A Family of Numerical Schemes for the Computation of Elastic Waves

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Abstract

A family of numerical schemes, based on finite difference operators is introduced for the computation of elastic waves. We use a displacement-stress formulation of the model. After deriving some stability results, we give an analysis of the computational cost. Imposing an accuracy criterion on the phase velocity, we derive the numerical parameters. We also show that an optimum order of approximation exists for a given precision.

Key Words. Elastic Wave Equation, Numerical Schemes, Stability, Computational Cost

AMS(MOS) subject classifications. 65M06, 65M12, 73D25

1 Introduction

High order finite-difference schemes for the acoustic wave equation, and the elastic wave equation have gained interest for some time, because of the computational cost of the simulation of such phenomena. Problems such as the inverse problem of seismology require also many of these simulations. General discussions on the application of finite-differences in seismology can be found in Alford *et al* [1], Kelly *et al* [2] and Dablain [3].
Recently different authors, Bayliss *et al* [4], Levander [5] for instance, have proposed schemes of order two in time and four in space for the elastic wave equation. Our goal in this paper is to see what could be gained, if anything, by using arbitrary accuracy in space in the finite difference scheme.
We introduce and analyse, for that purpose, a family of finite-difference schemes for the elastic wave equation. These schemes allow heterogeneous, possibly discontinuous elastic parameters.
After a presentation of the schemes in section 2, we show in section 3 that numerical stability is assured for these kinds of elastic parameters, that is to say for any Poisson ratio and Young modulus. In section 4 we turn to the plane wave analysis of the schemes. Then in section 5 we derive rules to control the cumulative error due to dispersion, by providing curves for the choice of the number of points per wavelength and the number of points per period. For a given accuracy on the phase velocity, we show that there exists an optimal order of approximation in terms of total arithmetical operations. Finally in section 6 we conclude and discuss some extension of the work presented here.
We recall in a bidimensionnal medium, the equations linking the displacement vector $\vec{U}(u, v)$ to the parameters defining the medium, the density ρ and the Lame parameters λ and μ. We have

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(cf [6]):

$$
(1.1) \qquad \begin{cases} \rho\dfrac{\partial^2 u}{\partial t^2} - \dfrac{\partial}{\partial x}\left((\lambda+2\mu)\dfrac{\partial u}{\partial x} + \lambda\dfrac{\partial v}{\partial z}\right) - \dfrac{\partial}{\partial z}\left(\mu(\dfrac{\partial u}{\partial z} + \dfrac{\partial v}{\partial x})\right) & = & 0 \\ \rho\dfrac{\partial^2 v}{\partial t^2} - \dfrac{\partial}{\partial x}\left(\mu(\dfrac{\partial u}{\partial z} + \dfrac{\partial v}{\partial x})\right) - \dfrac{\partial}{\partial z}\left(\lambda\dfrac{\partial u}{\partial x} + (\lambda+2\mu)\dfrac{\partial v}{\partial z}\right) & = & 0 \end{cases}
$$

We can write this system as a first order system in the spatial coordinates, using the stress tensor and Hooke's law. This gives :

$$
(1.2) \qquad \begin{cases} \rho\dfrac{\partial^2 u}{\partial t^2} - \left(\dfrac{\partial T^{xx}}{\partial x} + \dfrac{\partial T^{xz}}{\partial z}\right) & = & 0 \\ \rho\dfrac{\partial^2 v}{\partial t^2} - \left(\dfrac{\partial T^{xx}}{\partial x} + \dfrac{\partial T^{xz}}{\partial z}\right) & = & 0 \end{cases}
$$

with

$$
(1.3) \qquad \begin{cases} T^{xx} = (\lambda+2\mu)\dfrac{\partial u}{\partial x} + \mu\dfrac{\partial v}{\partial z} \\ T^{xz} = \mu\left(\dfrac{\partial u}{\partial z} + \dfrac{\partial v}{\partial x}\right) \\ T^{zz} = \mu\dfrac{\partial u}{\partial x} + (\lambda+2\mu)\dfrac{\partial v}{\partial z} \end{cases}
$$

A straightforward discretization of (1.2) of second order in space leads to :

$$
(1.4) \qquad \begin{cases} \rho\dfrac{\partial^2 u}{\partial t^2}(i,j) - \dfrac{T^{xx}_{i+\frac{1}{2},j} - T^{xx}_{i-\frac{1}{2},j}}{\Delta x} - \dfrac{T^{xz}_{i,j+\frac{1}{2}} - T^{xz}_{i,j-\frac{1}{2}}}{\Delta z} & = & 0 \\ \rho\dfrac{\partial^2 v}{\partial t^2}(i,j) - \dfrac{T^{xz}_{i+\frac{1}{2},j} - T^{xz}_{i-\frac{1}{2},j}}{\Delta x} - \dfrac{T^{zz}_{i,j+\frac{1}{2}} - T^{zz}_{i,j-\frac{1}{2}}}{\Delta z} & = & 0 \end{cases}
$$

We can see then that we need quantities like $T^{xz}_{i+\frac{1}{2},j}$ and $T^{xz}_{i,j+\frac{1}{2}}$. A straightforward discretization of T^{xz} gives :

$$
(1.5) \qquad \begin{cases} T^{xz}_{i+\frac{1}{2},j} & = & \mu\dfrac{u_{i+\frac{1}{2},j+\frac{1}{2}} - u_{i+\frac{1}{2},j-\frac{1}{2}}}{\Delta z} - \mu\dfrac{v_{i+1,j} - v_{i,j}}{\Delta x} \\ T^{xz}_{i,j+\frac{1}{2}} & = & \mu\dfrac{u_{i+1,j} - u_{i,j}}{\Delta z} - \mu\dfrac{v_{i+\frac{1}{2},j+\frac{1}{2}} - v_{i-\frac{1}{2},j+\frac{1}{2}}}{\Delta x} \end{cases}
$$

Therefore we need values of u and v at points $(i+\frac{1}{2}, j+\frac{1}{2})$ that we do not compute. A simple solution to this problem is to use a convex linear formula using $u_{i,j}$ and $v_{i,j}$ like :

$$
u_{i+\frac{1}{2},j+\frac{1}{2}} = \frac{1}{2}\left(\frac{u_{i+1,j}+u_{i,j}}{2} + \frac{u_{i,j+1}+u_{i,j}}{2}\right)
$$

This solution however is purely algorithmical and does not have a physical meaning. In addition, it leads to more operations per point and per time step.

2 Presentation of the Numerical Scheme

Another solution is to compute u and v on two different grids (cf [7], and more recently [8]), shifted of $\Delta x/2$ and $\Delta z/2$ from one another. Then we can replace (1.4) by

$$
\rho\frac{\partial^2 v}{\partial t^2}(i+\frac{1}{2}, j+\frac{1}{2}) - \frac{T^{xz}_{i+1,j+\frac{1}{2}} - T^{xz}_{i,j+\frac{1}{2}}}{\Delta x} - \frac{T^{zz}_{i+\frac{1}{2},j+1} - T^{zz}_{i+\frac{1}{2},j}}{\Delta z} \quad = \quad 0
$$

Now we compute only $T^{xz}_{i,j+\frac{1}{2}}$. Furthermore with (1.5) we only need u on the points (i,j) and v on the points $(i+\frac{1}{2}, j+\frac{1}{2})$. In relation to these two grids we introduce the following functional spaces

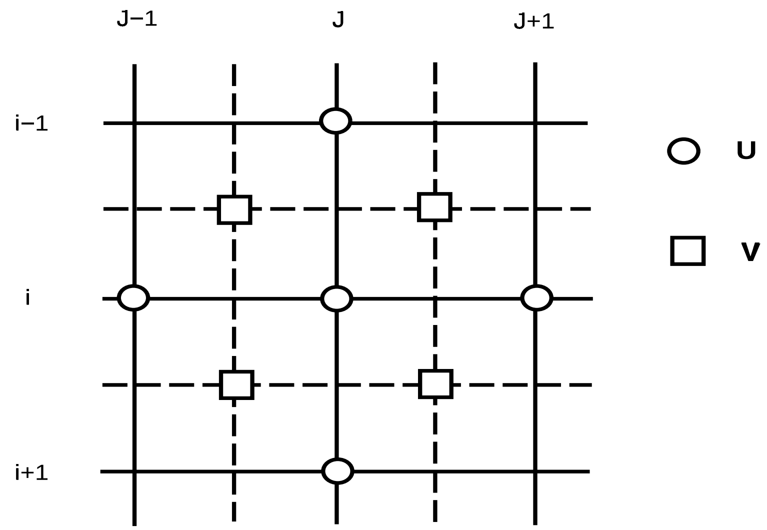

Figure 2.1: The Two Grids

$$L^2_{o,o} = \left\{ \varphi \in L^2(R^2) \ / \ \varphi = \sum_{i,j=-\infty}^{+\infty} \varphi_{i,j} \ 1_{[(i-\frac{1}{2})\Delta x,(i+\frac{1}{2})\Delta x]\times[(j-\frac{1}{2})\Delta z,(j+\frac{1}{2})\Delta z]}(x,z) \right\}$$

$$L^2_{*,*} = \left\{ \varphi \in L^2(R^2) \ / \ \varphi = \sum_{i,j=-\infty}^{+\infty} \varphi_{i+\frac{1}{2},j+\frac{1}{2}} \ 1_{[i\Delta x,(i+1)\Delta x]\times[j\Delta z,(j+1)\Delta z]}(x,z) \right\}$$

$$L^2_{o,*} = \left\{ \varphi \in L^2(R^2) \ / \ \varphi = \sum_{i,j=-\infty}^{+\infty} \varphi_{i,j+\frac{1}{2}} \ 1_{[(i-\frac{1}{2})\Delta x,(i+\frac{1}{2})\Delta x]\times[j\Delta z,(j+1)\Delta z]}(x,z) \right\}$$

$$L^2_{*,o} = \left\{ \varphi \in L^2(R^2) \ / \ \varphi = \sum_{i,j=-\infty}^{+\infty} \varphi_{i+\frac{1}{2},j} \ 1_{[i\Delta x,(i+1)\Delta x]\times[(j-\frac{1}{2})\Delta z,(j+\frac{1}{2})\Delta z]}(x,z) \right\}$$

where

$$1_{[a,b]\times[c,d]}(x,z) \begin{cases} = 1 & (x,z) \in [a,b]\times[c,d] \\ = 0 & (x,z) \notin [a,b]\times[c,d] \end{cases}$$

The symbol 'o' refers to the integer grid (i,j), the symbol '$*$' to the shifted grid $(i+\frac{1}{2}, j+\frac{1}{2})$. We set $L^2_{o,o}$ with the usual scalar product defined by :

$$(f,g)_{L^2_{o,o}} = (f,g)_{o,o} = \sum_{i,j=-\infty}^{+\infty} f_{i,j}\, g_{i,j} \ \ \Delta x \ \Delta z$$

We set analoguous scalar products on the spaces $L^2_{*,*}$, $L^2_{o,*}$ and $L^2_{*,o}$.
These spaces and notations will be useful for the stability results. They help define precisely and

concisely the different locations where we evaluate the displacement vector. Now we introduce the differentiation operator by :

$$\begin{array}{rcccl} A_x^o & : & L^2_{o,o} & \longrightarrow & L^2_{*,o} \\ & & u & \longmapsto & A_x^o u(i+\frac{1}{2},j) = \displaystyle\sum_{l=1}^{L} \frac{\beta_l}{\Delta x}[u_{i+l,j} - u_{i-l+1,j}] \end{array}$$

A_x^o is a finite difference approximation of order 2L in $((i+\frac{1}{2})\Delta x, j\Delta z)$ of the quantity $\frac{\partial u}{\partial x}$, with the coefficients $(\beta_l)_{l=1..L}$ defined in appendix 1. The exponant refers to the departure set $L^2_{o,o}$; the subscript to the direction of differentiation. Similarly we define

$$\begin{array}{rcccl} A_z^o & : & L^2_{o,o} & \longrightarrow & L^2_{o,*} \\ & & u & \longmapsto & A_z^o u(i,j+\frac{1}{2}) = \displaystyle\sum_{l=1}^{L} \frac{\beta_l}{\Delta z}[u_{i,j+l} - u_{i,j-l+1}] \\ A_x^* & : & L^2_{*,*} & \longrightarrow & L^2_{o,*} \\ & & v & \longmapsto & A_x^* v(i,j+\frac{1}{2}) = \displaystyle\sum_{l=1}^{L} \frac{\beta_l}{\Delta x}[v_{i+l+\frac{1}{2},j+\frac{1}{2}} - v_{i-l+\frac{1}{2},j+\frac{1}{2}}] \\ A_z^* & : & L^2_{*,*} & \longrightarrow & L^2_{*,o} \\ & & v & \longmapsto & A_z^* v(i+\frac{1}{2},j) = \displaystyle\sum_{l=1}^{L} \frac{\beta_l}{\Delta z}[v_{i+\frac{1}{2},j+l+\frac{1}{2}} - v_{i+\frac{1}{2},j-l+\frac{1}{2}}] \end{array}$$

We are now ready to introduce a semi-discrete scheme for the system of elasticity (1.1) as follows :

$$(2.1)\qquad \left\{ \begin{array}{l} \left(\rho\dfrac{\partial^2 u}{\partial t^2} +^t A_x^o\left((\lambda+2\mu)A_x^o u + \lambda A_z^* v\right) +^t A_z^o\left(\mu(A_z^o u + A_x^* v)\right)\right)(i,j) = 0 \\ \\ \left(\rho\dfrac{\partial^2 v}{\partial t^2} +^t A_x^*\left(\mu(A_z^o u + A_x^* v)\right) +^t A_z^*\left(\lambda A_x^o u + (\lambda+2\mu)A_z^* v\right)\right)(i+\frac{1}{2},j+\frac{1}{2}) = 0 \end{array} \right.$$

The way the numerical scheme is written, makes it easy to check consistency. Take the first equation for instance. We know that $u \in L^2_{o,o}$, therefore $A_x^o u \in L^2_{*,o}$, and $v \in L^2_{*,*}$, therefore $A_z^* v \in L^2_{*,o}$. Thus it makes sense to add $(\lambda+2\mu)A_x^o u$ and $\lambda A_z^* v$ (provided that λ and μ belong to $L^2_{*,o}$). Now we can apply ${}^tA_x^o$ to $(\lambda+2\mu)A_x^o u + \lambda A_z^* v$ and we in fact obtain an element of $L^2_{o,o}$, that is a quantity aproximated on the grid (i,j). This is consistent with the fact that $\rho\frac{\partial^2 u}{\partial t^2}$ is itself computed in (i,j).
To define the scheme we must have

$$\lambda \in L^2_{*,o} \qquad \mu \in L^2_{*,o} \bigcap L^2_{o,*} \qquad \rho \in L^2_{o,o} \bigcap L^2_{*,*}$$

that is we need the following quantities

$$\lambda_{i+1/2,j} \qquad \mu_{i+1/2,j} \quad \mu_{i,j+1/2} \qquad \rho_{i,j} \quad \rho_{i+1/2,j+1/2}$$

To define those quantities, we use an intermediate grid. On that grid we will suppose the different parameters constant. Then we will affect the different values on (i,j), $(i+1/2,j)$, $(i,j+1/2)$, $(i+1/2,j+1/2)$ according to this grid. We suppose given the space steps Δx and Δz, and the initial grid of points $(i\Delta x, j\Delta z)$. Let d be a real positive, such that $d < 1/2$. We shift the initial grid of $(-d.\Delta x)$ in the horizontal direction and $(-d.\Delta z)$ in the vertical direction.

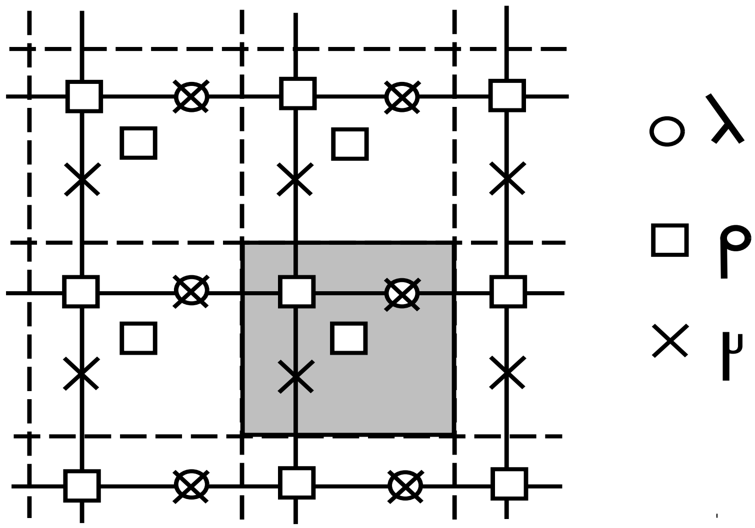

Figure 2.2: The original and shifted grids

Assuming λ, μ, ρ constant on each square of the shifted grid, we have the following simplifications :

$$\lambda_{i+1/2,j} = \lambda_{i,j} \qquad \rho_{i+1/2,j+1/2} = \rho_{i,j} \qquad \mu_{i+1/2,j} = \mu_{i,j+1/2} = \mu_{i,j}$$

This definition of the different quantities helps the implementarion of the scheme, avoids unnecessary computations and saves memory space.

3 Stability Results

We now turn to the study of the numerical stability of the scheme (2.1). We are going to proceed by the energy method (cf [10], [11]), in analogy with the continuous energy given by :

$$\left\{\begin{array}{lcl} E & = & E_c + E_p \\ E_c & = & \frac{1}{2}\int_{R^2} \rho \left((\frac{\partial u}{\partial t})^2 + (\frac{\partial v}{\partial t})^2 \right) dxdz \\ E_p & = & \frac{1}{2}\int_{R^2} \lambda \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial z}\right)^2 + \mu \left(\frac{\partial u}{\partial z} + \frac{\partial v}{\partial x}\right)^2 + \mu \left(\frac{\partial u}{\partial x}\right)^2 + \mu \left(\frac{\partial v}{\partial z}\right)^2 dxdz \end{array}\right.$$

In an infinite medium without source we have : $\frac{dE}{dt} = 0$, that is conservation of energy. The totally discretised equation is given by :

$$(3.1) \quad \left\{\begin{array}{l} \rho_{i,j} \dfrac{u_{i,j}^{n+1} - 2u_{i,j}^{n} + u_{i,j}^{n-1}}{\Delta t^2} + \left\{ {}^tA_x^o((\lambda + 2\mu)A_x^o u^n + \lambda A_z^* v^n) \right. \\ \qquad\qquad \left. + \ {}^tA_z^o(\mu(A_z^o u^n + A_x^* v^n)) \right\}(i,j) = 0 \\ \\ \rho_{i+\frac{1}{2},j+\frac{1}{2}} \dfrac{v_{i+\frac{1}{2},j+\frac{1}{2}}^{n+1} - 2v_{i+\frac{1}{2},j+\frac{1}{2}}^{n} + v_{i+\frac{1}{2},j+\frac{1}{2}}^{n-1}}{\Delta t^2} + \left\{ {}^tA_x^*(\mu(A_z^o u^n + A_x^* v^n)) \right. \\ \qquad\qquad \left. + \ {}^tA_z^*(\lambda A_x^o u^n + (\lambda + 2\mu)A_z^* v^n) \right\}(i+\frac{1}{2}, j+\frac{1}{2}) = 0 \end{array}\right.$$

By a discrete integration in time of the two equations of (3.1) with :

$$
\begin{aligned}
E^{n+1/2} &= E_c^{n+1/2} + E_p^{n+1/2} \\
E_c^{n+1/2} &= \frac{1}{2}\Bigg\{ \left(\rho \frac{u^{n+1}-u^n}{\Delta t}, \frac{u^{n+1}-u^n}{\Delta t}\right)_{oo} + \left(\rho \frac{v^{n+1}-v^n}{\Delta t}, \frac{v^{n+1}-v^n}{\Delta t}\right)_{**} \\
&- \frac{\Delta t^2}{4}\Bigg\{ \left(\lambda A_x^o \frac{u^{n+1}-u^n}{\Delta t} + \lambda A_z^* \frac{v^{n+1}-v^n}{\Delta t}, A_x^o \frac{u^{n+1}-u^n}{\Delta t} + A_z^* \frac{v^{n+1}-v^n}{\Delta t}\right)_{*o} \\
&+ \left(\mu A_z^o \frac{u^{n+1}-u^n}{\Delta t} + \mu A_x^* \frac{v^{n+1}-v^n}{\Delta t}, A_z^o \frac{u^{n+1}-u^n}{\Delta t} + A_x^* \frac{v^{n+1}-v^n}{\Delta t}\right)_{o*} \\
&+ \left(2\mu A_x^o \frac{u^{n+1}-u^n}{\Delta t}, A_x^o \frac{u^{n+1}-u^n}{\Delta t}\right)_{*o} + \left(2\mu A_z^* \frac{v^{n+1}-v^n}{\Delta t}, A_z^* \frac{v^{n+1}-v^n}{\Delta t}\right)_{*o}\Bigg\}\Bigg\} \\
E_p^{n+1/2} &= \frac{1}{2}\Bigg\{ \left(\lambda A_x^o \frac{u^{n+1}+u^n}{2} + \lambda A_z^* \frac{v^{n+1}+v^n}{2}, A_x^o \frac{u^{n+1}+u^n}{2} + A_z^* \frac{v^{n+1}+v^n}{2}\right)_{*o} \\
&+ \left(\mu A_z^o \frac{u^{n+1}+u^n}{2} + \mu A_x^* \frac{v^{n+1}+v^n}{2}, A_z^o \frac{u^{n+1}+u^n}{2} + A_x^* \frac{v^{n+1}+v^n}{2}\right)_{o*} \\
&+ \left(2\mu A_x^o \frac{u^{n+1}+u^n}{2}, A_x^o \frac{u^{n+1}+u^n}{2}\right)_{*o} + \left(2\mu A_z^* \frac{v^{n+1}+v^n}{2}, A_z^* \frac{v^{n+1}-v^n}{2}\right)_{*o}\Bigg\}
\end{aligned}
$$

we have conservation of the discrete energy, that is :

$$\frac{E^{n+1/2} - E^{n-1/2}}{\Delta t} = 0$$

The stability of the scheme will be proven if the potential energy $E_p^{n+1/2}$ and the kinetic energy $E_c^{n+1/2}$ are positive. Since $E_p^{n+1/2}$ is obviously positive, we need to find out under what conditions $E_c^{n+1/2}$ is positive. The problem can be reformaluted as : $\forall u \in L^2_{oo}, \forall v \in L^2_{**}$ with

$$
\begin{aligned}
I &= (\lambda A_x^o u + \lambda A_z^* v, A_x^o u + A_z^* v)_{*o} + (\mu A_z^o u + \mu A_x^* v, A_z^o u + A_x^* v)_{o*} \\
&\quad + (2\mu A_x^o u, A_x^o u)_{*o} + (2\mu A_z^* v, A_z^* v)_{*o}
\end{aligned}
$$

under what condition do we have :

$$\frac{\Delta t^2}{4} I \leq (\rho u, u)_{oo} + (\rho v, v)_{**}$$

We can bound I as follows :

$$
\begin{aligned}
I &\leq 2[(\lambda A_x^o u, A_x^o u)_{*o} + (\lambda A_z^* v, A_z^* v)_{*o} + (\mu A_z^o u, A_z^o u)_{o*} \\
&\quad + (\mu A_x^* v, A_x^* v)_{o*} + (\mu A_x^o u, A_x^o u)_{*o} + (2\mu A_z^* v, A_z^* v)_{*o}] \\
I &\leq 2[((\lambda+\mu) A_x^o u, A_x^o u)_{*o} + (\mu A_z^o u, A_z^o u)_{o*} \\
&\quad + ((\lambda+\mu) A_z^* v, A_z^* v)_{*o} + (\mu A_x^* v, A_x^* v)_{o*}]
\end{aligned}
$$

We set

$$
\begin{aligned}
I_1 &= ((\lambda+\mu) A_x^o u, A_x^o u)_{*o} + (\mu A_z^o u, A_z^o u)_{o*} \\
I_2 &= ((\lambda+\mu) A_z^* v, A_z^* v)_{*o} + (\mu A_x^* v, A_x^* v)_{o*}
\end{aligned}
$$

and we are now going to majorate I_1 and I_2. Lemma 1 proved in appendix 3, shows that

$$\forall u \in L^2_{oo} \qquad I_1 \leq (\frac{4}{\Delta x^2} + \frac{4}{\Delta z^2}) \left(\sum_{l=1}^{L} |\beta_l| \right)^2 .c_1^2.(\rho u, u)_{oo}$$

$$\begin{aligned} c_1^2 &= [\sum_{l=1}^{L} |\beta_l|]^{-1} [\max_{i,j} \sum_{l=1}^{L} |\beta_l| \frac{(\lambda+\mu)_{i+l-1/2,j}}{2\rho_{i,j}} + \frac{(\lambda+\mu)_{i-l+1/2,j}}{2\rho_{i,j}} \\ &\qquad + \frac{\mu_{i,j+l-1/2}}{2\rho_{i,j}} + \frac{\mu_{i,j-l+1/2}}{2\rho_{i,j}}] \end{aligned}$$

Using lemma 2, proved in appendix 3, we derive the following estimate for I_2

$$\forall v \in L^2_{**} \qquad I_2 \leq (\frac{4}{\Delta x^2} + \frac{4}{\Delta z^2}) \left(\sum_{l=1}^{L} |\beta_l| \right)^2 .c_1^2.(\rho v, v)_{**}$$

$$\begin{aligned} c_2^2 &= [\sum_{l=1}^{L} |\beta_l|]^{-1} [\max_{i,j} \sum_{l=1}^{L} |\beta_l| \frac{(\lambda+\mu)_{i+1/2,j+l}}{2\rho_{i+1/2,j+1/2}} + \frac{(\lambda+\mu)_{i+1/2,j-l}}{2\rho_{i+1/2,j+1/2}} \\ &\quad + \frac{\mu_{i+l,j+1/2}}{2\rho_{i+1/2,j+1/2}} + \frac{\mu_{i-l,j+1/2}}{2\rho_{i+1/2,j+1/2}}] \end{aligned}$$

We can now state the stability result in the following proposition.

Proposition 3.1 *A sufficient stability condition for the numerical scheme (3.1) is given by :*

(3.2)
$$\begin{aligned} C.\Delta t \sqrt{\frac{4}{\Delta x^2} + \frac{4}{\Delta z^2}} &\leq \frac{\sqrt{2}}{2} \left(\sum_{l=1}^{L} |\beta_l| \right)^{-1} \\ C &= \max(c_1, c_2) \end{aligned}$$

When $\Delta x = \Delta z = h$ we can derive a better stability condition given by :

Proposition 3.2 *A sufficient stability condition for the numerical scheme (3.1) with $\Delta x = \Delta z = h$ is given by :*

(3.3)
$$\begin{aligned} \frac{C.\Delta t}{h} &\leq \frac{\sqrt{2}}{2} \left(\sum_{l=1}^{L} |\beta_l| \right)^{-1} \\ C &= \max(c_1, c_2) \end{aligned}$$

We can write a more conventional stability condition involving the maximum P-wave velocity in the medium, by writing c_1 and c_2 differently. For instance for c_2 :

$$\begin{aligned} c_2^2 = [\sum_{l=1}^{L} |\beta_l|]^{-1} \quad . \quad \max_{i,j} \Bigg(\sum_{l=1}^{L} |\beta_l| \frac{(\lambda+\mu)_{i+1/2,j+l}}{2\rho_{i+1/2,j+1/2}} + \frac{(\lambda+\mu)_{i+1/2,j-l}}{2\rho_{i+1/2,j+1/2}} \\ + \frac{\mu_{i+l,j+1/2}}{2\rho_{i+1/2,j+1/2}} + \frac{\mu_{i-l,j+1/2}}{2\rho_{i+1/2,j+1/2}} \Bigg) \end{aligned}$$

$$c_2^2 = [\sum_{l=1}^{L} |\beta_l|]^{-1} \quad . \quad \max_{i,j} \left(\sum_{l=1}^{L} |\beta_l| \frac{(\lambda+\mu)_{i+1/2,j+l}}{\rho_{i+1/2,j+l}} \frac{\rho_{i+1/2,j+l}}{2\rho_{i+1/2,j+1/2}} + \frac{(\lambda+\mu)_{i+1/2,j-l}}{\rho_{i+1/2,j-l}} \right.$$
$$\left. . \frac{\rho_{i+1/2,j-l}}{2\rho_{i+1/2,j+1/2}} + \frac{\mu_{i+l,j+1/2}}{\rho_{i+l,j+1/2}} \frac{\rho_{i+l,j+1/2}}{2\rho_{i+1/2,j+1/2}} + \frac{\mu_{i-l,j+1/2}}{\rho_{i-l,j+1/2}} \frac{\rho_{i-l,j+1/2}}{2\rho_{i+1/2,j+1/2}} \right)$$

$$\begin{aligned} c_2^2 &\leq [\frac{(\lambda+\mu)}{\rho}]_\infty . \delta_1 + [\frac{\mu}{\rho}]_\infty . \delta_2 \\ c_2^2 &\leq [\frac{(\lambda+2\mu)}{\rho}]_\infty . \max(\delta_1, \delta_2) \end{aligned}$$

with

$$\begin{aligned} \delta_1 &= [\sum_{l=1}^{L} |\beta_l|]^{-1} \max_{i,j} \left(\sum_{l=1}^{L} |\beta_l| \frac{\rho_{i+1/2,j+l}}{2\rho_{i+1/2,j+1/2}} + \frac{\rho_{i+1/2,j-l}}{2\rho_{i+1/2,j+1/2}} \right) \\ \delta_2 &= [\sum_{l=1}^{L} |\beta_l|]^{-1} \max_{i,j} \left(\sum_{l=1}^{L} |\beta_l| \frac{\rho_{i+l,j+1/2}}{2\rho_{i+1/2,j+1/2}} + \frac{\rho_{i-l,j+1/2}}{2\rho_{i+1/2,j+1/2}} \right) \\ |f|_\infty &= \max_{i,j} f_{i,j} \end{aligned}$$

For c_1 we have

$$\begin{aligned} c_1^2 &\leq [\frac{(\lambda+\mu)}{\rho}]_\infty . \delta_3 + [\frac{\mu}{\rho}]_\infty . \delta_4 \\ c_1^2 &\leq [\frac{(\lambda+2\mu)}{\rho}]_\infty . \max(\delta_3, \delta_4) \end{aligned}$$

with

$$\begin{aligned} \delta_3 &= [\sum_{l=1}^{L} |\beta_l|]^{-1} \max_{i,j} \left(\sum_{l=1}^{L} |\beta_l| \frac{\rho_{i+l-1/2,j}}{2\rho_{i,j}} + \frac{\rho_{i-l+1/2,j}}{2\rho_{i,j}} \right) \\ \delta_4 &= [\sum_{l=1}^{L} |\beta_l|]^{-1} \max_{i,j} \left(\sum_{l=1}^{L} |\beta_l| \frac{\rho_{i,j+l-1/2}}{2\rho_{i,j}} + \frac{\rho_{i,j-l+1/2}}{2\rho_{i,j}} \right) \end{aligned}$$

whence

$$C = \max(c_1, c_2) \quad \leq \quad [\frac{(\lambda+2\mu)}{\rho}]_\infty \max(\delta_1, \delta_2, \delta_3, \delta_4)$$

With $C_{p,max} = [\frac{(\lambda+2\mu)}{\rho}]_\infty$ and $\delta = \max(\delta_1, \delta_2, \delta_3, \delta_4)$ we can then state the following result :

Corollary 1 *A sufficient stability condition for the numerical scheme (3.1) with $\Delta x = \Delta z = h$ is given by :*

$$\frac{C_{p,max}.\Delta t}{h} \quad \leq \quad \frac{\sqrt{2}}{2} \left(\delta \sum_{l=1}^{L} |\beta_l| \right)^{-1} \tag{3.4}$$

With no variation in density, $\rho = C^{te}$ we have $\delta = 1$. The parameter δ measures the "roughness" in density of the medium. The more heterogeneous the medium, the bigger δ and therefore the smaller the stability condition (CFL number).
In homogeneous medium, we have $c_{p,max} = \sqrt{\frac{\lambda+2\mu}{\rho}}$, and the stability condition is

$$\sqrt{\frac{\lambda+2\mu}{\rho}}.\frac{\Delta t}{h} \leq \frac{\sqrt{2}}{2}\left(\sum_{l=1}^{L}|\beta_l|\right)^{-1} \tag{3.5}$$

For a scheme of order 4 in space, that is with $L = 2$, we have in homogeneous media the following stability condition :

$$\sqrt{\frac{\lambda+2\mu}{\rho}}.\frac{\Delta t}{h} \leq 0.60 \tag{3.6}$$

A fourth order accurate sheme was derived differently by Bayliss *et al* (cf [4]). In homogeneous medium they obtained

$$\sqrt{\frac{\lambda+2\mu}{\rho}}.\frac{\Delta t}{h} \leq 0.67 \tag{3.7}$$

A priori the second scheme is the most appealing since it allows bigger time steps. But as we shall see the CFL number is a limit, and dispersion control implies to choose $C.\Delta t/h$ much lower than the CFL number. We are now going to study dispersion effects in the next section.

4 Plane Wave Analysis

We turn to the Fourier analysis of the scheme (cf [9]). We will derive the dispersion relation and by the Von Neumann criterion (cf [10]) we will get a necessary and sufficient stability condition. In homogeneous media the numerical scheme (3.1) can be written

$$\left\{\begin{aligned}
\rho\frac{u_{i,j}^{n+1} - 2u_{i,j}^{n} + u_{i,j}^{n-1}}{\Delta t^2} &+ (\lambda+2\mu)^t A_x^o A_x^o u^n + \lambda^t A_x^o A_z^* v^n \\
&+ \mu^t A_z^o A_z^o u^n + \mu^t A_z^o A_x^* v^n = 0 \\
\rho\frac{v_{i+\frac{1}{2},j+\frac{1}{2}}^{n+1} - 2v_{i+\frac{1}{2},j+\frac{1}{2}}^{n} + v_{i+\frac{1}{2},j+\frac{1}{2}}^{n-1}}{\Delta t^2} &+ \mu^t A_x^* A_z^o u^n + \mu^t A_x^* A_x^* v^n \\
&+ \lambda^t A_z^* A_x^o u^n + (\lambda+2\mu)^t A_z^* A_z^* v^n = 0
\end{aligned}\right. \tag{4.1}$$

We assume that $\vec{U} = \vec{d}\ \exp\ i(\omega t - \vec{k}.\vec{x})$ is a solution of (4.1), where $\vec{k} = (k_1, k_2)$ is the wave vector, ω is the pulsation, and $\vec{d} = (d_1, d_2)$ the vector giving the direction of movement (cf [12]). This gives the following relations :

$$
\begin{aligned}
d_1 \sin^2(\frac{\omega \Delta t}{2}) &= \frac{\Delta t^2}{h^2} \left\{ \frac{\lambda + 2\mu}{\rho} \left(\sum_{l=1}^{L} \beta_l \sin((2l-1)\frac{k_1 h}{2}) \right)^2 + \frac{\mu}{\rho} \left(\sum_{l=1}^{L} \beta_l \sin((2l-1)\frac{k_2 h}{2}) \right)^2 \right\} d_1 \\
&+ \frac{\Delta t^2}{h^2} \left\{ \frac{\lambda + \mu}{\rho} \left(\sum_{l=1}^{L} \beta_l \sin((2l-1)\frac{k_1 h}{2}) \right) \left(\sum_{l=1}^{L} \beta_l \sin((2l-1)\frac{k_2 h}{2}) \right) \right\} d_2
\end{aligned}
$$

$$
\begin{aligned}
d_2 \sin^2(\frac{\omega \Delta t}{2}) &= \frac{\Delta t^2}{h^2} \left\{ \frac{\lambda + 2\mu}{\rho} \left(\sum_{l=1}^{L} \beta_l \sin((2l-1)\frac{k_2 h}{2}) \right)^2 + \frac{\mu}{\rho} \left(\sum_{l=1}^{L} \beta_l \sin((2l-1)\frac{k_1 h}{2}) \right)^2 \right\} d_2 \\
&+ \frac{\Delta t^2}{h^2} \left\{ \frac{\lambda + \mu}{\rho} \left(\sum_{l=1}^{L} \beta_l \sin((2l-1)\frac{k_1 h}{2}) \right) \left(\sum_{l=1}^{L} \beta_l \sin((2l-1)\frac{k_2 h}{2}) \right) \right\} d_1
\end{aligned}
$$

By introducing the matrix $B = (b_{ij})$ defined by :

$$
\begin{aligned}
b_{11} &= \frac{\Delta t^2}{h^2} \left\{ \frac{\lambda + 2\mu}{\rho} \left(\sum_{l=1}^{L} \beta_l \sin((2l-1)\frac{k_1 h}{2}) \right)^2 + \frac{\mu}{\rho} \left(\sum_{l=1}^{L} \beta_l \sin((2l-1)\frac{k_2 h}{2}) \right)^2 \right\} \\
b_{12} &= b_{21} = \frac{\Delta t^2}{h^2} \left\{ \frac{\lambda + \mu}{\rho} \left(\sum_{l=1}^{L} \beta_l \sin((2l-1)\frac{k_1 h}{2}) \right) \left(\sum_{l=1}^{L} \beta_l \sin((2l-1)\frac{k_2 h}{2}) \right)^2 \right\} \\
b_{22} &= \frac{\Delta t^2}{h^2} \left\{ \frac{\lambda + 2\mu}{\rho} \left(\sum_{l=1}^{L} \beta_l \sin((2l-1)\frac{k_2 h}{2}) \right)^2 + \frac{\mu}{\rho} \left(\sum_{l=1}^{L} \beta_l \sin((2l-1)\frac{k_1 h}{2}) \right)^2 \right\}
\end{aligned}
$$

we can write the previous relation as :

$$
B.\vec{d} = \sin^2(\frac{\omega \Delta t}{2})\, \vec{d} \tag{4.2}
$$

The eigenvalues of B then express ω as a function of k, that is the dispersion relation. A quick calculation gives the following result

$$
\left\{
\begin{aligned}
\sin(\frac{\omega \Delta t}{2}) &= \frac{C_p.\Delta t}{h} \sqrt{A^2(k_1) + A^2(k_2)} \\
\sin(\frac{\omega \Delta t}{2}) &= \frac{C_s.\Delta t}{h} \sqrt{A^2(k_1) + A^2(k_2)}
\end{aligned}
\right. \tag{4.3}
$$

with

$$
A(k) = \sum_{l=1}^{L} \beta_l \sin((2l-1)\frac{kh}{2})
$$

$$
C_p = \frac{\lambda + 2\mu}{\rho} \qquad C_s = \frac{\mu}{\rho}
$$

We show below examples of dispersion curves for different operators. That is we plot the normalised phase E_φ error defined by

$$
E_\varphi = \frac{C_\varphi - C}{C} \qquad C_\varphi = \frac{\omega(k)}{k} \tag{4.4}
$$

as a function of $\frac{kh}{2\pi} = H$. Here C is either C_p or C_s depending on what kind of waves we are interested in.
The CFL number is baptised PMAX and is the bound of the stability interval for $p = C.\Delta t/h$. The first set of curves shows in 2D for a fixed direction of vibration θ the influence of the parameter p. The second set of curves shows for $p =$PMAX the angular dependence.
We notice on the first set of curves that when p increases the error grows due to dispersion in time. This is even more so as the operator gets more and more accurate in space. If we had used a Fourier operator to perform the space derivative the only error will be due to time dispersion. This will be a limiting factor for the choice of an optimal scheme.
The second set of curves shows the anisotropy of the numerical scheme. The worst angle of propagation is $\theta = \pi/4$ generally, except for the (2,2) scheme (L=1). In this case it is well known that for $\theta = \pi/4$, we are on the caracteristic and integrate the equation without error.

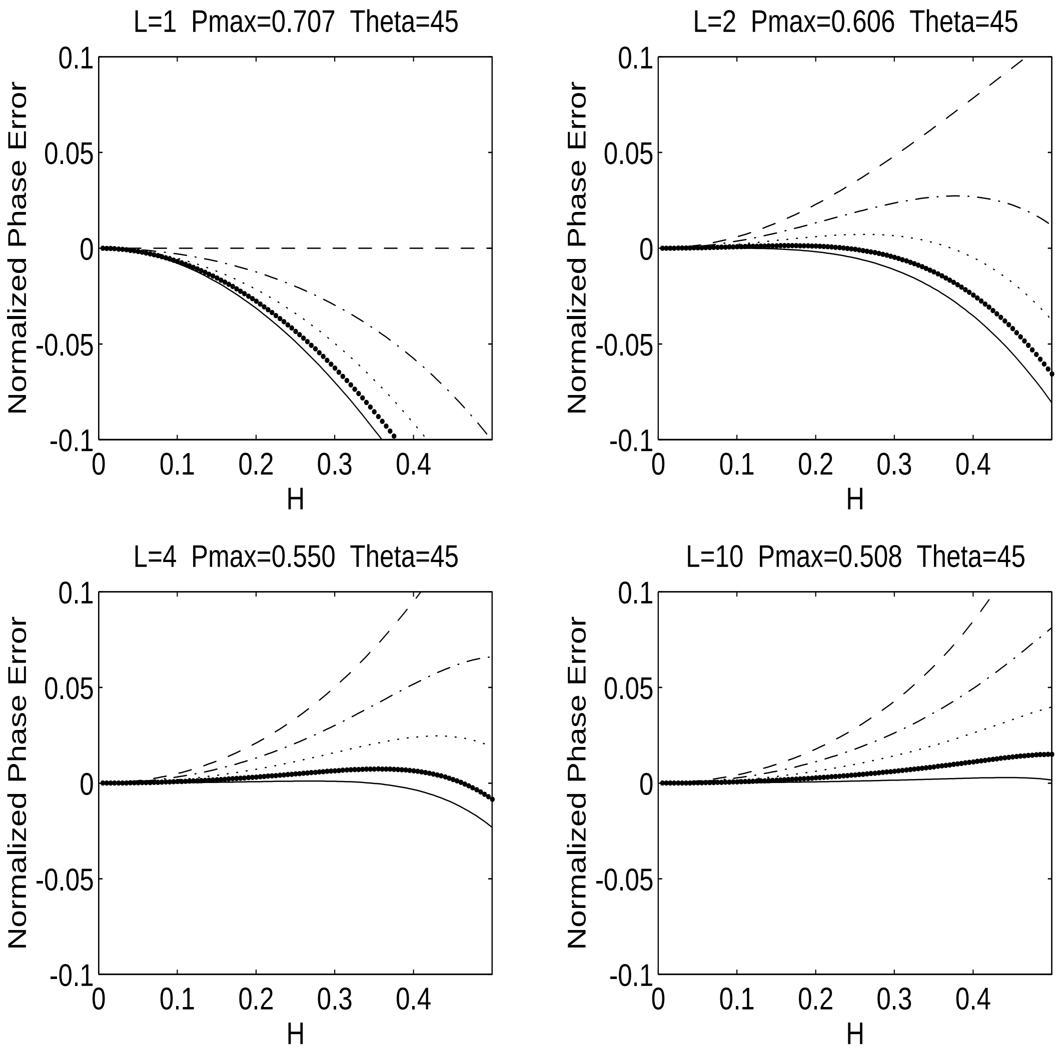

Figure 4.1: The normalised phase error for a propagation angle $\theta = 45$ as a function of the courant parameter $p = C.\Delta t/h$ for different operators. (solid) $p = 0.2Pmax$ (point) $p = 0.4Pmax$ (dotted) $p = 0.6Pmax$ (dashdot) $p = 0.8Pmax$ (dashed) $p = 1.0Pmax$

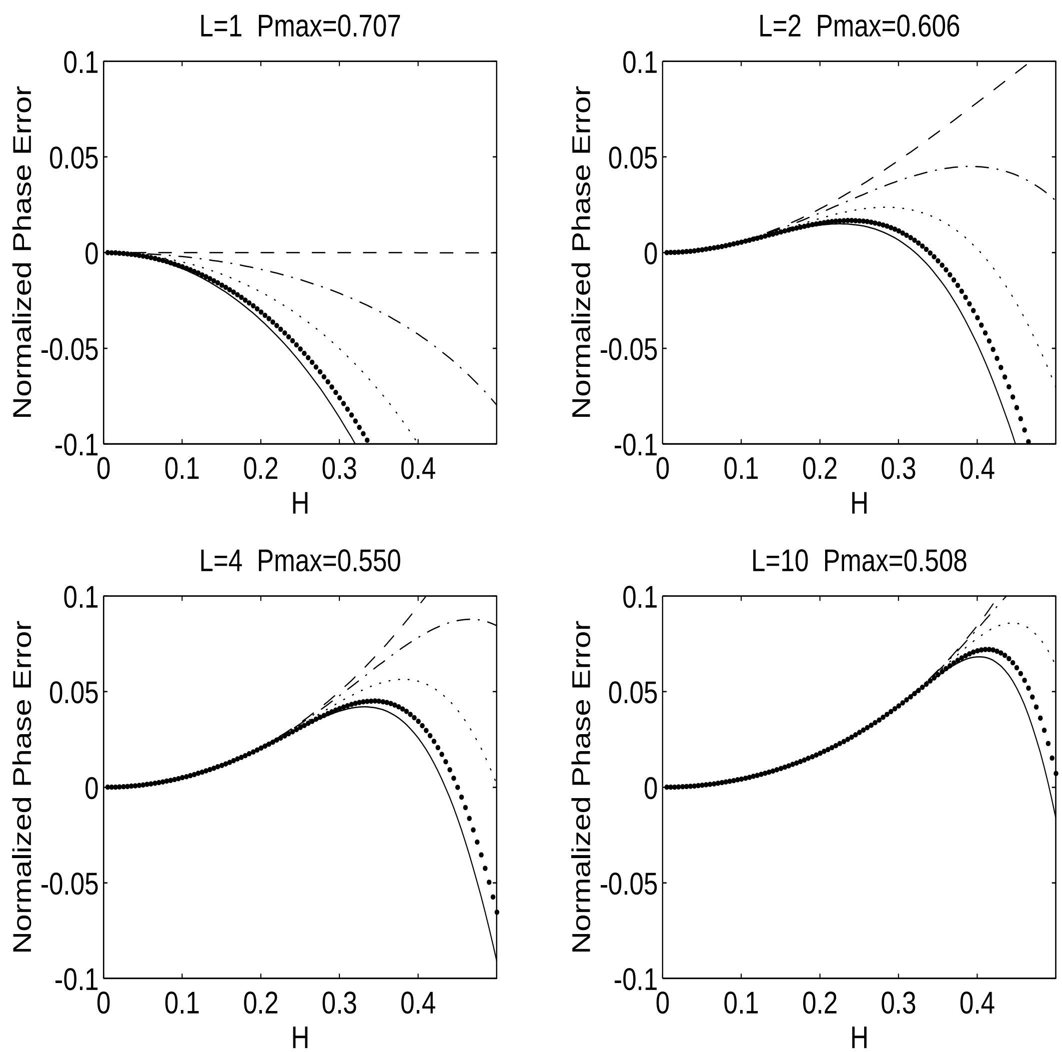

Figure 4.2: The normalised phase error for the courant parameter $p = C.\Delta t/h$ at the stability limit $Pmax$, as a function of a propagation angle θ for different operators. (solid) $\theta = 0$ (point) $\theta = 10$ (dotted) $\theta = 20$ (dashdot) $\theta = 30$ (dashed) $\theta = 45$

With the dispersion relation, we can use Von Neumann stability criterion. A necessary stability condition is that the pulsation ω is real. This implies that the eigenvalues of B must be lower than 1, therefore :

$$\forall \vec{k} \in R^2 \qquad \frac{C_p.\Delta t}{h}\sqrt{A^2(k_1)+A^2(k_2)} \leq 1 \qquad \frac{C_s.\Delta t}{h}\sqrt{A^2(k_1)+A^2(k_2)} \leq 1$$

It is easy to verify that

$$\max_{\vec{k}\in R^2} \sqrt{A^2(k_1)+A^2(k_2)} = \sqrt{A^2(\pi/h)+A^2(\pi/h)} = \sqrt{2}\left(\sum_{l=1}^{L}|\beta_l|\right)$$

therefore we can state

Proposition 4.1 *In homogeneous media, with $\Delta x = \Delta z = h$, a sufficient and necessary stability condition for the numerical scheme (3.1) is given by :*

$$\frac{C_p.\Delta t}{h} \leq \frac{\sqrt{2}}{2}\left(\sum_{l=1}^{L}|\beta_l|\right)^{-1} \tag{4.5}$$

This necessary and sufficient stability criterion is exactly the result of corollary 1 for homogeneous media. In that case the parameter δ, which measures the heterogeneity of the medium has its lowest value, that is $\delta = 1$. This lets us think that corollary 1 even though only a sufficient stablity result, must be quite close to a necessary condition as well.

5 Analysis of the Computational Cost

We impose a precision criterion on the phase velocity so that, at the end of the simulation, the phase velocity of the scheme is within a certain neighbourhood of the exact phase velocity. Then we choose the discrete parameters Δt and h (or equivalently the number of points per shortest wavelength and the number of points per shortest period) in order to respect this criterion. We can then derive the numerical cost of the simulation and see which operator minimize it. This operator will be the cheapest to fulfill the precision criterion stated above.
Equation (4.3) is composed of two dispersion relations, one for the P-waves and one for the S-waves. Like in the continous problem the two types of wave decouple into two acoustic wave equation. In order to study the quality of the approximation we use the phase velocity C_φ and the relative error E_φ defined in (4.4).
To control the dispersion effect we require that at the final time of simulation T_{max}, the phase shift between the exact wave (pulsation ω), and the numerical wave (pulsation $\omega(k)$) is less than $\pi/2$. That is to say, the two waves are not shifted more than half a wavelength, so :

$$|\omega - \omega(k)|.T_{max} \leq \frac{\pi}{2} \tag{5.1}$$

Assuming that we propagated the wave on J wavelength $\lambda = 2\pi/|\vec{k}|$, the time of propagation is $T_{max} = \frac{J\lambda}{C}$. Then (5.1) becomes

$$|E_\varphi| \leq \frac{1}{4.J} \tag{5.2}$$

We see here the cumulative effect of dispersion. To respect the constraint (5.1) at the final time T_{max} the error on the phase velocity has to be inversely proportional to the propagation time.
The computational cost of the simulation is obviously linked to the precision on the phase error. It is defined as the total number of arithmetical operations and is given by :

$$Cost = N_L \times N_x^2 \times N_t$$

where N_L is the number of operations per points and time step for the scheme of order $2L$ in space, N_x is the number of points in one direction (the domain is a square), N_t is the number of time steps. Assuming that the size S of the domain in one direction is $S = C.T_{max}/2$ (i.e we propagate the wave for a round trip to the bottom of the domain), we can write the cost as a function of J:

$$Cost(J,H,G,L) = N_L.(\frac{J}{2H})^2.(\frac{J}{G})$$

$$H = \frac{h}{\lambda} \qquad G = \frac{\Delta t}{\lambda/C}$$

H is the inverse of number of points per wavelength λ, and G is the inverse of the number of points per period $T = \lambda/C$.
Now H and G are going to depend on J, since we impose (5.1), that is $|E_\varphi| \leq 1/4J$. The dispersion curves displayed above show that the discretization in time gives a positive contribution to the phase error, whereas the discretization in space gives a negative contribution to the phase error. This can be shown analytically by a development around $|\vec{k}|h = 0$ of the phase error function. With $k = |\vec{k}|$ and $k_1 = k\cos(\theta)$ $\quad k_2 = k\sin(\theta)$ we have

$$E_\varphi(k,\theta,h,\Delta t) = \frac{2}{kc\Delta t}\arcsin\left(\frac{C\Delta t}{h}\sqrt{A^2(k_1)+A^2(k_2)}\right) - 1$$

When $kh \longrightarrow 0$ we have :

$$E_\varphi(k,\theta,h,\Delta t) = \underbrace{\frac{1}{6}(\frac{\omega\Delta t}{2})^2}_{\geq 0} - \underbrace{\frac{\prod_{l=1}^{L}(2l-1)^2}{(2L+1)!}}_{\leq 0}(\cos(\theta)+\sin(\theta))(\frac{kh}{2})^{2L} + O((kh)^{2L+2}$$

Therefore to minimize the cost we look for the biggest time step Δt respecting the constraint $E_\varphi \leq 1/4J$, and the biggest space step h respecting the constraint $E_\varphi \geq -1/4J$, for all directions of propagation θ. We show below different choices of Δt and h, for $J = 100$ that is $|E_\varphi| \leq 2.5\ 10^{-3}$. via the CFL parameter $p = C\Delta t/h$ and $H = \lambda/h$ for different operators of order 2 (L=1), 4 (L=2), 8 (L=4).
For the (2,2) scheme (L=1), the space error forces us to choose $H = 5.25\ 10^{-2}$ (that is 19.05 points per wavelength) and $p = \sqrt{2}/2$ (that is 26,8 points per period) to respect the precision on the phase error.
For the (2,4) scheme (L=2), we improve the space error and we have $H = 17.5\ 10^{-2}$ (that is 5.7 points per wavelength). For the time error however the special case of the (2,2) scheme where the phase error is always negative, does not happen anymore. Thus we can not choose like before $p =$ PMAX. The bound on the phase error allows us $p = 0.282$ (that is 20.2 points per period). We have therefore greatly decreased the cost of the simulation already by a factor of nearly 15.
For the (2,8) scheme (L=4), we still improve the space error and we have $H = 27.2\ 10^{-2}$ (that is 3.7 points per wavelength). For the time error the bound on the phase error allows us $p = 0.150$ (that is 24.7 points per period). Here we do not have the best deal anymore since we do not gain in space and in time. We improve the space accuracy, therefore decreasing the number of points per wavelength, but the limiting order 2 in time shows up here and we do not decrease the number of points per period.

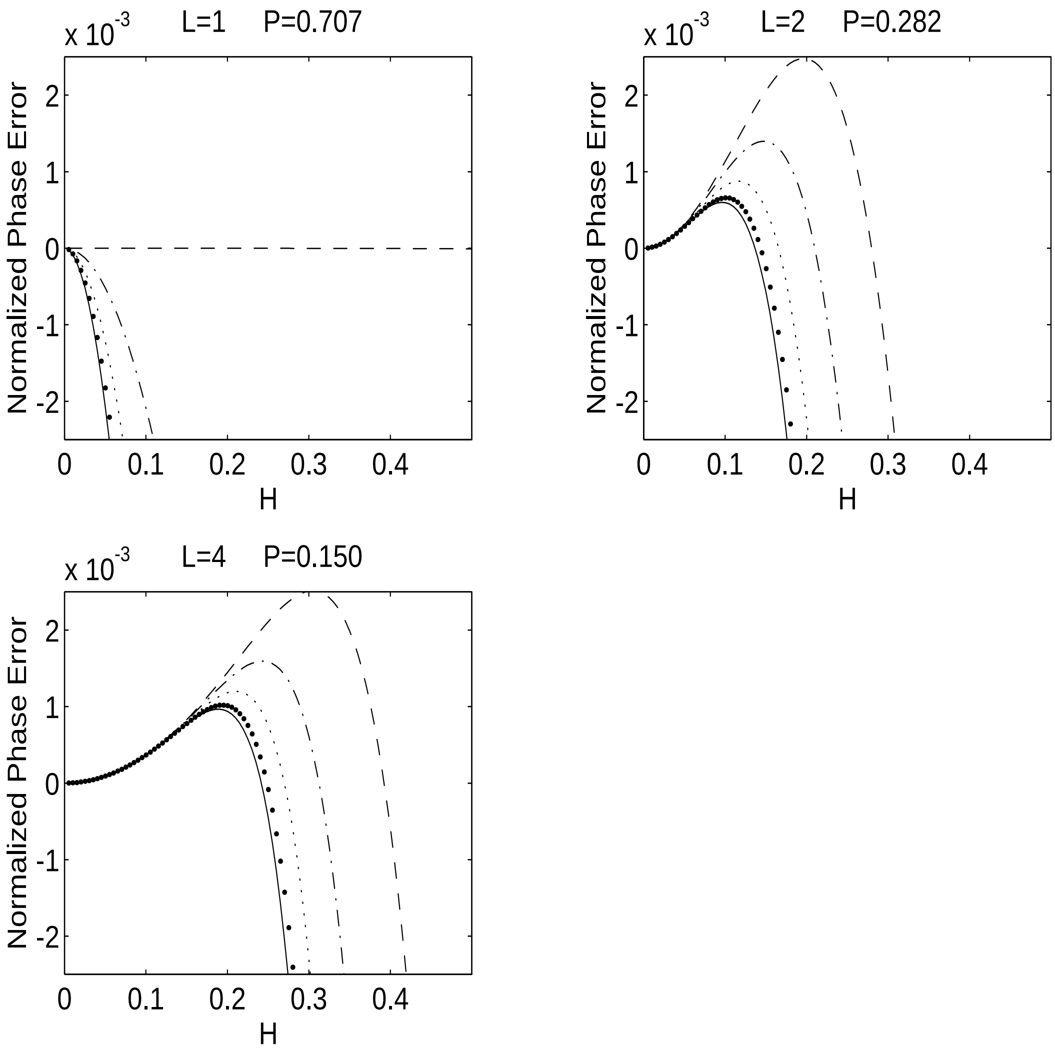

Figure 5.1: The normalised phase error for the maximum courant parameter $p = C.\Delta t/h$ respecting the constraint $E_\varphi \leq 2.5\ 10^{-3}$ as a function of a propagation angle θ for different operators. One can read on the curve the maximum admissible value of H respecting the constraint $E_\varphi \geq -2.5\ 10^{-3}$. (solid) $\theta = 0$ (point) $\theta = 10$ (dotted) $\theta = 20$ (dashdot) $\theta = 30$ (dashed) $\theta = 45$

For a given domain, that is a given J, it is possible to plot the functions $J \mapsto N_\lambda(J)$, $J \mapsto N_T(J)$. Therefore we get a table to choose the discrete parameters according to the medium size and the operator used.

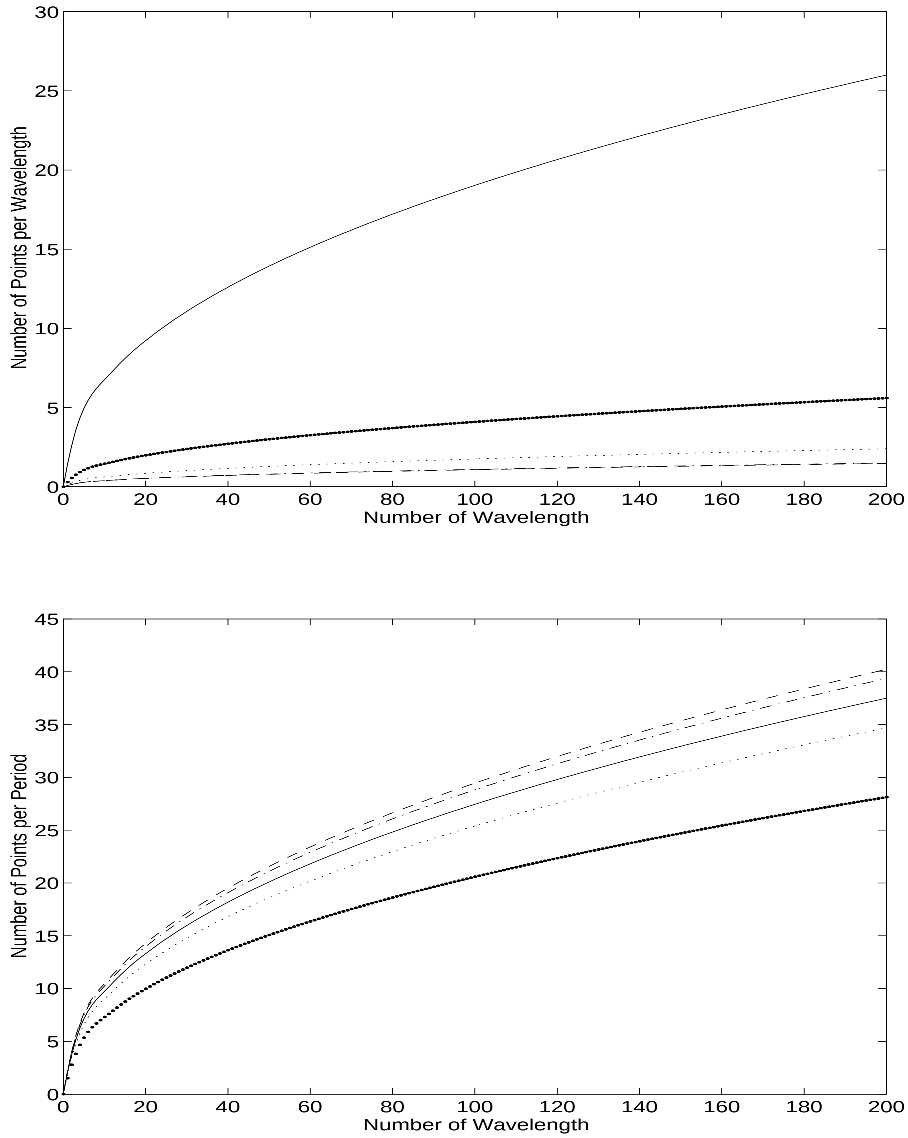

Figure 5.2: The number of points per wavelength and the number of points per period admissible to respect the constraint $|E_\varphi| \leq 1/4.J$.(solid) $L = 1$ (point) $L = 2$ (dotted) $L = 4$ (dashdot) $L = 8$ (dashed) $L = 10$

It is also interesting to plot the function $(J, L) \mapsto Cost(J, L)$. For an average geophysical medium, corresponding to the propagation of a hundred wavelength ($J = 100$) the function $L \mapsto Cost(100, L)/Cost(100, 1)$ shows the computational cost of the simulation compared to the cost of the (2,2) scheme. It shows that after a certain threshold corresponding at $L = 4$ (that is order 8 in space) the computational cost actually increases with the order of approximation in space.

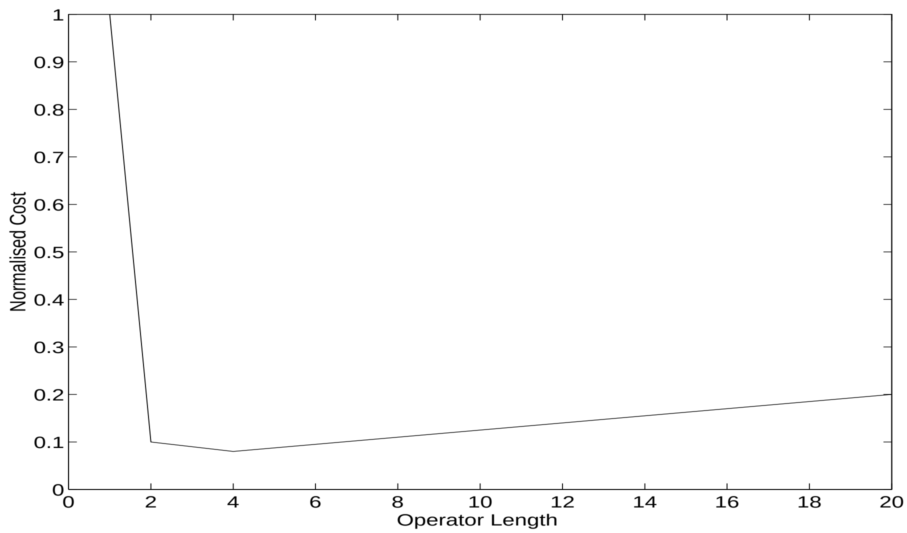

Figure 5.3: The normalised numerical cost of simulation respecting the constraint $|E_\varphi| \leq 1/4.J$ as a function of the order of the operator.

6 Discussion and Conclusions

The previous study shows that in order to control the phenomenon of dispersion in the numerical propagation of elastic waves, one must prescribe a number of points per wavelength and number of points per period, depending on the size of the domain. This is the cumulative effect of the dispersion error. Thus the widespread rule of thumb "10 points per wavelength" for a (2,2) scheme (cf Alford *et al*) is here to be understood in the sense "10 points per wavelength for an average geophysical medium" that is for the propagation of a hundred wavelength.

The study also shows that the computational cost is not a decreasing function of the order of approximation, and that after order 8 in space (with the finite difference operators introduced in this paper) the cost actually increases linearly with the order of approximation. The maximum gain for the computational cost is when we change from a (2,2) scheme to a (2,4) scheme. This justifies the endeavor for (2,4) schemes for the elastic wave equation (cf [4], [5]). It also proves that higher order schemes would not greatly improve the cost.

As a rule of thumb, for an average geophysical medium one must choose $N_\lambda = 6$, that is 6 points per shortest wavelength, and $N_T = 20$ that is 20 points per shortest period for the (2,4) scheme. The shortest wavelength is defined by the slowest velocity in the medium (the S-waves velocity) divided by the highest frequency in the source spectrum (which is the inverse of the shortest period T). Therefore

$$N_\lambda = \frac{C_{s,min} T}{h} \qquad\qquad N_T = \frac{T}{\Delta t}$$

and one can write the stability condition as

$$\frac{C_{p,max}\Delta t}{h} \leq C^{te} \iff \frac{N_\lambda}{N_T} \leq C^{te}\frac{C_{s,min}}{C_{p,max}}$$

Therefore we see that $N_\lambda = 6$ and $N_T = 20$, gives the parameter $p = C.\Delta t/h$. This shows that the stability criterion which gives the CFL number as the upper bound of the parameters $p = C.\Delta t/h$ is only indicative in practise.

The analysis presented here for the phase error velocity is transposable to the group velocity (cf [13]). The conclusions are qualitatively the same. The different functions giving the number of points per wavelegth, the number of points per period and the relative cost in the case of a criterion precision on the group velocity can be found in [14].

A natural extension of this paper would be to consider schemes of order 4 in time. This has been done in [14], and the results show that the phenomenal increase of computations to approximate a quantity like $Q = \frac{\partial}{\partial x}1/\rho\frac{\partial}{\partial x}\lambda\frac{\partial}{\partial x}1/\rho\frac{\partial u}{\partial x}$ is not balanced by the improvement of the time step. In fact we only need to approximate Q with a quantity of order two in space, to get order four in time and space. But when we approximate Q by ${}^tA_x^o(1/\rho(A_x^o(\lambda^t A_x^o(1/\rho(A_x^o)))))$ where A_x^o is the finite difference operator of order 2, we do not have stability results.

7 Appendix

7.1 Appendix 1

The coefficients β_l are defined by $\beta_l = \alpha_l/(2l-1)$. For consistency reasons α_l verify

$$\sum_{l=1}^{L}\alpha_l = 1$$

and to approximate the first derivative to order $2L$

$$\sum_{l=1}^{L}(2l-1)^{2p}\alpha_l = 0 \qquad p = 1..L$$

Solving this linear system in α_l gives :

$$\alpha_l = (-1)^{l+1}\frac{\prod_{m\neq l}^{L}(2m-1)}{\prod_{m\neq l}^{L}|(2m-1)^2-(2l-1)^2|}$$

7.2 Appendix 2

Lemma 7.1 *With*

$$c_1^2 = [\sum_{l=1}^{L}|\beta_l|]^{-1}[\max_{i,j}\sum_{l=1}^{L}|\beta_l|\frac{(\lambda+\mu)_{i+l-1/2,j}}{2\rho_{i,j}} + \frac{(\lambda+\mu)_{i-l+1/2,j}}{2\rho_{i,j}} + \frac{\mu_{i,j+l-1/2}}{2\rho_{i,j}} + \frac{\mu_{i,j-l+1/2}}{2\rho_{i,j}}]$$

we have

$$\forall u \in L^2_{oo} \qquad I_1 \le (\frac{4}{\Delta x^2} + \frac{4}{\Delta z^2}) \left(\sum_{l=1}^{L} |\beta_l| \right)^2 .c_1^2.(\rho u, u)_{oo}$$

Proof :

$$\begin{aligned}
I_1 &= \sum_{i,j} (\lambda+\mu)_{i+1/2,j} [\sum_{l=1}^{L} \frac{\beta_l}{\Delta x}(u_{i+l,j} - u_{i-l+1,j})]^2 \;\; \Delta x \, \Delta z \\
&+ \sum_{i,j} \mu_{i,j+1/2} [\sum_{l=1}^{L} \frac{\beta_l}{\Delta z}(u_{i,j+l} - u_{i,j-l+1})]^2 \;\; \Delta x \, \Delta z \\
I_1 &\le \sum_{l=1}^{L} |\frac{\beta_l}{\Delta x}| \sum_{i,j} (\lambda+\mu)_{i+1/2,j} \sum_{l=1}^{L} |\frac{\beta_l}{\Delta x}| (u_{i+l,j} - u_{i-l+1,j})^2 \;\; \Delta x \, \Delta z \\
&+ \sum_{l=1}^{L} |\frac{\beta_l}{\Delta z}| \sum_{i,j} \mu_{i,j+1/2} \sum_{l=1}^{L} |\frac{\beta_l}{\Delta z}| (u_{i,j+l} - u_{i,j-l+1})^2 \;\; \Delta x \, \Delta z \\
I_1 &\le 2[\sum_{l=1}^{L} |\frac{\beta_l}{\Delta x}|] \sum_{i,j} \sum_{l=1}^{L} |\frac{\beta_l}{\Delta x}| (\lambda+\mu)_{i+1/2,j} (u^2_{i+l,j} + u^2_{i-l+1,j}) \;\; \Delta x \, \Delta z \\
&+ 2[\sum_{l=1}^{L} |\frac{\beta_l}{\Delta z}|] \sum_{i,j} \sum_{l=1}^{L} |\frac{\beta_l}{\Delta z}| \mu_{i,j+1/2} (u^2_{i,j+l} + u^2_{i,j-l+1}) \;\; \Delta x \, \Delta z \\
I_1 &\le 4[\sum_{l=1}^{L} |\frac{\beta_l}{\Delta x}|][\sum_{i,j} \sum_{l=1}^{L} |\frac{\beta_l}{\Delta x}| \frac{(\lambda+\mu)_{i+l-1/2,j}}{2\rho_{i,j}} + \frac{(\lambda+\mu)_{i-l+1/2,j}}{2\rho_{i,j}}] \rho_{i,j} u^2_{i,j} \;\; \Delta x \, \Delta z \\
&+ 4[\sum_{l=1}^{L} |\frac{\beta_l}{\Delta z}|][\sum_{i,j} \sum_{l=1}^{L} |\frac{\beta_l}{\Delta z}| \frac{\mu_{i,j+l-1/2}}{2\rho_{i,j}} + \frac{\mu_{i,j-l+1/2}}{2\rho_{i,j}}] \rho_{i,j} u^2_{i,j} \;\; \Delta x \, \Delta z \\
I_1 &\le \frac{4}{\Delta x^2}[\sum_{l=1}^{L} |\beta_l|][\sum_{i,j} \sum_{l=1}^{L} |\beta_l| \frac{(\lambda+\mu)_{i+l-1/2,j}}{2\rho_{i,j}} + \frac{(\lambda+\mu)_{i-l+1/2,j}}{2\rho_{i,j}}](\rho u, u)_{oo} \\
&+ \frac{4}{\Delta z^2}[\sum_{l=1}^{L} |\beta_l|][\sum_{i,j} \sum_{l=1}^{L} |\beta_l| \frac{\mu_{i,j+l-1/2}}{2\rho_{i,j}} + \frac{\mu_{i,j-l+1/2}}{2\rho_{i,j}}](\rho u, u)_{oo} \\
I_1 &\le (\frac{4}{\Delta x^2} + \frac{4}{\Delta z^2})[\sum_{l=1}^{L} |\beta_l|][\max_{i,j} \sum_{l=1}^{L} |a_l| \frac{(\lambda+\mu)_{i+l-1/2,j}}{2\rho_{i,j}} + \frac{(\lambda+\mu)_{i-l+1/2,j}}{2\rho_{i,j}} \\
&+ \frac{\mu_{i,j+l-1/2}}{2\rho_{i,j}} + \frac{\mu_{i,j-l+1/2}}{2\rho_{i,j}}](\rho u, u)_{oo}
\end{aligned}$$

Lemma 7.2 *With*

$$\begin{aligned}
c_2^2 &= [\sum_{l=1}^{L} |\beta_l|]^{-1} [\max_{i,j} \sum_{l=1}^{L} |\beta_l| \frac{(\lambda+\mu)_{i+1/2,j+l}}{2\rho_{i+1/2,j+1/2}} + \frac{(\lambda+\mu)_{i+1/2,j-l}}{2\rho_{i+1/2,j+1/2}} \\
&\quad + \frac{\mu_{i+l,j+1/2}}{2\rho_{i+1/2,j+1/2}} + \frac{\mu_{i-l,j+1/2}}{2\rho_{i+1/2,j+1/2}}]
\end{aligned}$$

we have

$$\forall v \in L^2_{**} \qquad I_2 \leq (\frac{4}{\Delta x^2} + \frac{4}{\Delta z^2}) \left(\sum_{l=1}^{L} |\beta_l| \right)^2 .c_1^2.(\rho v, v)_{**}$$

Proof : Straightforward and totally parallel to lemma 1.

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