V}_h,\ t > 0. \tag{33}$$

We now show that the mixed finite element approximation $\{U(t), \mathbf{Z}(t)\}$ is close to the projection of $\{u(t), \mathbf{z}(t)\}$ introduced in Section 3. Let

$$\tau = U - P_h u, \quad \sigma = \mathbf{Z} - \Pi_h\mathbf{z}, \quad \eta = \mathbf{z} - \Pi_h\mathbf{z}.$$

Subtracting (24)–(28) from (29)–(33) and using properties (12) and (13) of the projections, we find that

$$(\tau(0), w) = 0, \qquad w \in W_h, \tag{34}$$

$$(A^{-1}\sigma(0), \mathbf{v}) = (A^{-1}\eta(0), \mathbf{v}), \qquad \mathbf{v} \in \mathbf{V}_h, \tag{35}$$

$$(\tau_t(0), w) = 0, \qquad w \in W_h, \tag{36}$$

$$(\tau_{tt}, w) + (\nabla\cdot\sigma, w) = 0, \qquad w \in W_h,\ t > 0, \tag{37}$$

$$(A^{-1}\sigma_t, \mathbf{v}) - (\tau_t, \nabla\cdot\mathbf{v}) + \ll \alpha\sigma, \mathbf{v} \gg = (A^{-1}\eta_t, \mathbf{v}) + \ll \alpha\eta, \mathbf{v} \gg, \qquad \mathbf{v} \in \mathbf{V}_h,\ t > 0. \tag{38}$$

3. "Raviart-Thomas" Mixed Finite Element Space. In this section, we specify the assumptions we use on the finite element spaces upon which we base our methods. The assumptions are directly motivated by the properties of the Raviart-Thomas-Nedelec mixed finite element spaces [12, 11, 10]. We consider a collection of finite dimensional subspaces $\mathbf{V}_h \times W_h$ of $\widehat{\mathbf{H}}(\Omega;\mathrm{div}) \times L^2(\Omega)$ which satisfy conditions that make them look like Raviart-Thomas-Nedelec spaces of order k+1, where k is a fixed non-negative integer (RT_k and $RT_{[k]}$ in the notation of [3]). For each $\mathbf{V}_h \times W_h$, we associate a positive parameter h which can be thought of as the element size.

For each of our finite element spaces, we require that the divergence operator is a map of $\mathbf{V}_h$ onto W_h and that there exists projections

$$\Pi_h \times P_h : \widetilde{\mathbf{V}} \times L^2(\Omega) \to \mathbf{V}_h \times W_h,$$

where $\widetilde{\mathbf{V}}$ is a subspace of $\mathbf{H}(\Omega;\mathrm{div})$ whose members are slightly more regular; for instance, if $q > 2$, then $\widetilde{\mathbf{V}}$ could be the space $\{\mathbf{v} \in \mathbf{H}(\Omega;\mathrm{div}) | \mathbf{v} \in (L^q(\Omega))^n\}$. These projections have the following properties:

(i) P_h is L^2 projection onto W_h, hence

$$(\nabla\cdot\mathbf{v}, u - P_h u) = 0, \quad \mathbf{v} \in \mathbf{V}_h; \tag{12}$$

(ii) div $\Pi_h = P_h$ div : $\widetilde{\mathbf{V}}@ >$ onto $>> W_h$, thus

$$(\nabla\cdot(\mathbf{z} - \Pi_h \mathbf{z}), w) = 0, \quad w \in W_h,\ \mathbf{z} \in \widetilde{\mathbf{V}}; \tag{13}$$

(iii) the following approximation properties hold:

$$\|\mathbf{z} - \Pi_h \mathbf{z}\| \le C_3 h^r \|\mathbf{z}\|_r, \quad 1 \le r \le k+1, \tag{14}$$

$$\|u - P_h u\| \le C_3 h^r \|u\|_r, \quad 0 \le r \le k+1; \tag{15}$$

(iv) there exists $\alpha_h^0 \in L^\infty(\partial\Omega)$ such that

$$\ll \alpha_h^0(\mathbf{z} - \Pi_h \mathbf{z}), \mathbf{v} \gg = 0, \quad \mathbf{v} \in \mathbf{V}_h, \tag{16}$$

and α_h^0 approximates α in the sense that

$$| < (\alpha - \alpha_h^0)\psi, \mathbf{v}\cdot\nu > | \le C_3 h^{\frac{1}{2}} \|\alpha^{\frac{1}{2}} \mathbf{v}\cdot\nu\|_{L^2(\partial\Omega)} \|\psi\|_{L^2(\partial\Omega)}, \quad \mathbf{v} \in \mathbf{V}_h, \psi \in L^2(\partial\Omega); \tag{17}$$

(v) if $\alpha - \alpha_h^0$ is not identically zero, we assume that

$$\|(\mathbf{z} - \Pi_h \mathbf{z})\cdot\nu\|_{L^2(\partial\Omega)} \le C_3 h^{r-\frac{1}{2}} \|\mathbf{z}\|_r, \quad 1 \le r \le k+1. \tag{18}$$

In the case of the explicit-in-time method introduced in Section 5, we use the following *inverse assumption*:

$$\|\nabla\cdot\mathbf{z}\| \le C_4 h^{-1} \|\mathbf{z}\|_A, \quad \mathbf{z} \in \mathbf{V}_h. \tag{19}$$

Except where explicitly stated otherwise, these assumptions are precisely the conditions we use to derive our results. We note, however, that the $h^{1/2}$ factor in (16) may be replaced by a full power of h at the expense of a possibly larger C_3 for a more regular α and that (15) is not always needed.

In the subsequent sections, we refer to spaces that satisfy the above assumptions as "R-T-N spaces". The description of the properties of the function spaces follows the exposition given by Arnold, Douglas and Roberts [5]; in their work, they extend (12)–(15) to hold for Raviart-Thomas elements with one curved edge in $\mathbb{R}^2$ and remarked on the extension to $\mathbb{R}^3$. The conditions (16)–(18) are not standard, but are easily checked in the case of the Raviart-Thomas-Nedelec elements as extended. In this case, α_h^0 may be taken to be a suitable projection of α into the space of piecewise constant functions. To see that the average of $(\mathbf{z} - \Pi_h \mathbf{z})\cdot\nu$ is zero on the curved edge one uses (13) with $w \equiv 1$ together with the fact that the average is zero on the straight edges.

Henceforth, C will denote a generic constant which may depend on the differential problem, k, C_3, and C_4, but is not dependent on the parameter h. In the remaining sections, r will be a fixed integer satisfying $1 \le r \le k+1$.

for a family of discrete-in-time schemes was also demonstrated in [4]. For Dirichlet boundary conditions, continuous-in-time and implicit-in-time results for displacement, velocity and stress can also be deduced directly from the work of Makridakis [9] for elastodynamics. The results contained herein for $\alpha \equiv 0$ differ from those recovered from [9] in two significant ways. First, the estimates for the approximation of displacement require less regularity of the solution. Second, the initial conditions are implemented as the L^2 projections of u_0 and u_1 instead of the computationally more expensive elliptic projections.

The paper consists of six additional sections. The next two sections are devoted to preliminaries. In Section 2, the notation used throughout this paper is defined, and in Section 3 some properties of the Raviart-Thomas mixed finite element spaces are recalled. A weak form of (1)–(4) suitable for approximation by mixed finite elements is formulated in Section 4. A continuous-in-time mixed finite element approximation is also presented and a priori error estimates are derived. Explicit-in-time and implicit-in-time mixed finite element approximations to (1)–(4) are formulated and analyzed in Section 5 and Section 6, respectively. Finally, in Section 7, estimates for approximation with BDM or BDFM mixed finite element spaces instead of the Raviart-Thomas spaces are discussed.

2. Notation. In this section we define some notation used in this paper. Denote by $L^2(\Omega)$, $(L^2(\Omega))^n$, $H^s(\Omega)$, $(H^s(\Omega))^n$ the standard Sobolev spaces of real- and real-vector-valued functions defined on Ω; see, e.g., [1, 8]. When it is clear from context, we denote $L^2(\Omega)$ and $H^s(\Omega)$ by L^2 and H^s respectively. Let the inner products on $L^2(\Omega)$ and $(L^2(\Omega))^n$ be denoted by $(\cdot,\cdot)$. And let the standard norms on $H^s(\Omega)$ and $(H^s(\Omega))^n$ be denoted by $\|\cdot\|_s$, dropping the subscript in the cases of $L^2(\Omega)$ and $(L^2(\Omega))^n$. Also, define the A-weighted L^2-norm $\|\cdot\|_A$ on $(L^2(\Omega))^n$ by

$$\|\mathbf{v}\|_A = (A^{-1}\mathbf{v},\mathbf{v})^{\frac{1}{2}}, \quad \mathbf{v} \in (L^2(\Omega))^n.$$

Define $\mathbf{H}(\Omega;\mathrm{div})$, a subspace of $(L^2(\Omega))^n$, by

$$\mathbf{H}(\Omega;\mathrm{div}) = \{\mathbf{v} \in (L^2(\Omega))^n | \nabla\cdot\mathbf{v} \in L^2(\Omega)\},$$

with associated norm

$$\|\mathbf{v}\|_H^2 = \|\mathbf{v}\|^2 + \|\nabla\cdot\mathbf{v}\|^2.$$

Let $L^2(\partial\Omega)$ be the space of Lebesgue measurable functions defined on $\partial\Omega$ which are square integrable; equip this space with the inner product $<\cdot,\cdot>$. For convenience, denote by $\ll,\gg$ the bilinear form

$$\ll \mathbf{u},\mathbf{v} \gg = < \mathbf{u}\cdot\nu, \mathbf{v}\cdot\nu > .$$

Let $\widehat{\mathbf{H}}(\Omega;\mathrm{div})$ be the subspace of $\mathbf{H}(\Omega;\mathrm{div})$ such that for all $\mathbf{v} \in \widehat{\mathbf{H}}(\Omega;\mathrm{div})$, $\alpha^{1/2}\mathbf{v}\cdot\nu$ is in $L^2(\partial\Omega)$, and norm this space with

$$\|\mathbf{v}\|_{\widehat{H}}^2 = \|\mathbf{v}\|_H^2 + \|\alpha^{\frac{1}{2}}\mathbf{v}\cdot\nu\|_{L^2(\partial\Omega)}^2.$$

Let $\mathcal{X}$ be a normed space with norm $\|\cdot\|_{\mathcal{X}}$. Take $C^k(0,T;\mathcal{X})$ to be the space of k-times continuously differentiable maps of $[0,T]$ into $\mathcal{X}$, and define the following norms for $1 \le p < \infty$ and suitable functions $v : [0:T] \to \mathcal{X}$

$$\|v\|_{L^p(0,T;\mathcal{X})} = (\int_0^T \|v(t)\|_{\mathcal{X}}^p dt)^{1/p}.$$

For $p = \infty$, the usual modification is made.

Adopt the following notation related to functions defined at discrete time levels. Let N be a positive integer, $\Delta t = T/N$, and $t^n = n\Delta t$. Set

$$v^n = v(t^n), \tag{5}$$

$$v^{n+\frac{1}{2}} = \frac{1}{2}(v^{n+1} + v^n), \tag{6}$$

$$\partial_t v^{n+\frac{1}{2}} = (v^{n+1} - v^n)/\Delta t, \tag{7}$$

$$\partial_t^2 v^n = (\partial_t v^{n+\frac{1}{2}} - \partial_t v^{n-\frac{1}{2}})/\Delta t, \tag{8}$$

$$v^{n;\frac{1}{4}} = \frac{1}{4}v^{n+1} + \frac{1}{2}v^n + \frac{1}{4}v^{n-1}. \tag{9}$$

Define two discrete L^∞-norms for time-discrete functions by

$$\|v\|_{L^\infty_{\Delta t}(0,T;\mathcal{X})} = \max_{0\le n\le N} \|v^n\|_{\mathcal{X}}, \tag{10}$$

$$\|v\|_{\widetilde{L}^\infty_{\Delta t}(0,T;\mathcal{X})} = \max_{0\le n\le N-1} \|v^{n+\frac{1}{2}}\|_{\mathcal{X}}. \tag{11}$$

A PRIORI ESTIMATES FOR MIXED FINITE ELEMENT APPROXIMATIONS OF SECOND ORDER HYPERBOLIC EQUATIONS WITH ABSORBING BOUNDARY CONDITIONS *

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Abstract. Optimal order L^∞-in-time, L^2-in-space a priori error estimates are derived for mixed finite element approximations for both displacement and stress for a second order hyperbolic equation with first order absorbing boundary conditions. Continuous-in-time, explicit-in-time, and implicit-in-time procedures are formulated and analyzed.

Key Words. mixed finite element methods, second order hyperbolic equations, absorbing boundary conditions

AMS(MOS) subject classification. 65M12, 65M15

1. Introduction. Let Ω be a bounded domain in $\mathbb{R}^n$ with Lipschitz boundary, $\partial\Omega$, and unit outward normal ν. For fixed $0 < T < \infty$, we discuss mixed finite element approximations of the second order hyperbolic equation with first order absorbing boundary conditions:

$$u_{tt} - \nabla\cdot A\nabla u = f \quad \text{in } \Omega \times (0,T), \tag{1}$$

$$u_t + \alpha(A\nabla u)\cdot\nu = g \quad \text{on } \partial\Omega \times (0,T), \tag{2}$$

$$u(\cdot,0) = u_0 \quad \text{in } \Omega, \tag{3}$$

$$u_t(\cdot,0) = u_1 \quad \text{in } \Omega. \tag{4}$$

A is a symmetric matrix with elements that are uniformly bounded and measurable. Additionally, we assume that the spatial operator is uniformly elliptic, i.e. there exists a constants $C_1, C_2 > 0$ such that

$$C_1\xi^t\xi \le \xi^t A(x)\xi \le C_2\xi^t\xi, \quad x \in \overline{\Omega},\ \xi \in \mathbb{R}^n.$$

We assume that α is a bounded, non-negative function independent of time. When α is identically zero, (2) is a just a non-standard way of specifying Dirichlet boundary conditions with g being the time derivative of the standard Dirichlet data. Consequently, the case of Dirichlet boundary conditions will be a special case of the analysis presented in this paper. The functions f, g, u_0, u_1 are given data for the problem and will be assumed as regular as necessary. The solution u and its time derivate u_t will be referred to as the displacement and velocity, respectively. We refer to $A\nabla u$ as the stress.

A priori error estimates for Galerkin approximations for this problem have been previously derived by Dupont [6] using a standard energy argument. In the case of Dirichlet boundary conditions, these estimates were improved by Baker [2] using a technique that can be interpreted as a non-standard energy argument. In this paper, we formulate a mixed finite element scheme for the approximation of (1)–(4) and establish optimal order L^∞-in-time, L^2-in-space error bounds for mixed finite element approximations to displacement, velocity and stress in the cases of continuous-in-time, explicit-in-time, and implicit-in-time methods. The error estimates for displacement are analogues of the estimates of Baker, while those for velocity and stress follow from natural energy arguments. Particular attention is given to the explicit-in-time scheme since the numerical experiments of [4] seem to suggest that this may be the preferred formulation in terms of both computational effort and numerical accuracy.

Mixed finite element approximations to second order hyperbolic equations with Dirichlet boundary conditions have been previously considered by Geveci [7] and the authors [4]. In [7], Geveci derives L^∞-in-time, L^2-in-space error bounds for the continuous-in-time mixed finite element approximations of velocity and stress. The authors in [4] derive bounds for the continuous-in-time mixed finite element approximation for displacement which require less regularity than was needed in [7]. Stability

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A Priori Estimates for Mixed Finite Element Approximations of Second Order Hyperbolic Equations with Absorbing Boundary Conditions

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